Fast Diffusion of a Mutant in Controlled Evolutionary Dynamics

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Abstract: In this paper we present a novel approach to model the diffusion of a mutant in a geographical network. In the last few years, this became a key issue in control of epidemics, being many diseases transmitted by intermediate hosts that could be substituted with genetically modified organisms (GMOs) with a mutation that prevents them from spreading the pathogen. The main strength of our model lies in making analytically tractable the estimation of the expected time needed for the mutant to replace the native species all over the geographical network. Notably, our main results consist in providing an upper-bound and two lower-bounds on this quantity, depending on the network topology. Their use is presented through some simple examples, for which Monte Carlo simulations corroborate our analytical results. Finally, we propose a non-trivial feedback control policy that, using few knowledge on the network topology and on the evolution of the dynamics, allows to substantially speed up the diffusion.

Keywords: Multi-agent systems; Control over networks

1. INTRODUCTION

The study of spreading processes on networks has been absolutely effective in improving our understanding in social sciences concerning how ideas, opinions and innovations diffuse as well as in widening our comprehension of spread of epidemics and mutations. Thanks to the increased awareness of the spreading mechanisms obtained through these studies, many improvements in controlling diffusion processes have been possible, with inestimable potential benefits for the society.

Notably, as epidemics are considered, the deep improvement in the understanding of their diffusion on a generic network, in Ganesh et al. (2005), paved the way in the last few years for the design and the analysis of control policies that have a direct applications to the health system. Among all we recall Borgs et al. (2010) and Drakopoulos et al. (2014), while a fairly complete survey of analysis and control of epidemics can be found in Nowzari et al. (2016). A well-known model for diffusion of opinions is the voter model, for which an extensive analysis in terms of long-run behavior and convergence time to consensus on lattices has been provided in Ligget (1985) and Frachebourg and Krapivsky (1996). Recently, more general game-based opinions models have been proposed and analyzed in Montanari and Saberi (2010) and Young (2011). From all these analysis, many control policies and applications have been proposed. Among the others, we cite an analytical study in Kempe et al. (2003) to maximize the spreading effect of the opinion of an individual, depending on its position in the network, and an analysis on the spread of hoaxes and false news in Tambuscio et al. (2015). For both these cases it is straightforward to comprehend the range of these results for possible applications in social sciences.

Another example of diffusion processes on network are the evolutionary dynamics presented in Lieberman et al. (2005) and analyzed, among the others, therein and in Broom et al. (2011). These models study the spread of mutants in a geographic area, analyzing the probability that the mutants diffuse widely in the area (the so called fixation probability) and, at least numerically, the duration of the spreading process depending on the topology of the network. One of the main applications is that of studying the effect of the introduction of a genetically modified organism (GMO) in a certain area, in order to replace a native species. This problem is a hot topic in epidemics control, since many epidemics are transmitted by intermediate hosts (e.g. dengue, malaria, and, more recently, Zika are Mosquito-borne diseases). In fact, in the last few years, many efforts have been done by researchers to create GMOs similar to the intermediate hosts that can not transmit the pathogen and some trials in which mutants are introduced in nature have been done with different outcomes. See Harris et al. (2011) and Carvalho et al. (2015). However, very few analytical results are available for these models. As a consequence of this lack of results in the analysis, at now, very few control policies for evolutionary dynamics have been considered and analyzed.

In this paper, inspired by Lieberman et al. (2005), we propose a new model for mutant diffusion which presents two basic different features: first, the diffusion process is
modeled through edge-based (instead of node-based as in the cited work) activation mechanisms; second, an explicit control action is incorporated. This change of prospective allows us to obtain, on one hand, new and strong analytical results, notably concerning the expected duration of the spreading process, that, in the literature, was usually only analyzed through extensive Monte Carlo simulations. On the other hand, feedback control strategies in this setting can also be considered and analyzed: a first attempt in this direction is proposed in a final section. We believe that the capability to incorporate and analyze control architecture in these diffusive models, will give remarkable insight to better plan the introduction of GMOs to achieve a faster and more efficient replacement of the disease-spreading species. Moreover, due to the generality of our model, this evolutionary interpretation has to be intended as one of the possible applications, besides more classical uses in social sciences to model opinion dynamics and diffusion of innovation.

2. THE MODEL AND OUR MAIN RESULTS

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a time-invariant undirected connected graph with $|\mathcal{V}| = n$ nodes representing locations of a geographical network and links $\{u, v\} \in \mathcal{E}$ representing adjacency of nodes $u$ and $v$. Let $\mathcal{N}_u$ denote the neighborhood of node $u$, i.e., the set of nodes adjacent to $u$. This notion can be extended defining the neighborhood of a subset $S \subseteq V$ as $\mathcal{N}_S = \bigcup_{u \in S} \mathcal{N}_u$. In order to study the spread of the mutants in the network, each node is given a state $X_u(t) \in \{0, 1\}$, representing whether at time $t \in \mathbb{R}^+$ mutants occupy location $u$ ($X_u(t) = 1$) or the native species does ($X_u(t) = 0$). Assembling all the states in a vector $X(t) \in \{0, 1\}^V$ we define the configuration of the system, that is the state variable of the spreading dynamics.

The system evolves according to two dynamics: an external control and the spreading process. The external control consists in the choice of a target set $M_0 \subset \mathcal{V}$ and in the introduction of mutants in its nodes with a rate $r$ (overall) throughout the spreading process. The set $M_0$ as well as the rate $r$ are, in the first part of the paper, kept constant in time. In a final section, we will instead consider when these two parameters are feedback control laws and thus time-varying. As the spreading process is concerned, a non-negative symmetric matrix $W \in \mathbb{R}_+^{V \times V}$ strongly adapted to $\mathcal{G}$ and a probability $\beta$ are fixed. Hereinafter, we refer to graph $\mathcal{G}$ as the weighted graph $\mathcal{G} = (\mathcal{V}, W)$. Each edge $\{u, v\}$ is associated with an independent Poisson clock with rate $w_{uv} := (W)_{uv}$, modeling the time the two species in nodes $u$ and $v$ come in touch. When the clock associated with the link $\{u, v\}$ clicks, if both the locations $u$ and $v$ are occupied by the same species, nothing happens. Otherwise, if the two species in $u$ and $v$ differs, then a conflict takes place and the winner occupies both location. Conflict are solved according to a stochastic law: the constant $\beta$ gives the probability for mutants to win a conflict.

We observe that, in the special case without external control, i.e. $r = 0$, and when all $w_{uv} = w$, $\beta = 1/2$ leads to an isothermal voter model, introduced in Liggett (1985), while $\beta \neq 1/2$ leads to a biased isothermal voter model, as in Ferreira (1990). Therefore our model encompasses and generalizes isothermal voter models (both fair and biased) through the inclusion of external control and heterogeneity of link activation rates. Moreover, an (homogeneous) agent-based version of our model without control, where clocks are associated with nodes instead of links, has been proposed in Lieberman et al. (2005) as an evolutionary model. Notice that isothermal models and homogeneous agent-based ones coincides on regular networks. However, as we already said in the introduction, the very few analytical results on evolutionary models are limited to the computation (or at least to the estimation) of the fixation probability (i.e. the probability that the whole network is eventually occupied by mutants, before the native species do) and to the analysis on some very specific network topologies, as in Rychtář and Stadler (2008) and in Ohtsuki and Nowak (2006), while convergence times are tackled only through extensive Monte Carlo simulations.

Finally, a different interpretation of our model consists in thinking the external control as the addition of a fictitious stubborn node $s$ with fixed state $X_s(t) = 1$, $\forall t \in \mathbb{R}^+$, linked to each one of the nodes of the target set $M_0$ with $w_{us} = w_{su} = r/\beta |M_0|$. Models with stubborn nodes have been deeply analyzed as the long-run behavior is considered, usually with consensus-like dynamics where a plurality of stubborn nodes with different state are involved, see Acemoglu et al. (2013). However, despite this interesting different point of view, the presence of a single stubborn node in our model poses different issues with respect to those usually analyzed in the literature.

In order to model the evolutionary advantage given to the modified species with respect to the native one, we consider $\beta \in (1/2, 1]$ and we will stick to the case $r > 0$, that models the presence of an external control on $M_0$.

Let us define the following notation, useful to give a formal definition of our model: $\delta^{(u)}$ denotes a vector of all zeros but a 1 in the $u$-th position and, accordingly, for any subset $S \subseteq \mathcal{V}$, $\delta(S) = \sum_{u \in S} \delta^{(u)}$, and $\mathds{1} = \delta^{(V)}$.

The dynamics described above induce a Markov jump process $X(t)$ on the configuration space $\{0, 1\}^V$, whose initial condition is the mutant-free pure configuration $\mathds{0}$. The only transitions that can take place from a generic state $X = \mathbf{x} = (x_1, \ldots, x_n)$ are the one to states that differs from $\mathbf{x}$ in a single entry. Their rates are, $\forall u \in \mathcal{V}$:

$$
\lambda^w_u(\mathbf{x}) = (1 - x_u) \left( \beta (W \mathbf{x})_u + \frac{r}{|M_0|} \delta^{(M_0)}_u \right) \quad \lambda^w_u(\mathbf{x}) = x_u (1 - \beta) |W(1 - \mathbf{x})|_u,
$$

where $\lambda^w_u(\mathbf{x})$ denotes the transition rate from $\mathbf{x}$ to $\mathbf{x} \pm \delta^{(u)}$.

From (1), being $r > 0$, it is straightforward that the pure configuration $\mathds{0}$, the one with the whole network occupied by mutants, is the only absorbing state of the process and it can be reached from any other state. Therefore, in the long run, mutants will almost surely occupy the whole network. However, in view of an application of this model to real-world situations, two main issues arise. At first, one can be interested in estimating the number of times

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1. This dynamics is in accordance with the procedure used to introduce GMO mosquitoes in trials and case studies, see Harris et al. (2011).

2. A matrix $W$ is said to be strongly adapted to a graph $G = (V, E)$ when $w_{uv} > 0 \iff \{u, v\} \in E$. 
the spreading process dies out and only the presence of the external control prevents the mutants to extinguish. As we will see in Section 4, this issue is strongly related to the computation of the fixation probability in Lieberman et al. (2005). The second issue is the estimation of the expected time required for the process from its inception to reach the absorbing state, named expected absorbing time, in formula

\[ \tau = \mathbb{E} \left[ \min_{t \in \mathbb{N}^+} \{t | X(t) = 1\} \right], \]

which is the main theoretical contribution of this work.

Notably, our contributions consist in the computation of an upper-bound on \( \tau \) (Theorem 4), depending on the topology of \( \mathcal{G} \), and a lower-bound (Theorem 7), strictly related with the choice of the external control. In the remaining part of this section we briefly state these two results. The tools used to tackle the problem and the detailed proofs will be extensively presented in Section 4.

In order to present our results, let us introduce some notion related to the conductance of a symmetric weighted graph, similarly to Goel et al. (2006).

**Definition 1.** Given a subset of nodes \( S \subseteq V \) of a weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \), its weighted boundary is defined as

\[ \mathcal{B}[S] = \sum_{u \in S} \sum_{w \notin S} w_{uw}. \]

Notice that, since \( \mathcal{W} \) is symmetric, then \( \mathcal{B}[S] = \mathcal{B}[\mathcal{V} \setminus S] \).

**Definition 2.** Given a symmetric weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \) with \( |\mathcal{V}| = n \), its conductance profile is a function \( \phi: \{1, \ldots, n-1\} \rightarrow \mathbb{R} \), defined as

\[ \phi(z) = \min_{S \subseteq \mathcal{V}, |S| = z} \mathcal{B}[S]. \]

**Definition 3.** Given a symmetric weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \) and a subset \( S \subseteq \mathcal{V} \), its S-conductance is defined as

\[ \bar{\phi}(S) = \min_{U \supseteq S} \mathcal{B}[U]. \]

We have the following upper-bound to the absorbing time.

**Theorem 4.** Given a weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \) with conductance profile \( \phi \), let

\[ \Phi = \sum_{z=1}^{n-1} \frac{1}{\phi(z)}. \]

Then it holds

\[ \tau \leq \frac{2\beta}{2\beta - 1} \Phi + \frac{\beta}{(2\beta - 1)r}. \]

On the other hand, a simple lower-bound is the following.

**Proposition 5.** Given a weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \) and a target set \( M_0 \), it holds \( \tau \geq \bar{\phi}(M_0)^{-1} \).

A more sophisticated lower-bound, useful to get tighter estimations in certain graphs, uses the notion of neighborhood monotone crusades, defined as follows.

**Definition 6.** Given a target set \( M_0 \) and a graph \( \mathcal{G} \), a neighborhood monotone crusade (NMC) controlled in \( M_0 \), denoted by \( \omega = (\omega_1, \ldots, \omega_{n+1}) \), is a sequence of subsets of \( V \) such that:

- \( \omega_1 = \emptyset \);
- \( \omega_i = \omega_{i-1} \cup \{v_i\} \), with \( v_i \in (\mathcal{N}_{\omega_{i-1}} \setminus M_0) \setminus \omega_{i-1} \);
- \( \omega_{n+1} = \mathcal{V} \).

Let \( \Omega_{\omega_{n+1}}^{(r)} \) be the set of all NMCs controlled in \( M_0 \).

**Theorem 7.** Given a weighted graph \( \mathcal{G} = (\mathcal{V}, \mathcal{W}) \) and a target set \( M_0 \), it holds

\[ \tau \geq \min_{\omega \in \Omega_{\omega_{n+1}}^{(r)}} \sum_{i=1}^{n} \frac{1}{|\omega_i| + r|M_0|^{-1}|M_0 \setminus \omega_i|}. \]

Section 3 presents application of the above bounds to some popular families of graphs (e.g. expanders, grids, barbel). Section 4 is the main technical part where all results are proven. Finally, Section 5 is devoted to the presentation and the analysis of a feedback control policy for his dynamics and to the discussion of the results obtained under this policy with an effective example, providing insights for future research seeking for an optimal control.

3. SOME EXAMPLES

In this section we present some examples of the application of our analytical bounds on well known topologies, commonly used in statistical physics, see Castellano et al. (2009). Notably, we analyze the spread of mutants on expander graphs (that can be considered as a fast-diffusive benchmark), cycle graphs, barbell graphs and square lattices, noticing that in some of these cases the upper-bound is (order-)tight to one of the two lower-bounds, in other cases this does not happen. Along with these analytical results, we present Monte Carlo numerical estimations of \( \tau \), over 200 simulations of the process. For the sake of simplicity, we set \( w_{uv} = w, \forall (u, v) \in \mathcal{E} \) (i.e. we consider an isothermal model), \( r = 1 \), and we choose the target set the be the singleton \( M_0 = \{1\} \).

3.1 Expander graphs

Let \( \mathcal{G} \) be an expander graph with \( n \) nodes. Expander graphs are characterized by having high conductance profile in the sense that \( \exists \gamma > 0 \) constant such that \( \phi(z) \geq \gamma \min \{z, n - z\} \). Complete graphs, Erdős-Rényi random graphs, small-world networks (Watts and Strogatz (1998)) and Ramamujan graphs (Lubotzky et al. (1988)) are well-known examples of expander graphs. Let \( \Delta \) be the maximum degree, then \( \forall S \subset \mathcal{V} \) it clearly holds \( \mathcal{B}[S] \leq \Delta w\|S\| \), which gives a natural bound on the solution of the minimization problem (8). Hence, Theorems 4 and 7 can be straightforward applied, obtaining the following (order-)tight bounds:

\[ \frac{1}{w\Delta} \ln n + 1 \leq \tau \leq \frac{2\beta}{(2\beta - 1)} \ln n + \frac{\beta}{2\beta - 1}. \]

Since in the large-scale case (i.e. when \( n \to \infty \)) most of the expander graphs are not bounded degree, a standard

![Fig. 1. Examples of the topologies analyzed in Section 3.](image-url)
choice to prevent the first denominator in (9) from blowing up and the expander condition on $\phi$ to be verified consists in defining a proper parametrization for $w$ (e.g. $w = \alpha/\Delta$).

We remark that expander graphs have more a theoretical interest than a practical one, since geographic networks usually do not belong to this family. However, we can use (9) as a benchmark for fast-diffusive topologies in large-scale networks, since its lower-bound clearly holds true for all topologies and the upper-bound, for expander graphs, is (order-)tight, leading to $\tau \in \Theta(\ln n)$.

3.2 Cycle graphs

Let $G = C_n$ be a cycle graph with $n$ nodes. Computation of the conductance profile and solution of (8) are straightforward: on the one hand it holds $\phi(z) = 2w$. On the other hand the minimum in (8) is realized by every NMC in $\Omega(M_0)$, having all $B[\omega|] = 2w$. This leads to the following (order-)tight bounds:

$$\frac{n - 1}{2w} + 1 \leq \tau \leq \frac{\beta}{(2\beta - 1)w} n + \frac{\beta}{2\beta - 1}$$ (10)

We observe that, for large scale networks, $\tau \in \Theta(n)$. Numerical simulations in Fig. 2 are consistent with our analytical result.

3.3 Barbell graphs

Let $G = B_n$ be a barbell graph with $n$ (even) nodes equally divided into two complete sub-communities linked by a single edge. Since, as $n \to \infty$, degrees are not bounded, we parametrize $w = \alpha/n$. Since the two sub-communities form complete graphs, the conductance profile is

$$\phi(z) = \begin{cases} \alpha z \left( \frac{1}{2} - \frac{z}{n} \right) & \text{if } z < n/2 \\ \alpha/n & \text{if } z = n/2 \\ \alpha(n - z) \left( \frac{z}{n} - \frac{1}{2} \right) & \text{if } z > n/2, \end{cases}$$ (11)

therefore $\Phi = (n + 4\ln(n/2))/\alpha$. On the other hand, Theorem 7 does not provide an (order-)tight bound, while Proposition 5 does, since $\phi(M_0) = n/\alpha$. Hence we obtain the following (order-)tight bounds, leading to $\tau \in \Theta(n)$:

$$\frac{n}{\alpha} \leq \tau \leq \frac{2\beta}{\alpha(2\beta - 1)}(n + 4\ln(n/2)) + \frac{\beta}{2\beta - 1}.$$ (12)

4. ANALYSIS OF THE PROCESS

In order to analyze the issues exposed in Section 2, we define a stochastic process

$$Z(t) := 1^TX(t),$$ (15)

counting the number of locations occupied by mutants. Moreover, we also define $M(t) = \{u | X_u(t) = 1\}$ as the set of nodes occupied by mutants at time $t$ and the boundary of the process,

$$B(t) := X(t)^TW(1 - X(t)).$$ (16)

We notice that $B(t) = B[M(t)]$.

In general, the process $Z(t)$, taking values in $S_n = \{0, 1, \ldots, n\}$, is not Markovian since its transition rates, governed by the clocks associated with the edges, depend on the whole process $X(t)$ and not only on $Z(t)$. However, each transition can only increase or decrease $Z(t)$ by 1. Thus, using (1), we can compute its increasing and decreasing rates.

In [1] the authors at first spread in the first line and in the first columns, then they invade all the other odd lines (columns) in increasing order, finally they occupy the remaining nodes.
Lemma 8. In order to prove the stochastic domination, we exhibit a coupling \( (Z(t), \tilde{Z}(t)) \) in which \( Z(t) \geq \tilde{Z}(t) \). This coupling comes naturally since, as noticed above, \( \lambda^+ \geq \lambda^+ \), while \( \lambda^- = \lambda^- \), similarly to Ganesh et al. (2005).

The structure of the transitions of process \( \tilde{Z}(t) \) can be seen in Fig. 5. We notice that \( \tilde{Z}(t) \) is still not Markovian, but all its transition rates depends on \( X(t) \) only through \( B(t) \), that will be very useful in our further analysis of the process.

At this stage, we prove a technical Lemma that will be used further on, combined with the stochastic domination from Lemma 8, to compute our estimation of \( \tau \).

**Fig. 4.** Transition graph of the continuous-time processes \( Z(t) \).

descending transition rates conditioned to \( X(t) = x = (x_1, \ldots, x_n) \). Notably, the increasing rate, that is
\[
\lambda^+(z|x) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}[Z(t+h) = Z(t)+1|X(t) = x] = \sum_{u \in V} \lambda^+(x)
= \beta(1-x)^T W x + \sum_{u \in V} (1-x_u) \frac{r}{|M_0|} \delta_u(M_0)
= \beta B(t) + \frac{r}{|M_0|} (1-x)^T \delta(M_0),
\]
depends on \( X(t) \) only through \( B(t) \) and through the state \( X_u(t) \) of the nodes \( u \in M_0 \). Similarly,
\[
\lambda^-(z|x) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}[Z(t+h) = Z(t)-1|X(t) = x] = (1-\beta)B(t),
\]
depends on \( X(t) \) only through \( B(t) \). The structure of the transitions of process \( Z(t) \) can be seen in Fig. 4.

4.1 A stochastic domination

The dependency of the process \( Z(t) \) on \( X(t) \) restricts the possibility to analyze it directly. We tackle this issue defining an ancillary stochastic process \( \tilde{Z}(t) \), obtained by dropping the dependence on the state of nodes in the target set \( M_0 \), whose transition rates now depend only on \( B(t) \) in a linear way. \( \tilde{Z}(t) \) is a stochastic process with the following increasing and decreasing transition rates from state \( \tilde{Z}(t) = z \), conditioned to \( B(t) = b \):
\[
\begin{align*}
\lambda^+(z|b) &= r & \text{if } z = 0 \\
\lambda^+(z|b) &= \beta b & \text{if } z = [1, \ldots, n-1] \\
\lambda^-(z|b) &= (1-\beta)b & \text{if } z = [1, \ldots, n-1].
\end{align*}
\]

Recalling that, given two stochastic processes \( Y(t) \) and \( \tilde{Y}(t) \) taking values in the same totally ordered set \( \mathcal{A} \), \( Y(t) \) stochastically dominates \( \tilde{Y}(t) \) (denoted by \( Y(t) \geq \tilde{Y}(t) \)) if and only if \( \mathbb{P}[Y(t) \geq a] \geq \mathbb{P}[\tilde{Y}(t) \geq a], \forall a \in \mathcal{A} \) and \( \forall t > 0 \). Then, we state the following Lemma.

**Lemma 8.** It holds \( Z(t) \geq \tilde{Z}(t) \).

**Proof.** In order to prove the stochastic domination, we exhibit a coupling \( (Z(t), \tilde{Z}(t)) \) in which \( Z(t) \geq \tilde{Z}(t) \). This coupling comes naturally since, as noticed above, \( \lambda^+ \geq \lambda^+ \), while \( \lambda^- = \lambda^- \), similarly to Ganesh et al. (2005).

The structure of the transitions of process \( \tilde{Z}(t) \) can be seen in Fig. 5. We notice that \( \tilde{Z}(t) \) is still not Markovian, but all its transition rates depends on \( X(t) \) only through \( B(t) \), that will be very useful in our further analysis of the process.

At this stage, we prove a technical Lemma that will be used further on, combined with the stochastic domination from Lemma 8, to compute our estimation of \( \tau \).

**Fig. 5.** The transition graph of the continuous-time process \( \tilde{Z}(t) \), conditioned to \( B(t) = b \).

**Fig. 6.** The transition graph of the embedded discrete-time process \( Y(k) = \tilde{Z}(T_k) \).

**Lemma 9.** Let \( N_z \) be the random variable counting the number of times the process \( \tilde{Z}(t) \) enters in state \( z \). Then, non-depending on \( n \), it holds
\[
\mathbb{E}[N_z] \leq \frac{\beta}{2\beta-1}.
\]
and, \( \forall z \in \mathcal{S}_n \setminus \{0, n\} \), it holds
\[
\mathbb{E}[N_z] \leq \frac{2\beta}{2\beta-1}.
\]

**Proof.** Let \( \{T_k\} \) be the set of random times at which the transitions of the process \( \tilde{Z}(t) \) occur. We can define the embedded discrete-time process \( Y(k) = \tilde{Z}(T_k) \), that evolves coupled with \( \tilde{Z}(t) \) and has a discrete time step for each transition of the continuous-time process. Differently form the continuous-time process \( Y(k) \) is a birth-death Markov chain, since in its transition probabilities we get rid of the dependence on the boundary \( B(t) \). Notably, the non-zero increasing (decreasing) probabilities of \( Y(k) \) are
\[
\begin{align*}
q^+(y) &= 1 & \text{if } z = 0, \\
q^+(y) &= \beta & \text{if } z \in [1, \ldots, n-1], \\
q^-(y) &= 1-\beta & \text{if } z \in [1, \ldots, n-1].
\end{align*}
\]
Moreover, being \( n \) an absorbing state, a self-loop with probability 1 is added to it. The process \( Y(k) \), whose structure is represented in Fig. 6, is a well known birth-death Markov chain. Notably, between two consecutive entrances in 0, it acts as a Moran process, extensively studied in Moran (1958). Therefore, from the relative literature, we know \( N_0 \) to be a geometrically distributed random variable with success probability \( (2\beta-1)/\beta \), whose expected value is upper-bounded by \( \beta/(2\beta-1) \).

To compute \( \mathbb{E}[N_z] \) we fix \( z \) and we define
\[
e_h = \mathbb{P}_h[\exists k \geq 1 : Y(k) = z],
\]
where \( \mathbb{P}_h[] := \mathbb{P}[]|Y(0) = h \), being the probability that the chain ever enters in node \( z \), starting from \( h \). Since the chain is almost surely absorbed in \( n \), it comes straightforward that \( e_h = 1 \) for all \( h < z \). Moreover, using again the fixation probability argument, we obtain
\[
e_{z+1} \leq 1 - \beta \frac{1}{\beta}.
\]
Finally, \( e_h \) satisfies Laplace discrete equation (see Levin et al. (2009)), that, combined with (24) and with the boundary condition \( e_{z-1} = 1 \), leads to
\[
e_z = (1-\beta)e_{z-1} + \beta e_{z+1} \leq 2(1-\beta).
\]
with failure probability $c_z \leq 2(1 - \beta)$. Therefore it holds $E[R_z] \leq (2\beta - 1)^{-1}$. In the end, $N_z = R_z + 1$, since the process starting in $0$ and ending in $n$, enters at least once in all the states, hence considering the expected value,
\[
E[N_z] \leq 1 + E[R_z] \leq 1 + \frac{1}{2\beta - 1} = \frac{2\beta}{2\beta - 1}.
\]
(26)
Notice that result in (26) does not depend on the specific $z$ chosen, therefore it holds true $\forall z \not= \{0, n\}$, completing the proof.

We observe that the upper-bound provided in Lemma 9 does not depend on the size of the network $n$.

4.2 Proof of Theorem 4

In this subsection we present the proof of Theorem 4, stating our upper-bound on $\tau$, that is actually a direct consequence of a more general result we prove in the following.

**Theorem 10.** Let suppose that, during the spreading process, it holds $B(t) \geq f(Z(t))$, $\forall t$, where $f$ is a positive function, and let
\[
F = \sum_{z=1}^{n-1} \frac{1}{f(z)}.
\]
(27)
Then it holds
\[
\tau \leq \frac{2\beta}{2\beta - 1} F + \frac{\beta}{(2\beta - 1)r}.
\]
(28)

**Proof.** Stochastic domination in Lemma 8 ensures that $\tau \leq \tilde{\tau}$, where
\[
\tilde{\tau} = \mathbb{E} \left[ \min_{t \in \mathbb{R}^+} \left\{ t : \tilde{Z}(t) = n \right\} \right].
\]
(29)
Therefore, in this proof, we will focus on $\tilde{\tau}$. Using the property of linearity of the expected value, $\tilde{\tau}$ can be expressed as the sum of the expected amounts of time the process $\tilde{Z}(t)$ spends in each one of the non-absorbing states. In formula $\tilde{\tau} = \sum_{z=0}^{n-1} E[T_z]$, where $T_z$ is the amount of time the process $\tilde{Z}(t)$ spends in state $z$. Let $S_z^i$ be the $i$-th sojourn-time in state $z$ (i.e. the time spent in $z$ the $i$-th time it enters in it). $S_z^i$ is an exponentially distributed random variable with parameter $B(t)$. Finally, denote by $t_i$ the time of the $i$-th entrance in state $z$ of the process $\tilde{Z}(t)$. Using Lemma 9 and the fact that $N_z$ is independent from the various $S_z^i$’s, we can now estimate, $\forall z = \{1, \ldots, n-1\}$,
\[
E[T_z] = E \left[ \sum_{i=1}^{N_z} E \left[ S_z^i \right] \right] = E \left[ \sum_{i=1}^{N_z} \frac{1}{B(t_i)} \right] \leq \frac{E[N_z]}{f(z)} \leq \frac{1}{(2\beta - 1)f(z)},
\]
(30)
while the expected amount of time the process spends in state 0 can similarly be computed noticing that all the sojourn-times in 0 are independent exponentially distributed random variables $S_0$ with parameter $r$. Therefore,
\[
E[T_0] = E \left[ \sum_{i=1}^{N_0} E \left[ S_0^i \right] \right] = E \left[ S_0 \right] E \left[ N_0 \right] \leq \frac{\beta}{(2\beta - 1)r}.
\]
(31)
Thus we complete the proof by computing,
\[
\tau \leq \tilde{\tau} = \sum_{z=0}^{n-1} E[T_z] \leq \frac{2\beta}{2\beta - 1} \sum_{z=0}^{n-1} \frac{1}{f(z)} + \frac{\beta}{(2\beta - 1)r}
\leq \frac{2\beta}{2\beta - 1} F + \frac{\beta}{(2\beta - 1)r}.
\]
(32)

**Proof.** [of Theorem 4] Let $\mathcal{G}$ have conductance profile $\phi$. For any $z \in \{1, \ldots, n\}$, for all subsets $S \subset V$ with $|S| = z$, it holds $\phi(z) \leq B[S]$. Therefore Theorem 10 can be applied with $f = \phi$, for which $F = \Phi$.

We observe that, in situations in which more informations about the evolution of the process is given, Theorem 10 might be used to obtain a tighter upper-bound on $\tau$. For example, in order to speed up the diffusion process, a control policy that guarantees $B(t)$ to be larger than a given function $f(Z(t))$ could be defined through a thoughtful choice of the target set $M_0$.

4.3 Proof of Theorem 7 and Proposition 5

In this subsection we present the proofs of our lower-bounds.

**Proof.** [of Proposition 5] In order to reach the absorbing state $1$, one of the out-going edges of $S \supseteq M_0$ has to activate. This event occurs according to an exponentially distributed random variable with rate $B[S]$. The bound comes straightforward by minimizing $B[S]$ over $S \supseteq M_0$.

Theorem 7 lies on the notion of NMC which differ from the one of monotone crusade used in Drakopoulos et al. (2014), since monotone crusades does not require the new added node to be in the neighborhood of some node already in the set.

**Proof.** [of Theorem 7] Let $\omega = (\omega_1, \ldots, \omega_k)$ be a feasible trajectory of the stochastic process $M(t)$, intended as the sequence of sets of mutants after each transition from the inception of the process to the entrance in the absorbing state. Let $\Omega$ be the set of all possible trajectories, $P(\omega)$ the probability that trajectory $\omega$ occurs and $T[\omega]$ the random variable modeling the absorbing time conditioned to the event that $M(t)$ follows the trajectory $\omega$. Then, for $\forall \omega \in \Omega \cap \mathbb{E}[T[\omega]]$ it clearly holds
\[
\tau = \sum_{\omega \in \Omega} P[\omega] \mathbb{E}[T[\omega]] \geq \mathbb{E}[T[\omega]].
\]
(33)
At this stage we want to show that, any trajectory $\omega \in \Omega$ can be coupled with a trajectory $\bar{\omega} \in \Omega^{(M_0)}_e$ such that $T[\bar{\omega}] \geq T[\omega]$. The coupling comes naturally: given the trajectory $\omega$ of the process $X(t)$, a process $\bar{X}(t)$ in which is set $\beta = 1$ (i.e mutants win all the conflicts) can be coupled to it and $\bar{\omega}$ is the respective trajectory. It is straightforward to see that $\bar{X}(t) \geq X(t)$ component-wise, leading to prove the stochastic domination. As a consequence $\bar{\omega} \in \Omega^{(M_0)}_e$.

To conclude the proof, the sojourn-time of process $M(t)$ in state $S$ is known to be exponentially distributed with rate $B[S] + r|S \cap M_0|/|M_0|$. Therefore, given a trajectory $\omega = (\omega_1, \ldots, \omega_k)$, the expected absorbing time conditioned to the event that $M(t)$ follows the trajectory $\omega$, $\mathbb{E}[T[\omega]]$, is given by the sum of the expected sojourn-times in
the states the process $M(t)$ passes through, that are $ω_1, \ldots, ω_k$. This summation is the objective function of the minimization problem (8).

5. FEEDBACK CONTROL POLICY

In this section we propose and discuss a feedback control policy for the dynamics. At first, we notice that it is useless to insert mutants in nodes already occupied by them, therefore we move the target set $M_0$ during the spreading process in such a way that the nodes in it have always state 0. Less trivially, we can also act on the control rate $r = r(t)$ by setting it as a function of the boundary of the process $B(t)$ and of the number of occupied nodes $Z(t)$ as follows

$$r(t) = \begin{cases} 
u_0 & \text{if } Z(t) = 0 \\ B(t)u(Z(t)) & \text{if } Z(t) = n \\ 0 & \text{else,} \end{cases} \tag{34}$$

where $u(z)$ is the actual control function that depends only on the number of occupied nodes $Z(t)$. Having a control rate that is proportional to the boundary of the process is intuitively important, since, from transition rates (17) and (18) we notice that the speed of the spreading process is proportional to the boundary $B(t)$. We notice that, if $u_0 = 0$, the mutant-free configuration is an absorbing state coinciding with the initial state of the process. Therefore, in order to spread the mutant, the condition $u_0 > 0$ has to be verified. Under this control policy, transition rates from state $Z(t) = z$ in (17) and (18) conditioned to $X(t) = x$, reads

$$\begin{cases} \lambda^+(z|x) = \lambda^+(z|B(t), u(t)) = B(t)(\beta + u(z)) \\ \lambda^-(z|x) = \lambda^-(z|B(t)) = (1 - \beta)B(t). \end{cases} \tag{35}$$

In this new setting, we want to estimate the absorbing time $τ$ to the state $X(t) = 1$ as well the expected cost $\mathbb{E}[J[r]]$ where

$$J[r] = \int r(t)dt \tag{36}$$

Since (35) depends on $x$ only through $B(t)$, we tackle the analysis of $Z(t)$ with arguments similar to those used in Section 4 to analyze $Z(t)$, yielding to the following results.

**Theorem 11.** Given a weighted graph $G = (V, W)$ with conductance profile $φ$ and control policy (34). Let

$$g(u(z)) = \frac{(3\beta - 1)u(z) + 2β^2}{(2β - 1)(\beta + u(z))(1 + u(z))}. \tag{37}$$

Then it holds

$$τ ≤ \frac{β}{(2β - 1)u_0} + \sum_{z=1}^{n-1} \frac{g(u(z))}{φ(z)}. \tag{38}$$

While, the expected cost of the policy can be upper-bounded by

$$\mathbb{E}[J[r]] ≤ \frac{β}{(2β - 1)u_0} + \sum_{z=1}^{n-1} g(u(z))u(z). \tag{39}$$

**Proof.** First, to prove (38), an argument similar to the one in Lemma 9 leads to upper-bound the expected number of entrances $N_z$ in each state of $Z(t)$ with (20) as state 0 is considered, and $∀z ∈ \{1, \ldots, n - 1\}$,

$$\mathbb{E}[N_z] ≤ \frac{(3\beta - 1)u(z) + 2β^2}{(2β - 1)(\beta + u(z))} = g(u(z))(1 + u(z)). \tag{40}$$

Notice that, if $u(z) = 0$, (40) and (21) coincide, while in general (21) upper-bounds (40). At this stage, we complete the proof following the arguments used in Theorems 10 and 4, considering that, named $t^*$ the moment $Z(t)$ enters in $z$ for the $i$-th time, the sojourn-time in that state is exponentially distributed with rate $B(t^*)(1 + u(z)) ≥ Φ(z)(1 + u(z))$. To prove (39), the sojourn-time in node $z$ at time $t^*$ is exponentially distributed with rate $B(t^*)(1 + u(z))$, while the rate of the control policy is $B(t^*)u(z)$, hence the total expected cost of the sojourn does not depend on the boundary of the process and is given by $u(z)/(1 + u(z))$. Similarly, the sojourn-time in node 0 is exponentially distributed with rate $u_0$ and also the rate of the control policy is $u_0$, hence the total expected cost of each sojourn in node 0 is 1. Finally, the result is obtained by summing over all the sojourns in all the non-absorbing states and substituting the expression of $g(u(z))$.

From these propositions, we notice that the upper-bound of the absorbing time in Proposition 38 is a function of the control policy $u(z)$ and of the conductance profile $φ$, so, ultimately, of the network topology. On the contrary, the cost of the control policy can be upper-bounded in Proposition 39 by a function that only depends on the control $u(z)$, and not on the topology. This very interesting remark, suggests and paves the way for the definition and the study of an optimization problem in which the absorbing time is minimized, subject to some constraints on $J[u]$, estimated from Proposition 39. We notice that, despite the very myopic choice of our control policy in which no optimization is done on the choice of $M_0$, we obtain an interesting control policy that can be used in optimization over networks, far beyond the usually analyzed mean field case, with few a-priori knowledge on the topology (only the conductance profile) and with a feedback function that requires only two observables, namely $Z(t)$ and $B(t)$.

5.1 Example of application on barbell graph

A simple example in which this feedback control policy provides an important improvement with respect to the standard control policy initially defined, is the case of a barbell graph. In Section 3.3 we proved that, under a fixed control policy ($M_0$ fixed and $r$ constant), $τ ∈ Θ(n)$, while the expected cost of that policy, is clearly proportional to the expected absorbing time, i.e. $\mathbb{E}[J[r]] ∈ Θ(n)$.

Consider a feedback control policy of type (34) with $u_0 > 0$ and

$$u(z) = \begin{cases} 0 & \text{if } z = n/2 \\ n & \text{else.} \end{cases} \tag{41}$$

Theorem 11 can be applied. As the time is considered we can compute (38) using the computations from Section 3.3, noticing that the only difference is in the component $z = n/2$. Hence we can write

$$τ ≤ \frac{β}{(2β - 1)u_0} + \frac{2β}{\alpha(2β - 1)} \left(Φ - φ \left(\frac{n}{2}\right)^{-1}\right) + \frac{β}{\alpha(2β - 1)(β + n)(1 + n)} \tag{42}$$

$$≤ \frac{β}{(2β - 1)u_0} + \frac{8β}{\alpha(2β - 1)}\ln \left(\frac{n}{2}\right) + \frac{2β}{\alpha(2β - 1)},$$
i.e. \( \tau \in \Theta(n \ln n) \). The expected cost can be bounded with (39), considering that all the terms in the summation are null but the one with \( z = n/2 \). Therefore
\[
\mathbb{E}[J] \leq \frac{\beta}{(2\beta - 1)} + \frac{(3\beta - 1)n + 2\beta^2n}{2\beta} \left(1 + \frac{1}{\beta} \right) \left(\frac{1}{\beta} - \frac{1}{\beta^2}\right). 
\]
We notice that, using our feedback control policy we greatly improve the speed of the mutants spread, guaranteeing the expected absorbing time to be \( \Theta(n \ln n) \) with an expected cost that is \( \Theta(1) \). On the contrary, using the fixed control, we can only have \( \tau \in \Theta(n) \) with an expected cost that is \( \Theta(n) \).

6. CONCLUSION

In this work, inspired by previous models for the diffusion of mutants, we have proposed a diffusive dynamical model (encompassing isothermal voter model) which allows an analytical treatment of evolutionary dynamics and incorporates exogenous control input mechanisms. With respect to previous literature, we deepen an analysis of the expected duration of the spreading process, providing two bounds (Theorems 4 and 7) that can be easily applied (see the examples in Section 3). Thereafter, we notice that the results obtained, in particular through the generalization of the upper-bound in Theorem 10, are robust to modifications of the control policy, paving the way for their use in future researches in which optimization of the control policy in order to speed up the spreading process is discussed. Notably, we are currently working on a refining of Theorem 4 for networks that can be partitioned in a set of communities having at least node of the target set in each of them, from this refining, control policy may pass through the detection of the communities in the network. Finally, we proposed a feedback control policy for which is possible to upper-bound both the duration of the spreading process (that depends both on the control and on the network topology) and the cost of the control policy (that does not depend on the topology), presenting an example for which this control policy works significantly better that the fixed one, paving the way for further research seeking for an optimal control policy.

REFERENCES


