FUNCTIONAL ANALYSIS LECTURE NOTES

CHAPTER 1. HILBERT SPACES

CHRISTOPHER HEIL

1. Elementary Properties of Hilbert Spaces

Notation 1.1. Throughout, $\mathbb{F}$ will denote either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. All vector spaces are assumed to be over the field $\mathbb{F}$.

Definition 1.2 (Semi-Inner Product, Inner Product). If $X$ is a vector space over the field $\mathbb{F}$, then a semi-inner product on $X$ is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ such that

(a) $\langle x, x \rangle \geq 0$ for all $x \in X$,
(b) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$, and
(b) Linearity in the first variable: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$.

Exercise 1.3. Immediate consequences are:

(a) Anti-linearity in the second variable: $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$.
(b) $\langle x, 0 \rangle = 0 = \langle 0, y \rangle$.
(c) $\langle 0, 0 \rangle = 0$.

Definition 1.4. If a semi-inner product $\langle \cdot, \cdot \rangle$ satisfies:

$\langle x, x \rangle = 0 \implies x = 0$,

then it is called an inner product, and $X$ is called an inner product space or a pre-Hilbert space.

Notation 1.5. There are many different standard notations for a semi-inner product, e.g.,

$\langle x, y \rangle = [x, y] = (x, y) = \langle x | y \rangle$,

to name a few. We shall prefer the notation $\langle x, y \rangle$.

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Exercise 1.6. Prove the following.

(a) The dot product $x \cdot y = x_1y_1 + \cdots + x_ny_n$ is an inner product on $\mathbb{C}^n$.

(b) If $w_1, \ldots, w_n \geq 0$ are fixed scalars, then the weighted dot product $\langle x, y \rangle = x_1w_1 + \cdots + x_nw_n$ is a semi-inner product on $\mathbb{C}^n$. It is an inner product if $w_i > 0$ for each $i$.

(c) If $A$ is an $n \times n$ positive semi-definite matrix ($Ax \cdot x \geq 0$ for all $x \in \mathbb{C}^n$), then $\langle x, y \rangle = Ax \cdot y$ is a semi-inner product on $\mathbb{C}^n$, and it is an inner product if $A$ is positive definite (meaning $Ax \cdot x > 0$ for all $x \neq 0$). The weighted dot product is just the special case where $A$ is diagonal.

(d) Show that if $\langle \cdot, \cdot \rangle$ is an arbitrary inner product on $\mathbb{C}^n$, then there exists a positive definite matrix $A$ such that $\langle x, y \rangle = Ax \cdot y$. Hint: Let $e_1, \ldots, e_n$ be the standard basis. Then $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ for each $k$, the mapping $x \mapsto \langle x, e_k \rangle$ is linear.

Exercise 1.7. Let $I$ be a countable index set (e.g., $I = \mathbb{N}$ or $\mathbb{Z}$). Let $w : I \to [0, \infty)$. Let $\ell^2_w = \ell^2_w(I)$ be the weighted $\ell^2$ space defined by

$$
\ell^2_w = \ell^2_w(I) = \left\{ x = (x_i)_{i \in I} : \sum_{i \in I} |x_i|^2 w(i) < \infty \right\}.
$$

Show that

$$
\langle x, y \rangle = \sum_{i \in I} x_i \overline{y}_i w(i)
$$

defines a semi-inner product on $\ell^2_w$. If $w(i) > 0$ for all $i$ then it is an inner product.

If $w(i) = 1$ for all $i$, then we simply call this space $\ell^2$.

The series defining $\langle x, y \rangle$ converges because of the Cauchy–Schwarz inequality.

Example 1.8. Let $(X, \Omega, \mu)$ be a measure space ($X$ is a set, $\Omega$ is a $\sigma$-algebra of subsets of of $X$, and $\mu : \Omega \to [0, \infty]$ is a countably additive measure). Define

$$
L^2(X) = \left\{ f : X \to \mathbb{F} : \int_X |f(x)|^2 d\mu(x) < \infty \right\},
$$

where we identify functions that are equal almost everywhere, i.e.,

$$
f = g \text{ a.e.} \iff \mu \{ x \in X : f(x) \neq g(x) \} = 0.
$$

Then

$$
\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu(x)
$$
defines an inner product on $L^2(X)$.

Other notations for $L^2(X)$ are $L^2(\mu)$, $L^2(X, \mu)$, $L^2(d\mu)$, $L^2(X, d\mu)$, etc.

The space $\ell^2_w(I)$ is a special case of $\ell^2(X)$, where $X = I$ and $\mu$ is a weighted counting measure on $I$. 
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Exercise 1.9. Every subspace of an inner product space is itself an inner product space (using the same inner product).

Hence every subspace of $\ell^2_w$ or $L^2(X)$ is also an inner product space. For example,

$$V = L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : f \text{ is continuous} \}$$

is a subspace of $L^2(\mathbb{R}^n)$ and hence is also an inner product space. List some other subspaces of some particular $\ell^2_w$ or $L^2(X)$ spaces.

Notation 1.10 (Associated Semi-Norm or Norm). If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $X$, then we will write

$$\|x\| = \langle x, x \rangle^{1/2}.$$  

We will see later that $\| \cdot \|$ defines a semi-norm on $X$, and that it is a norm on $X$ if $\langle \cdot, \cdot \rangle$ is an inner product on $X$.

Lemma 1.11 (Polar Identity). If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $X$, then for all $x, y \in X$ we have

$$\|x + y\|^2 = \|x\|^2 + 2 \text{Re} \langle x, y \rangle + \|y\|^2.$$  

Proof. Using the fact that $z + \bar{z} = 2 \text{Re} z$, we have

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$$

$$= \|x\|^2 + 2 \text{Re} \langle x, y \rangle + \|y\|^2. \quad \square$$

Theorem 1.12 (Cauchy-Bunyakowski-Schwarz Inequality). If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $X$, then

$$\forall x, y \in X, \quad |\langle x, y \rangle| \leq \|x\| \|y\|.$$  

Moreover, equality holds if and only if there exist scalars $\alpha, \beta \in \mathbb{F}$, not both zero, such that $\|\alpha x + \beta y\| = 0$.

Proof. If $x, y \in X$ and $\alpha \in \mathbb{F}$, then

$$0 \leq \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y \rangle$$

$$= \langle x, x \rangle - \alpha \langle y, x \rangle - \bar{\alpha} \langle x, y \rangle + \alpha \bar{\alpha} \langle y, y \rangle$$

$$= \|x\|^2 - \alpha \langle y, x \rangle - \bar{\alpha} \langle x, y \rangle + |\alpha|^2 \|y\|^2.$$  

Write $\langle y, x \rangle = be^{i\theta}$ where $b \geq 0$ and $\theta \in \mathbb{R}$ ($\theta = 0$ if $\mathbb{F} = \mathbb{R}$). Set $\alpha = e^{-it}t$ where $t \in \mathbb{R}$. Then for this $\alpha$ we have

$$0 \leq \|x\|^2 - e^{-it}t be^{i\theta} - e^{it}t be^{-i\theta} + t^2 \|y\|^2 = \|x\|^2 - 2bt + \|y\|^2 t^2.$$  

This is a quadratic polynomial in $t$. In order for this polynomial to be everywhere nonnegative, it can have at most one real root, which means that the discriminant must be $\leq 0$, i.e.,

$$(-2b)^2 - 4 \|y\|^2 \|x\|^2 \leq 0.$$
Hence
\[ |\langle x, y \rangle|^2 = |\langle y, x \rangle|^2 = b^2 \leq \|x\|^2 \|y\|^2. \]

Exercise: Supply the proof for the case of equality. \qed

**Corollary 1.13.** If \( \langle \cdot, \cdot \rangle \) is a semi-inner product on \( X \), then \( \|x\| = \langle x, x \rangle^{1/2} \) defines a semi-norm on \( X \), which means that:

(a) \( \|x\| \geq 0 \) for all \( x \in X \),
(b) \( \|\alpha x\| = |\alpha| \|x\| \) for all \( x \in X \) and \( \alpha \in \mathbb{F} \),
(c) Triangle Inequality: \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x, y \in X \).

If \( \langle \cdot, \cdot \rangle \) is an inner product on \( X \) then \( \| \cdot \| \) is a norm on \( X \), which means that in addition to (a)–(c) above, we also have:

(d) \( \|x\| = 0 \implies x = 0. \)

**Proof.** (a), (b), (d) Exercises.

(c) Using the Polar Identity, we have
\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \text{Re} \langle x, y \rangle + \|y\|^2 \\
\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \\
\leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 \\
= \left(\|x\| + \|y\|\right)^2. \]
\qed

**Definition 1.14** (Distance). Let \( \langle \cdot, \cdot \rangle \) be an inner product on \( X \). Then the distance from \( x \) to \( y \) in \( X \) is
\[
d(x, y) = \|x - y\|. \]

**Exercise 1.15.** \( d(\cdot, \cdot) \) defines a metric on \( X \).

**Definition 1.16.** Many of the results in this chapter are valid not only for inner product spaces, but for any space which possesses a norm. Assume that \( X \) is a vector space. A semi-norm on \( X \) is a function \( \| \cdot \| : X \to [0, \infty) \) such that statements (a)-(c) above hold. If, in addition, statement (d) holds, then \( \| \cdot \| \) is a norm, and \( X \) is called a normed space, normed linear space, or normed vector space.

**Definition 1.17** (Convergence). Let \( X \) be a normed linear space (such as an inner product space), and let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of elements of \( X \).
(a) We say that \( \{f_n\} \) converges to \( f \in X \), and write \( f_n \to f \), if
\[
\lim_{n \to \infty} \|f - f_n\| = 0,
\]
i.e.,
\[
\forall \varepsilon > 0, \ \exists N > 0 \text{ such that } n > N \implies \|f - f_n\| < \varepsilon.
\]
(b) We say that \( \{f_n\} \) is Cauchy if
\[
\forall \varepsilon > 0, \ \exists N > 0 \text{ such that } m, n > N \implies \|f_m - f_n\| < \varepsilon.
\]

**Exercise 1.18.** Let \( X \) be a normed linear space. Prove the following.

(a) Reverse Triangle Inequality: \( \|x\| - \|y\| \leq \|x - y\| \).

(b) Continuity of the norm: \( x_n \to x \implies \|x_n\| \to \|x\| \).

(c) Continuity of the inner product: If \( X \) is an inner product space then
\[
x_n \to x, \ y_n \to y \implies \langle x_n, y_n \rangle \to \langle x, y \rangle.
\]

(d) All convergent sequences are bounded, and the limit of a convergent sequence is unique.

(e) Cauchy sequences are bounded.

(f) Every convergent sequence is Cauchy.

(g) There exist inner product spaces for which not every Cauchy sequence is convergent.

**Definition 1.19** (Hilbert Space). An inner product space \( H \) is called a Hilbert space if it is complete, i.e., if every Cauchy sequence is convergent. That is,
\[
\{f_n\}_{n=1}^\infty \text{ is Cauchy in } H \implies \exists f \in H \text{ such that } f_n \to f.
\]
The letter \( H \) will always denote a Hilbert space.

**Definition 1.20** (Banach Space). A normed linear space \( X \) is called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent. We make no assumptions about the meaning of the symbol \( X \), i.e., it need not denote a Banach space. A Hilbert space is thus a Banach space whose norm is associated with an inner product.

**Example 1.21.** \( \mathbb{C}^n \), \( \ell_w^2 \) (\( w \) strictly positive), and \( L^2(X) \) are all Hilbert spaces (using the default inner products).

**Exercise 1.22.** Prove that \( \mathbb{C}^n \) and \( \ell_w^2 \) are Hilbert spaces.
Definition 1.23 (Convergent Series). Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of elements of a normed linear space \( X \). Then the series \( \sum_{n=1}^{\infty} f_n \) converges and equals \( f \in X \) if the partial sums \( s_N = \sum_{n=1}^{N} f_n \) converge to \( f \). That is,

\[
\|f - s_N\| = \left\| f - \sum_{n=1}^{N} f_n \right\| \to 0 \quad \text{as } N \to \infty.
\]

Exercise 1.24. Let \( X \) be an inner product space, and suppose that the series \( \sum_{n=1}^{\infty} f_n \) converges in \( X \). Show that if \( g \in X \), then

\[
\langle \sum_{n=1}^{\infty} f_n, g \rangle = \sum_{n=1}^{\infty} \langle f_n, g \rangle.
\]

Note that this is NOT an immediate consequence of the definition of inner product. You must use both the fact that the inner product is linear in the first variable AND the continuity of the inner product (Exercise 1.18) to prove this result.

Exercise 1.25 (Absolutely Convergent Series). Let \( X \) be a Banach space and let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of elements of \( X \). Prove that if \( \sum_{n=1}^{\infty} \|f_n\| < \infty \) then the series \( \sum_{n=1}^{\infty} f_n \) converges in \( X \). We say that such a series is absolutely convergent.

Hint: You must show that the sequence of partial sums \( \{s_N\} \) converges. Since \( X \) is a Banach space, you just have to show that this sequence is Cauchy.

Exercise 1.26 (Unconditionally Convergent Series). Let \( X \) be a Banach space and let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of elements of \( X \). The series \( f = \sum_{n=1}^{\infty} f_n \) is said to converge unconditionally if every rearrangement of the series converges. That is, \( f = \sum_{n=1}^{\infty} f_n \) converges unconditionally if for each bijection \( \sigma: \mathbb{N} \to \mathbb{N} \) the series

\[
\sum_{n=1}^{\infty} f_{\sigma(n)}
\]

converges. It can be shown that in this case, the series will converge to \( f \), i.e., \( f = \sum_{n=1}^{\infty} f_{\sigma(n)} \) for every permutation \( \sigma \).

Prove that if a series \( f = \sum_{n=1}^{\infty} f_n \) converges absolutely, then it converges unconditionally. In finite dimensions the converse is true, but we will see later that the converse fails in infinite dimensions (see Exercise 4.16).

Proposition 1.27. Let \( X \) be a Banach space and let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of elements of \( X \). Then the series \( f = \sum_{n=1}^{\infty} f_n \) converges unconditionally if and only if it converges with respect to the net of finite subsets of \( \mathbb{N} \), i.e., if

\[
\forall \varepsilon > 0, \quad \exists \text{ finite } F_0 \subseteq \mathbb{N} \text{ such that } \forall \text{ finite } F \supseteq F_0, \quad \left\| f - \sum_{n \in F} f_n \right\| < \varepsilon.
\]
Definition 1.28 (Topology). Let $X$ be a normed linear space.

(a) The open ball in $X$ centered at $x \in X$ with radius $r > 0$ is
\[ B(x, r) = B_r(x) = \{ y \in X : \|x - y\| < r \} . \]

(b) A subset $U \subseteq X$ is open if
\[ \forall x \in U, \quad \exists r > 0 \text{ such that } B(x, r) \subseteq U. \]

(c) A subset $F \subseteq X$ is closed if $X \setminus F$ is open.

Definition 1.29 (Limit Points, Closure, Density). Let $X$ be a normed linear space and let $A \subseteq X$.

(a) A point $f \in A$ is called a limit point of $A$ if there exist $f_n \in A$ with $f_n \neq f$ such that $f_n \to f$.

(b) The closure of $A$ is the smallest closed set $\bar{A}$ such that $A \subseteq \bar{A}$. Specifically,
\[ \bar{A} = \bigcap \{ F \subseteq X : F \text{ is closed and } F \supseteq A \} . \]

(c) We say that $A$ is dense in $X$ if $\bar{A} = X$.

Exercise 1.30. (a) The closure of $A$ equals the union of $A$ and all limit points of $A$:
\[ \bar{A} = A \cup \{ x \in X : x \text{ is a limit point of } A \} = \{ z \in X : \exists y_n \in A \text{ such that } y_n \to z \} . \]

(b) If $X$ is a normed linear space, then the closure of an open ball $B(x, r)$ is the closed ball $\bar{B}(x, r) = \{ y \in X : \|x - y\| \leq r \} . \]

(c) Prove that $A$ is dense if and only if
\[ \forall x \in X, \quad \forall \varepsilon > 0, \quad \exists y \in A \text{ such that } \|x - y\| < \varepsilon . \]

Example 1.31. The set of rationals $\mathbb{Q}$ is dense in the real line $\mathbb{R}$.

Proposition 1.32. Let $X$ be a normed linear space and let $F \subseteq X$. Then $F$ is closed $\iff$ $F$ contains all its limit points.

Proof. $\Rightarrow$. Suppose that $F$ is closed but that there exists a limit point $f$ that does not belong to $F$. By definition, there must exist $f_n \in F$ such that $f_n \to f$. However, $f \in X \setminus F$, which is open, so there exists some $r > 0$ such that $B(f, r) \subseteq X \setminus F$. Yet there must exist some $f_n$ with $\|f - f_n\| < r$, so this $f_n$ will belong to $X \setminus F$, which is a contradiction.

$\Leftarrow$. Exercise. \qed
Exercise 1.33. In finite dimensions, all subspaces are closed sets. This is not true in infinite dimensions.

(a) Prove that 
\[ c_{00} = \{ x = (x_1, \ldots, x_N, 0, 0, \ldots) : N > 0, x_1, \ldots, x_N \in \mathbb{F} \} \]
is a subspace of \( \ell^2(\mathbb{N}) \) that is not closed. Prove that \( c_{00} \) is dense in \( \ell^2(\mathbb{N}) \).

(b) Prove that 
\[ c_0 = \{ x = (x_k)_{k=1}^{\infty} : \lim_{k \to \infty} x_k = 0 \} \]
is a dense subspace of \( \ell^2(\mathbb{N}) \).

(c) Prove that \( C_c(\mathbb{R}) \), the space of continuous, compactly supported functions on \( \mathbb{R} \), is a subspace of \( L^2(\mathbb{R}) \) that is dense and not closed. The support of a function \( f : \mathbb{R} \to \mathbb{C} \) is the closure in \( \mathbb{R} \) of the set \( \{ x \in \mathbb{R} : f(x) \neq 0 \} \). Thus a function is compactly supported if nonzero only within a bounded subset of \( \mathbb{R} \).

(d) Let \( E \) be a (Lebesgue) measurable subset of \( \mathbb{R}^n \). Let \( M = \{ f \in L^2(\mathbb{R}^n) : \text{supp}(f) \subseteq E \} \). Prove that \( M \) is a closed subspace of \( L^2(\mathbb{R}^n) \).

Proposition 1.34. Let \( H \) be a Hilbert space and let \( M \) be a subspace of \( H \). Then \( M \) is itself a Hilbert space (using the inner product from \( H \)) if and only if \( M \) is closed.

Let \( X \) be a Banach space and let \( M \) be a subspace of \( X \). Then \( M \) is itself a Banach space (using the norm from \( X \)) if and only if \( M \) is closed.

Proof. \( \Rightarrow \). Suppose that \( M \) is a Hilbert space. Let \( f \) be any limit point of \( M \), i.e., suppose \( f_n \in M \) and \( f_n \to f \). Then \( \{ f_n \} \) is a convergent sequence in \( H \), and hence is Cauchy in \( H \). Since each \( f_n \) belongs to \( M \), it is also Cauchy in \( M \). Since \( M \) is a Hilbert space, \( \{ f_n \} \) must therefore converge in \( M \), i.e., \( f_n \to g \) for some \( g \in M \). However, limits are unique, so \( f = g \in M \). Therefore \( M \) contains all its limit points and hence is closed.

\( \Leftarrow \). Exercise. \qed

Exercise 1.35. Find examples of inner product spaces that are not Hilbert spaces.

Remark 1.36. In constructing the examples for the preceding exercise, you will probably look for examples of inner product spaces \( X \) which are subspaces of a larger Hilbert space \( H \). Is every inner product space a subspace of a larger Hilbert space? The answer is yes, it is also possible to construct a Hilbert space \( H \supseteq X \), called the completion of \( X \).

Theorem 1.37 (Parallelogram Law). If \( X \) is an inner product space, then
\[ \forall f, g \in X, \quad \| f + g \|^2 + \| f - g \|^2 = 2(\| f \|^2 + \| g \|^2). \]

Proof. Exercise. \qed
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Exercise 1.38. Show that $\ell^p$ for $p \neq 2$ and $L^p(\mathbb{R})$ for $p \neq 2$ are not inner product spaces under the default norms (show that the Parallelogram Law fails).

Exercise 1.39. Suppose that $X$ is a Banach space over the complex field $\mathbb{C}$, and the norm $\| \cdot \|$ of $X$ satisfies the Parallelogram Law. Prove that

$$\langle f, g \rangle = \frac{1}{4} \left( \| f + g \|^2 - \| f - g \|^2 + i\| f + ig \|^2 - i\| f - ig \|^2 \right)$$

is an inner product on $X$, and that $\| f \|^2 = \langle f, f \rangle$.

2. Orthogonality

Definition 2.1. Let $H$ be a Hilbert space.

(a) Two vectors $f, g \in H$ are orthogonal (written $f \perp g$) if $\langle f, g \rangle = 0$.

(b) A sequence of vectors $\{f_i\}_{i \in I}$ is an orthogonal sequence if $\langle f_i, f_j \rangle = 0$ whenever $i \neq j$.

(c) A sequence of vectors $\{f_i\}_{i \in I}$ is an orthonormal sequence if it is orthogonal and each vector is a unit vector, i.e.,

$$\langle f_i, f_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Exercise 2.2. (a) If $\{f_i\}_{i \in I}$ is an orthogonal sequence of nonzero vectors, then it is finitely linearly independent, i.e., every finite subset is linearly independent.

(b) Let $I = \mathbb{N}$. Define $e_n = (\delta_{mn})_{m=1}^{\infty}$, i.e., $e_n$ is the sequence having a 1 in the $n$th component and zeros elsewhere. Show that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $\ell^2$ (called the standard basis of $\ell^2$).

(c) Let $X = [0, 1]$ and let $\mu$ be Lebesgue measure. Define $e_n(x) = e^{2\pi inx}$. Show that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal sequence in $L^2[0, 1]$.

Theorem 2.3 (Pythagorean Theorem). If $f_1, \ldots, f_n \in H$ are orthogonal, then

$$\left\| \sum_{k=1}^{n} f_k \right\|^2 = \sum_{k=1}^{n} \| f_k \|^2.$$

Proof. Exercise. \qed

Definition 2.4 (Convex Set). Let $X$ be a vector space and $K \subseteq X$, then $K$ is convex if $x, y \in K$, $0 \leq t \leq 1$ implies $tx + (1 - t)y \in K$.

Thus, the entire line segment between $x$ and $y$ is contained in $K$ (including the midpoint $\frac{1}{2}x + \frac{1}{2}y$ in particular).
Exercise 2.5. (a) Every subspace of a vector space $X$ is convex.

(b) If $X$ is a normed linear space, then open and closed balls in $X$ are convex.

Definition 2.6. Let $X$ be a normed linear space and let $A \subseteq H$. The distance from a point $x \in H$ to the set $A$ is
\[
dist(x, A) = \inf\{\|x - y\| : y \in A\}.
\]

Theorem 2.7 (Closest Point Property). If $H$ is a Hilbert space and $K$ is a nonempty, closed, convex subset of $H$, then given any $h \in H$ there exists a unique point $k_0 \in K$ that is closest to $h$. That is, there is a unique point $k_0 \in K$ such that
\[
\|h - k_0\| = \text{dist}(h, K) = \inf\{\|h - k\| : k \in K\}.
\]

Proof. Let
\[
d = \text{dist}(h, K) = \inf\{\|h - k\| : k \in K\}.
\]
By definition, there exist $k_n \in K$ such that $\|h - k_n\| \to d$, and furthermore we have $d \leq \|h - k_n\|$ for each $n$. Therefore, if we fix any $\varepsilon > 0$ then we can find an $N$ such that
\[
n > N \implies d^2 \leq \|h - k_n\|^2 \leq d^2 + \varepsilon^2.
\]
By the Parallelogram Law,
\[
\|(h - k_n) - (h - k_m)\|^2 + \|(h - k_n) + (h - k_m)\|^2 = 2\left(\|h - k_n\|^2 + \|h - k_m\|^2\right).
\]
Hence,
\[
\left\|\frac{k_m - k_n}{2}\right\|^2 = \frac{1}{4}\|(h - k_n) - (h - k_m)\|^2 = \frac{\|h - k_n\|^2}{2} + \frac{\|h - k_m\|^2}{2} - \left\|\frac{k_m + k_n}{2}\right\|^2.
\]
However, $\frac{k_m + k_n}{2} \in K$ since $K$ is convex, so $\|h - \frac{k_m + k_n}{2}\| \geq d$ and therefore
\[
-\left\|\frac{k_m + k_n}{2}\right\|^2 \leq -d^2.
\]
Also, if $m, n > N$ then
\[
\|h - k_n\|^2, \|h - k_m\|^2 \leq d^2 + \varepsilon^2.
\]
Therefore,
\[
\left\|\frac{k_m - k_n}{2}\right\|^2 \leq \frac{d^2 + \varepsilon^2}{2} + \frac{d^2 + \varepsilon^2}{2} - d^2 = \varepsilon^2.
\]
So, $\|k_m - k_n\| \leq 2\varepsilon$ for all $m, n > N$, which says that the sequence $\{k_n\}$ is Cauchy. Since $H$ is complete, this sequence must converge, i.e., $k_n \to k_0$ for some $k_0 \in H$. But $k_n \in K$ for all $n$ and $K$ is closed, so we must have $k_0 \in K$.

Since $h - k_n \to h - k_0$, we have
\[
\|h - k_0\| = \lim_{n \to \infty} \|h - k_n\| = d,
\]
and thus $\|h - k_0\| \leq \|h - k\|$ for every $k \in K$.

Exercise: Prove that $k_0$ is the unique point closest to $h$ by assuming that there exists another point $k'_0$ that satisfies $\|h - k'_0\| \leq \|h - k\|$ for every $k \in K$, and derive a contradiction. Draw a picture and think about the midpoint $\frac{k_0 + k'_0}{2}$. \qed
**Notation 2.8** (Notation for Closed Subspaces). Since we will often deal with closed subspaces of a Hilbert space, we declare that the notation

\[ M \subseteq H \]

means that \( M \) is a closed subspace of the Hilbert space \( H \). The letter \( H \) will always denote a Hilbert space.

**Theorem 2.9.** Let \( M \subseteq H \), and fix \( h \in H \). Then the following statements are equivalent.

(a) \( h = p + e \) where \( p \) is the (unique) point in \( M \) closest to \( h \).

(b) \( h = p + e \) where \( p \in M \) and \( e \perp M \) (i.e., \( e \perp f \) for every \( f \in M \)).

**Proof.** (a) \( \Rightarrow \) (b). Let \( p \) be the (unique) point in \( M \) closest to \( h \), and let \( e = p - h \). Choose any \( f \in M \). We must show that \( \langle f, e \rangle = 0 \).

Since \( M \) is a subspace, \( p + \lambda f \in M \) for any scalar \( \lambda \in \mathbb{F} \). Hence,

\[
\|h - p\|^2 \leq \|h - (p + \lambda f)\|^2 = \|(h - p) - \lambda f\|^2 \\
= \|h - p\|^2 - 2 \Re \langle \lambda f, h - p \rangle + |\lambda|^2 \|f\|^2 \\
= \|h - p\|^2 - 2 \Re \lambda \langle f, h - p \rangle + |\lambda|^2 \|f\|^2.
\]

Therefore,

\[
\forall \lambda \in \mathbb{F}, \quad 2 \Re \lambda \langle f, e \rangle \leq |\lambda|^2 \|f\|^2.
\]

If we consider \( \lambda = t > 0 \), then we can divide through by \( t \) to get

\[
\forall t > 0, \quad 2 \Re \langle f, e \rangle \leq t \|f\|^2.
\]

Letting \( t \to 0^+ \), we conclude that \( \Re \langle f, e \rangle \leq 0 \). Similarly, taking \( \lambda = t < 0 \) and letting \( t \to 0^- \), we obtain \( \Re \langle f, e \rangle \geq 0 \).

If \( \mathbb{F} = \mathbb{R} \) then we are done. If \( \mathbb{F} = \mathbb{C} \), then we take \( \lambda = it \) with \( t > 0 \) and \( \lambda = it \) with \( t < 0 \) to show that \( \Im \langle f, e \rangle = 0 \) as well.

(b) \( \Rightarrow \) (a). Suppose that \( h = p + e \) where \( p \in M \) and \( e \perp M \). Choose any \( f \in M \). Then \( p - f \in M \), so \( h - p = e \perp p - f \). Therefore, by the Pythagorean Theorem,

\[
\|h - f\|^2 = \|(h - p) + (p - f)\|^2 = \|h - p\|^2 + \|p - f\|^2 \geq \|h - p\|^2.
\]

Hence \( p \) is the point in \( M \) that is closest to \( h \).

**Definition 2.10** (Orthogonal Projection). Let \( M \subseteq H \).

(a) For each \( h \in H \), the point \( p \in M \) closest to \( H \) is called the orthogonally projection of \( h \) onto \( M \).

(b) Define \( P: H \to H \) by \( Ph = p \), the orthogonal projection of \( h \) onto \( M \). Then the operator \( P \) is called the orthogonal projection of \( H \) onto \( M \). Note that by definition and Theorem 2.9, \( h = Ph + e \) where \( e = h - Ph \perp M \).
Definition 2.11 (Orthogonal Complement). Let $A$ be a subset (not necessarily a subspace) of a Hilbert space $H$. The orthogonal complement of $A$ is

$$A^\perp = \{ x \in H : x \perp A \} = \{ x \in H : \langle x, y \rangle = 0 \forall y \in A \}.$$ 

Exercise 2.12. (a) If $A \subseteq H$, then $A^\perp$ is a closed subspace of $H$ (even if $A$ itself is not).
(b) $\{0\}^\perp = H$ and $H^\perp = \{0\}$.

Definition 2.13 (Notation for Operators). Let $X, Y$ be normed linear spaces. Let $T : X \to Y$ be a function (= operator = transformation). We write either $T(x)$ or $Tx$ to denote the image of an element $x \in X$.

(a) $T$ is linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for every $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$.
(b) $T$ is injective if $T(x) = T(y)$ implies $x = y$.
(c) The kernel or nullspace of $T$ is $\ker(T) = \{ x \in X : T(x) = 0 \}$.
(d) The range of $T$ is $\operatorname{range}(T) = \{ T(x) : x \in X \}$.
(e) $T$ is surjective if $\operatorname{range}(T) = Y$.
(f) We use the notation $I$ or $I_H$ to denote the identity map of a Hilbert space $H$ onto itself.

Exercise 2.14. Show that if $T : X \to Y$ is linear and continuous, then $\ker(T)$ is a closed subspace of $X$ and that $\operatorname{range}(T)$ is a subspace of $Y$. Must $\operatorname{range}(T)$ be a closed subspace?

Theorem 2.15 (Properties of Orthogonal Projections). Let $M \leq H$, and let $P$ be the orthogonal projection of $H$ onto $M$. Then the following statements hold.

(a) $h - Ph \perp M$ for every $h \in H$.
(b) $h - Ph \in M^\perp$ for every $h \in H$.
(c) $\| h - Ph \| = \operatorname{dist}(h, M)$ for every $h \in H$.
(d) $P$ is a linear transformation of $H$ into itself.
(e) $\| Ph \| \leq \| h \|$ for every $h \in H$.
(f) $P$ is idempotent, i.e., $P^2 = P$.
(g) $\ker(P) = M^\perp$.
(h) $\operatorname{range}(P) = M$.
(i) $I - P$ is the orthogonal projection of $H$ onto $M^\perp$. 
Proof. (a) Follows from the definition of orthogonal projection and Theorem 2.9.

(b) Follows from (a) and the definition of $M^\perp$.

(e) Choose any $h \in H$. Then $h = Ph + e$ with $Ph \in M$ and $e \in M^\perp$. Hence $Ph \perp e$, so by the Pythagorean Theorem, $\|h\|^2 = \|Ph\|^2 + \|e\|^2 \geq \|Ph\|^2$.

(g) Suppose that $h \in \ker(P)$, i.e., $Ph = 0$. Then $h = h - Ph \perp M$, so $h \in M^\perp$.

Conversely, suppose that $h \in M^\perp$. Then $h = 0 + h$ with $0 \in M$ and $h \in M^\perp$, so we must have $Ph = 0$ and hence $h \in \ker(P)$.

The remaining parts are left as exercises. \hfill \Box

Corollary 2.16 (Double Complements). If $M \subseteq H$, then $(M^\perp)^\perp = M$.

Proof. Choose any $x \in M$. Then $\langle x, y \rangle = 0$ for every $y \in M^\perp$. Thus $x \perp M^\perp$, so $x \in (M^\perp)^\perp$.

Conversely, suppose that $x \in (M^\perp)^\perp$. Since $M$ is closed, we can write $x = p + e$ with $p \in M$ and $e \in M^\perp$. Since $x \in (M^\perp)^\perp$ we have $\langle x, e \rangle = 0$. But we also have $p \in M \subseteq (M^\perp)^\perp$, so $\langle e, p \rangle = 0$. Therefore $\|e\|^2 = \langle e, e \rangle = \langle x - p, e \rangle = \langle x, e \rangle - \langle p, e \rangle = 0$. Hence $e = 0$, so $x = p \in M$. \hfill \Box

Definition 2.17 (Span, Closed Span). Let $A$ be a subset of a normed linear space $X$.

(a) The finite span (or linear span or just the span) of $A$, denoted $\text{span}(A)$, is the set of all finite linear combinations of elements of $A$:

$$\text{span}(A) = \left\{ \sum_{k=1}^{n} \alpha_k x_k : n > 0, x_k \in A, \alpha_k \in \mathbb{F} \right\}.$$ 

In particular, if $A$ is a countable sequence, say $A = \{x_k\}_{k=1}^{\infty}$, then

$$\text{span}(\{x_k\}_{k=1}^{\infty}) = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k : n > 0, \alpha_k \in \mathbb{F} \right\}.$$ 

(b) The closed finite span (or closed linear span or closed span) of $A$, denoted $\overline{\text{span}}(A)$ or $\forall A$, is the closure of the set of all finite linear combinations of elements of $A$:

$$\forall A = \overline{\text{span}}(A) = \overline{\text{span}}(A) = \{ z \in X : \exists y_n \in \text{span}(A) \text{ such that } y_n \to z \}.$$ 

Beware: This does NOT say that

$$\overline{\text{span}}(A) = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k : x_k \in A, \alpha_k \in \mathbb{F} \right\}.$$ 

It does not even imply that every element of $\overline{\text{span}}(A)$ can be written $x = \sum_{k=1}^{\infty} \alpha_k x_k$ for some $x_k \in A$, $\alpha_k \in \mathbb{F}$. What does it say about $\overline{\text{span}}(A)$? If we consider the case of a countable sequence $A = \{x_k\}_{k=1}^{\infty}$, then we have

$$\overline{\text{span}}(\{x_k\}_{k=1}^{\infty}) = \left\{ z \in X : \exists \alpha_{k,n} \in \mathbb{F} \text{ such that } \sum_{k=1}^{n} \alpha_{k,n} x_k \to z \text{ as } n \to \infty \right\}.$$
That is, an element $z$ lies in the closed span if there exist $\alpha_{k,n} \in \mathbb{F}$ such that
\[ \sum_{k=1}^{n} \alpha_{k,n} x_k \to z \text{ as } n \to \infty. \]

In contrast, to say that $x = \sum_{k=1}^{\infty} \alpha_k x_k$ means that
\[ \sum_{k=1}^{n} \alpha_k x_k \to x \text{ as } n \to \infty, \]
i.e., the scalars $\alpha_k$ must be independent of $n$.

(c) If $X$ is a Banach space then we say that a subset $A \subseteq X$ is complete if $\text{span}(A)$ is dense in $X$, or equivalently, if $\overline{\text{span}(A)} = X$.

Beware of the two different meanings that we have assigned to the word complete!

**Exercise 2.18.** Let $\{e_n\}_{n=1}^{\infty}$ be the standard basis for $\ell^2(\mathbb{N})$. Prove that $c_{00} = \text{span}(\{e_n\}_{n=1}^{\infty})$ and conclude that $\{e_n\}_{n=1}^{\infty}$ is complete in $\ell^2(\mathbb{N})$.

**Exercise 2.19.** Let $A$ be any subset of a Hilbert space $H$.

(a) $(A^\perp)^\perp = \overline{\text{span}}(A)$.

(b) $A$ is complete if and only if $A^\perp = \{0\}$.

Note that, by definition, $A^\perp = \{0\}$ is equivalent to the statement that $x \perp A \implies x = 0$. Thus, this exercise says that $A$ is complete if and only if the only element orthogonal to every element of $A$ is 0.

### 3. The Riesz Representation Theorem

**Definition 3.1** (Continuous and Bounded Operators). Let $X$, $Y$ be normed linear spaces, and let $L: X \to Y$ be a linear operator.

(a) $L$ is continuous at a point $x \in X$ if $x_n \to x$ in $X$ implies $Lx_n \to Lx$ in $Y$.

(b) $L$ is continuous if it is continuous at every point, i.e., if $x_n \to x$ in $X$ implies $Lx_n \to Lx$ in $Y$ for every $x$.

(c) $L$ is bounded if there exists a finite $K \geq 0$ such that
\[ \forall x \in X, \quad \|Lx\| \leq K \|x\|. \]

Note that $\|Lx\|$ is the norm of $Lx$ in $Y$, while $\|x\|$ is the norm of $x$ in $X$.

(d) The operator norm of $L$ is
\[ \|L\| = \sup_{\|x\|=1} \|Lx\|. \]
(e) We let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators mapping $X$ into $Y$, i.e.,
$$\mathcal{B}(X, Y) = \{ L : X \to Y : L \text{ is bounded and linear} \}.$$ 
If $X = Y = X$ then we write $\mathcal{B}(X) = \mathcal{B}(X, X)$.

(f) If $Y = \mathbb{F}$ then we say that $L$ is a functional. The set of all bounded linear functionals on $X$ is the dual space of $X$, and is denoted
$$X' = \mathcal{B}(X, \mathbb{F}) = \{ L : X \to \mathbb{F} : L \text{ is bounded and linear} \}.$$

Exercise 3.2. Let $X$, $X$, $Y$ be normed linear spaces. Let $L : X \to Y$ be a linear operator.

(a) $L$ is injective if and only if $\ker L = \{0\}$.
(b) $L$ is bounded if and only if $\|L\| < \infty$.
(c) If $L$ is bounded then $\|Lx\| \leq \|L\| \|x\|$ for every $x \in X$ (note that three different meanings of the symbol $\| \cdot \|$ appear in this statement!).
(d) If $L$ is bounded then $\|L\|$ is the smallest value of $K$ such that $\|Lx\| \leq K\|x\|$ holds for all $x \in X$.
(e) $\|L\| = \sup_{\|x\| \leq 1} \|Lx\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|}$.

(f) $\mathcal{B}(X, Y)$ is a subspace of the vector space $V$ containing ALL functions $A : X \to Y$. Moreover, the operator norm is a norm on the space $\mathcal{B}(X, Y)$, i.e.,
  i. $\|L\| \geq 0$ for all $L \in \mathcal{B}(X, Y)$.
  ii. $\|L\| = 0$ if and only if $L = 0$ (the zero operator that sends every element of $X$ to the zero vector in $Y$).
  iii. $\|\alpha L\| = |\alpha| \|L\|$ for every $L \in \mathcal{B}(X, Y)$ and every $\alpha \in \mathbb{F}$.
  iv. $\|L + K\| \leq \|L\| + \|K\|$ for every $L, K \in \mathcal{B}(X, Y)$.

(g) Part (f) shows that $\mathcal{B}(X, Y)$ is itself a normed linear space, and we will see later that it is a Banach space if $Y$ is a Banach space. However, in addition to vector addition and scalar multiplication operations, there is a third operation that we can perform with functions: composition. Prove that the operator norm is submultiplicative, i.e., if $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$, then $BA \in \mathcal{B}(X, Z)$, and furthermore,
$$\|BA\| \leq \|B\| \|A\|.$$ 
In particular, when $X = Y = Z$, we see that $\mathcal{B}(X)$ is closed under composition. The space $\mathcal{B}(X)$ is an example of an algebra.

(h) If $L \in X' = \mathcal{B}(X, \mathbb{F})$ then $\|L\| = \sup_{\|x\|=1} |Lx|$.

Exercise 3.3. Let $M \leq H$, and let $P$ be the operator of orthogonal projection onto $M$. Find $\|P\|$.
Proof. $k$ there must exist are equivalent. Thus $Lx$ normed linear spaces, and let $Lx$.

Lemma 3.5. Let $X$, $Y$ be normed linear spaces. Let $L : X \to Y$ be linear. Then $L$ is continuous if and only if $U$ open in $Y$ implies $L^{-1}(U)$ open in $X$.

Proof. $\Rightarrow$. Suppose that $L$ is continuous and that $U$ is an open subset of $Y$. We will show that $X \setminus L^{-1}(U)$ is closed by showing that it contains all its limit points.

Suppose that $x$ is a limit point of $X \setminus L^{-1}(U)$. Then there exist $x_n \in X \setminus L^{-1}(U)$ such that $x_n \to x$. Since $L$ is continuous, this implies $Lx_n \to Lx$. However, $x_n \notin L^{-1}(U)$, so $Lx_n \notin U$, i.e., $Lx_n \in Y \setminus U$, which is a closed set. Therefore $Lx \in Y \setminus U$, and hence $x \in X \setminus L^{-1}(U)$. Thus $X \setminus L^{-1}(U)$ is closed, so $L^{-1}(U)$ is open.

$\Leftarrow$. Exercise.

Theorem 3.6 (Equivalence of Bounded and Continuous Linear Operators). Let $X$, $Y$ be normed linear spaces, and let $L : X \to Y$ be a linear mapping. Then the following statements are equivalent.

(a) $L$ is continuous at some $x \in X$.
(b) $L$ is continuous at $x = 0$.
(c) $L$ is continuous.
(d) $L$ is bounded.

Proof. (c) implies (d). Suppose that $L$ is continuous but unbounded. Then $\|L\| = \infty$, so there must exist $x_n \in X$ with $\|x_n\| = 1$ such that $\|Lx_n\| \geq n$. Set $y_n = x_n/n$. Then $\|y_n - 0\| = \|y_n\| = \|x_n\|/n \to 0$, so $y_n \to 0$. Since $L$ is continuous and linear, this implies $Ly_n \to L0 = 0$, and therefore $\|Ly_n\| \to \|0\| = 0$. But $\|Ly_n\| = \frac{1}{n} \|Lx_n\| \geq \frac{1}{n} \cdot n = 1$

for all $n$, which is a contradiction. Hence $L$ must be bounded.

(d) implies (c). Suppose that $L$ is bounded, so $\|L\| < \infty$. Suppose that $x \in X$ and that $x_n \to x$. Then $\|x_n - x\| \to 0$, so

$$\|Lx_n - Lx\| = \|L(x_n - x)\| \leq \|L\| \cdot \|x_n - x\| \to 0,$$

i.e., $Lx_n \to Lx$. Thus $L$ is continuous.

The remaining implications are left as exercises.
Exercise 3.7. Let $X$, $Y$ be normed linear spaces. Suppose that $L: X \to Y$ is linear and satisfies $\|Lx\| = \|x\|$ for all $x \in X$. Such an operator is said to be an isometry or norm-preserving. Prove that $L$ is injective and find $\|L\|$. Find an example of an isometry that is not surjective. Contrast this with the fact that if $A: \mathbb{C}^n \to \mathbb{C}^n$ is linear, then $A$ is injective if and only if it is surjective.

Exercise 3.8. (a) Define $L: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $L(x) = (x_2, x_3, \ldots)$ for $x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N})$. Prove that this left-shift operator is bounded, linear, surjective, not injective, and is not an isometry. Find $\|L\|$.

(b) Define $R: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $R(x) = (0, x_1, x_2, x_3, \ldots)$ for $x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N})$. Prove that this right-shift operator is bounded, linear, injective, not surjective, and is an isometry. Find $\|R\|$.

(c) Compute $LR$ and $RL$. Contrast this computation with the fact that in finite dimensions, if $A, B: \mathbb{C}^n \to \mathbb{C}^n$ are linear maps (hence correspond to multiplication by $n \times n$ matrices), then $AB = I$ implies $BA = I$ and conversely.

Exercise 3.9. Let $X$ be a Banach space and $Y$ a normed linear space. Suppose that $L: X \to Y$ is bounded and linear. Prove that if there exists $c > 0$ such that $\|Lx\| \geq c\|x\|$ for all $x \in X$, then $L$ is injective and range($L$) is closed.

Exercise 3.10. For each $h \in H$, define a linear functional $L_h: H \to \mathbb{F}$ by

$$L_h(x) = \langle x, h \rangle, \quad x \in H.$$  

Prove the following.

(a) $L_h$ is linear.
(b) $\ker(L_h) = \{h\}^\perp$.
(c) $\|L_h\| = \|h\|$, so $L_h$ is a bounded linear functional on $H$. Therefore $L_h \in H' = \mathcal{B}(H, \mathbb{F})$.

Thus, each element $h$ of $H$ determines an element $L_h$ of $H'$. In the Riesz Representation Theorem we will prove that these are all the elements of $H'$.

Example 3.11 (Dual Space of $\mathbb{C}^n$). Consider $H = \mathbb{C}^n$ (with $\mathbb{F} = \mathbb{C}$). Let $h = (h_1, \ldots, h_n)^T \in H = \mathbb{C}_n$ (think of $h$ as a column vector). Then $L_h(x) = x \cdot h = x_1h_1 + \cdots + x_nh_n$ defines a bounded linear functional on $\mathbb{C}^n$.

Suppose that $L: \mathbb{C}^n \to \mathbb{C}$ is any linear functional on $\mathbb{C}^n$ (it will necessarily be bounded since $\mathbb{C}_n$ is finite-dimensional, see Exercise 3.4). Any linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$ is given by multiplication by an $m \times n$ matrix. So in this case there must be a $1 \times m$ matrix (i.e., a row vector) $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ such that

$$Lx = Ax = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + \cdots + x_na_n = x \cdot h = L_h(x)$$
where \( h = (a_1, \ldots, a_n)^T \).

Thus we see that every element of \((\mathbb{C}^n)'\) is an \( L_h \) for some \( h \in \mathbb{C}^n \).

**Exercise 3.12.** Define \( T : \mathbb{C}^n \to (\mathbb{C}^n)' \) by
\[
T(h) = L_h, \quad h \in \mathbb{C}^n.
\]
Prove directly that \( T \) is anti-linear, i.e.,
\[
T(\alpha h + \beta k) = \alpha T(h) + \beta T(k),
\]
and that \( T \) is an isometry, i.e., \( \|T(h)\| = \|h\| \) for each \( h \). All isometries are injective, and in the preceding example we showed that \( T \) is surjective, so \( T \) is an anti-linear isometric bijection of \( \mathbb{C}^n \) onto \((\mathbb{C}^n)'\).

If we define \( S : \mathbb{C}^n \to (\mathbb{C}^n)' \) by \( S(h) = L_h \), then \( S \) a linear isometric bijection of \( \mathbb{C}^n \) onto \((\mathbb{C}^n)'\). Such an \( S \) is called an isomorphism, and thus \( \mathbb{C}^n \) and \((\mathbb{C}^n)'\) are isomorphic.

**Theorem 3.13** (Riesz Representation Theorem). For each \( h \in H \), define \( L_h : H \to \mathbb{F} \) by
\[
L_h(x) = \langle x, h \rangle, \quad x \in H.
\]

(a) For each \( h \in H \), we have \( L_h \in H' \) and \( \|L_h\| = \|h\| \).
(b) If \( L \in H' \), then there exists a unique \( h \in H \) such that \( L = L_h \).
(c) The mapping \( T : H \to H' \) defined by \( T(h) = L_h \) is an anti-linear isometric bijection of \( H \) onto \( H' \).

**Proof.** (a) See Exercise 3.10.

(b) Choose any \( L \in H' \). If \( L = 0 \) then \( h = 0 \) is the required vector, so assume that \( L \neq 0 \) (i.e., \( L \) is not the zero operator). Then \( L \) does not map every vector to zero, so the kernel of \( L \) is a closed subspace of \( H \) that is not all of \( H \).

Choose any \( z \notin \ker(L) \), and write
\[
z = p + e, \quad p \in \ker(L), \quad e \in \ker(L)^\perp.
\]
Note that \( L(p) = 0 \) by definition, and therefore we must have \( L(e) \neq 0 \) (since \( L(z) \neq 0 \)).

Set \( u = e/L(e) \), and note that \( u \in \ker(L)^\perp \) and \( L(u) = 1 \).

Given \( x \in H \), since \( L \) is linear and \( L(u) = 1 \) we have
\[
L(x - L(x) u) = L(x) - L(x) L(u) = 0.
\]
Therefore \( x - L(x) u \in \ker(L) \). However, \( u \perp \ker(L) \), so
\[
0 = \langle x - L(x)u, u \rangle = \langle x, u \rangle - \langle L(x)u, u \rangle = \langle x, u \rangle - L(x) \|u\|^2.
\]
Hence,
\[
L(x) = \frac{1}{\|u\|^2} \langle x, u \rangle = \left\langle x, \frac{u}{\|u\|^2} \right\rangle = \langle x, h \rangle = L_h(x)
\]
where
\[
h = \frac{u}{\|u\|^2}.
Thus $L = L_h$, and from part (a), we have that $\|L\| = \|L_h\| = \|h\|$. It remains only to show that $h$ is unique. Suppose that we also had $L = L_{h'}$. Then for every $x \in H$ we have

$$
\langle x, h - h' \rangle = \langle x, h \rangle - \langle x, h' \rangle = L_h(x) - L_{h'}(x) = L(x) - L(x) = 0.
$$

Consequently, $h - h' = 0$.

(c) Parts (a) and (b) show that $T$ is surjective and that $T$ is norm-preserving. Therefore, we just have to show that $T$ is anti-linear. Let $h \in H$ and let $c \in \mathbb{F}$. We must show that $T(ch) = cT(h)$, i.e., that $L_{ch} = cL_h$. This follows immediately from the fact that for each $x \in H$, we have

$$
L_{ch}(x) = \langle x, ch \rangle = c \langle x, h \rangle = cL_h(x).
$$

The proof that $L_{h+k} = T(h + k) = T(h) + T(k) = L_h + L_k$ is left as an exercise.

Corollary 3.14. If $L: \ell^2(I) \to \mathbb{F}$ is a bounded linear functional, then there exists a unique $h = (h_i)_{i \in I} \in \ell^2(I)$ such that

$$
L(x) = \sum_{i \in I} x_i \bar{h}_i, \quad x = (x_k)_{i \in I} \in \ell^2(I).
$$

Corollary 3.15. If $(X, \Omega, \mu)$ is a measure space and $L: L^2(X) \to \mathbb{F}$ is a bounded linear functional, then there exists a unique $h \in L^2(X)$ such that

$$
L(f) = \int_X f(x) \overline{h(x)} \, d\mu(x), \quad f \in L^2(X).
$$

4. Orthogonal Sets of Vectors and Bases

Definition 4.1 (Hamel Basis). Let $V$ be a vector space. We say that $\{f_i\}_{i \in I}$ is a basis for $V$ if it is both finitely linearly independent and its finite span is all of $V$. That is, every vector in $V$ equals a unique finite linear combination of the $f_i$ (aside from trivial zero terms), i.e., every $f \neq 0$ can be written $f = \sum_{k=1}^N c_k f_{i_k}$ for a unique choice of indices $i_1, \ldots, i_N$ and nonzero scalars $c_1, \ldots, c_N$. Because the word “basis” is heavily overused, we shall refer to such a basis as a Hamel basis or a vector space basis. For most vector spaces, Hamel bases are only known to exist because of the Axiom of Choice; in fact, the statement “Every vector space has a Hamel basis” is equivalent to the Axiom of Choice.

As in finite dimensions, it can be shown that any two Hamel bases for a vector space $V$ must have the same cardinality. This cardinality is called the (vector space) dimension of $V$.

Example 4.2 (Existence of Unbounded Linear Functionals). We can use the Axiom of Choice to show that there exist unbounded linear functionals $L: X \to \mathbb{F}$ whenever $X$ is an infinite-dimensional normed linear space.

Let $\{f_i\}_{i \in I}$ be a Hamel basis for an infinite-dimensional normed linear space $X$, normalized so that $\|f_i\| = 1$ for every $i \in I$. Let $J_0 = \{j_1, j_2, \ldots\}$ be any countable subsequence of $I$. 

Define $L: X \to \mathbb{C}$ by setting $L(f_{j_n}) = n$ for $n \in \mathbb{N}$ and $L(f_i) = 0$ for $i \in I \setminus J_0$. Then extend $L$ linearly to all of $X$: if $f = \sum_{k=1}^{N} c_k f_{j_k}$ is the unique representation of $f$ using nonzero scalars $c_1, \ldots, c_N$, then define $L(f) = \sum_{k=1}^{N} c_k L(f_{j_k})$. This $L$ is a linear functional on $X$, but since $\|f_{j_n}\| = 1$ yet $|L(f_{j_n})| = n$, we have $\|L\| = \sup_{\|f\|=1} |L(f)| = \infty$.

Thus, if $L: X \to Y$ and $Y$ is finite-dimensional, we cannot conclude that $L$ must necessarily be bounded. Contrast this with the fact that if $L: X \to Y$ and $X$ is finite-dimensional, then $L$ must be bounded (see Exercise 3.4).

**Remark 4.3.** Aside from the preceding result, Hamel bases are not going to be much use to us. For example, consider the space $\ell^2 = \ell^2(\mathbb{N})$, and let $\{e_n\}_{n \in \mathbb{N}}$ be the standard basis for $\ell^2$. Note that $\{e_n\}_{n \in \mathbb{N}}$ is NOT a Hamel basis for $\ell^2$! Its finite linear span is the proper subspace $c_{00}$. Thus $\{e_n\}_{n \in \mathbb{N}}$ is a Hamel basis for $c_{00}$, but not for $\ell^2$. In fact, a Hamel basis for an infinite-dimensional Hilbert or Banach space must be uncountable.

The main point is that since we have a norm, there is no reason to restrict to only finite linear combinations. Given a sequence of vectors $\{f_i\}_{i \in \mathbb{N}}$ and scalars $(c_i)_{i \in \mathbb{N}}$, we can consider “infinite linear combinations”

$$\sum_{i=1}^{\infty} c_i f_i.$$ 

HOWEVER, we must be very careful about convergence issues: the series above will not converge for EVERY choice of scalars $c_i$. On the other hand, if our sequence $\{f_i\}$ is orthogonal, or even better yet, orthonormal, then we will see that the convergence issues become easy to deal with, and we can make sense of what it means to have a “basis” which allows “infinite linear combinations.”

HOWEVER, there still remains an issue of how many vectors we need in this “basis”—for some Hilbert spaces (the separable Hilbert spaces) we will be able to use a basis that requires only countably many vectors, as was implicitly assumed in the discussion in the preceding paragraph. But for others, a “basis” may need uncountably many vectors, even allowing “infinite (but still countable) linear combinations.” In some fields (e.g., the geometry of Banach spaces literature), it is customary to only use the word basis for the case of countable bases. Conway does not follow this custom, which is fine as long as you realize that other authors use the term “basis” differently.

Conway’s definition of a basis for a Hilbert space follows. The definition does not seem to say anything about linear combinations, infinite or otherwise—we shall see the connection later. To emphasize that Conway’s definition is not universally accepted, I will refer to his definition of basis as a *Conway basis*.

**Definition 4.4 (Conway Basis).** Let $H$ be a Hilbert space. Then a *Conway basis* for $H$ is a maximal orthonormal set. That is, a subset $E \subseteq H$ is a Conway basis if

(a) $E$ is orthonormal, and

(b) there does not exist an orthonormal set $E' \subseteq H$ such that $E \subseteq E'$.
Exercise 4.5. Zorn’s Lemma is an equivalent formulation of the Axiom of Choice. Learn what Zorn’s Lemma says, and use it to prove that every Hilbert space has a Conway basis. Then ask all your friends the classic riddle: What is yellow and equivalent to the Axiom of Choice?\footnote{Zorn’s Lemon!} That one is nearly as good as the classic: What is purple and commutes?\footnote{An abelian grape.}

Exercise 4.6. Show that \( \{e_n\}_{n \in \mathbb{N}} \) is a Conway basis for \( \ell^2(\mathbb{N}) \).

Exercise 4.7. Let \( I \) be any set. Define \( \ell^2(I) \) to be the set of all functions \( x: I \to \mathbb{F} \) such that \( x(i) \neq 0 \) for at most a countable number of \( i \) and such that

\[
\|x\|^2 = \sum_{i \in I} |x(i)|^2 < \infty
\]

there are at most countably many nonzero terms in the above sum, so this series with a possibly uncountable index set just means to sum the countably many nonzero terms. Then \( \ell^2(I) \) is a Hilbert space with respect to the inner product

\[
\langle x, y \rangle = \sum_{i \in I} x(i) \overline{y(i)}.
\]

For each \( i \in I \), define \( e_i: I \to \mathbb{F} \) by

\[
e_i(j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Then \( \{e_i\}_{i \in I} \) is called the standard basis for \( \ell^2(I) \). Prove that it is a Conway basis for \( \ell^2(I) \).

Instead of “Conway basis,” the following gives us a terminology that does not require us to use the word basis.

Lemma 4.8. Let \( \{f_i\}_{i \in I} \) be a sequence of elements of \( H \). Then:

\( \{f_i\}_{i \in I} \) is orthonormal and complete \( \iff \) \( \{f_i\}_{i \in I} \) is a Conway basis.

Proof. \( \Rightarrow \). Suppose that \( \{f_i\}_{i \in I} \) is orthonormal and complete. Then by Exercise 2.19, there is no nonzero \( g \in H \) that is orthogonal to every \( f_i \). Therefore there can be no orthonormal set that properly contains \( \{f_i\}_{i \in I} \).

\( \Leftarrow \). Exercise. \( \square \)

Definition 4.9 (Schauder basis). Now we give the “correct” definition of a basis.

Let \( X \) be a Banach space. Then a (countable!) sequence \( \{f_i\}_{i \in \mathbb{N}} \) is a Schauder basis, or just a basis, for \( X \) if for each \( f \in X \) there exist unique scalars \( \{c_i\}_{i \in \mathbb{N}} \) such that

\[
f = \sum_{i=1}^{\infty} c_i f_i.
\]
Note in particular that this means that $\sum_{i=1}^{N} c_i f_i \to f$ as $N \to \infty$.

To be a little more precise, the definition of Schauder basis really includes the requirement that if we let $c_i(f)$ denote the unique scalars such that $f = \sum_{i=1}^{\infty} c_i(f) f_i$, then the mapping $f \mapsto c_i(f)$ must be a continuous linear functional on $X$ for each $i$. However, the Strong Basis Theorem proves that this is automatically true when $X$ is a Banach space, so we have omitted that requirement in our definition.

**Exercise 4.10.** The standard basis $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal Schauder basis for $\ell^2(\mathbb{N})$.

**Exercise 4.11.** (See also Exercise 10 on p. 98 of Conway.) Let $\{f_i\}_{i \in \mathbb{N}}$ be a Schauder basis for a Banach space $X$.

(a) Prove that $\{f_i\}_{i \in \mathbb{N}}$ is complete in $X$, i.e., that the set of finite linear combinations is dense in $X$. Note: The converse is false, i.e., a complete sequence need not be a Schauder basis (see Example 4.18). On the other hand, we will see that in a separable Hilbert space, a complete orthonormal sequence IS a Schauder basis.

(b) Let

$$S = \left\{ \sum_{i=1}^{N} r_i f_i : N > 0, \, \text{Re}(r_i), \, \text{Im}(r_i) \in \mathbb{Q} \right\}.$$

Prove that $S$ is a countable, dense subset of $X$. A Banach space which has a countable, dense subset is said to be separable.

**Remark 4.12.** Thus, if a Banach space has a Schauder basis, then it must be separable. Does every separable Banach space have a Schauder basis? This was a longstanding open problem in Banach space theory, called the Basis Problem. It was settled by Enflo in 1973: there exist separable, reflexive Banach spaces which do not possess any Schauder bases.

**Exercise 4.13.** Let $X$ be a Banach space. Suppose there exists an uncountable sequence $\{x_i\}_{i \in I}$ of elements of $X$ such that $\|x_i - x_j\| \geq \varepsilon > 0$ for all $i \neq j \in I$. Prove that $X$ is not separable.

**Exercise 4.14.** Use the preceding exercise to prove the following.

(a) Prove that $\ell^\infty(\mathbb{N})$ is not separable.

(b) Prove that $L^\infty(\mathbb{R})$ is not separable.

(c) Prove that if $I$ is uncountable then $\ell^2(I)$ is not separable. More generally, any Hilbert space which contains an uncountable orthonormal sequence is not separable.

We will show that, in a separable Hilbert space, any complete orthonormal sequence is a Schauder basis for $H$. If $H$ is nonseparable, something very similar happens; we just have to be careful with what we mean by infinite series with uncountably many terms (only countably many terms can be nonzero). First, however, we will present the results for separable Hilbert spaces.
Theorem 4.15. Let \( \{e_n\}_{n \in \mathbb{N}} \) be any orthonormal sequence in a Hilbert space \( H \). Then the following statements hold.

(a) Bessel’s Inequality: \( \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 \leq \|f\|^2 \).

(b) If \( f = \sum_{n=1}^{\infty} c_n e_n \) converges, then \( c_n = \langle f, e_n \rangle \).

(c) \( \sum_{n=1}^{\infty} c_n e_n \) converges \( \iff \) \( \sum_{n=1}^{\infty} |c_n|^2 < \infty \).

(d) If \( \sum_{n=1}^{\infty} c_n e_n \) converges then it converges \emph{unconditionally}, i.e., it converges regardless of the ordering of the indices.

(e) \( f \in \text{span}(\{e_n\}_{n \in \mathbb{N}}) \iff f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \).

(f) If \( f \in H \), then

\[
P_f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n
\]

is the orthogonal projection of \( f \) onto \( \text{span}(\{e_n\}_{n \in \mathbb{N}}) \).

Proof. (a) Choose \( f \in H \). For each \( N \) define

\[
f_N = f - \sum_{n=1}^{N} \langle f, e_n \rangle e_n.
\]

Show that \( f_N \perp e_1, \ldots, e_N \) (exercise). Therefore, by the Pythagorean Theorem, we have

\[
\|f\|^2 = \|f_N + \sum_{n=1}^{N} \langle f, e_n \rangle e_n \|^2 = \|f_N\|^2 + \sum_{n=1}^{N} \|\langle f, e_n \rangle e_n\|^2 \geq \sum_{n=1}^{N} |\langle f, e_n \rangle|^2.
\]

Since this is true for each finite \( n \), Bessel’s inequality follows.

(b) Exercise.

(c) \( \iff \). Suppose that \( \sum_{n=1}^{\infty} |c_n|^2 < \infty \). Set

\[
s_N = \sum_{n=1}^{N} c_n e_n, \quad t_N = \sum_{n=1}^{N} |c_n|^2.
\]

We know that \( \{t_N\} \) is a convergent (hence Cauchy) sequence of scalars, and we must show that \( \{s_N\} \) is a convergent sequence of vectors. We have for \( N > M \) that

\[
\|s_N - s_M\|^2 = \left\| \sum_{n=M+1}^{N} c_n e_n \right\|^2 = \sum_{n=M+1}^{N} \|c_n e_n\|^2 = \sum_{n=M+1}^{N} |c_n|^2 = |t_N - t_M|.
\]
Since \( \{t_N\} \) is Cauchy, we conclude that \( \{s_N\} \) is Cauchy and hence converges.

\( \Rightarrow \). Exercise.

(d), (e) Exercise.

(f) By Bessel’s inequality and part (c), we know that the series defining \( Pf \) converges, so we just have to show that it is the orthogonal projection of \( f \) onto \( \overline{\text{span}}(\{e_n\}) \). Check that \( \langle f - Pf, e_n \rangle = 0 \) for every \( n \) (exercise). By linearity, conclude that \( f - Pf \perp \overline{\text{span}}(\{e_n\}) \). By continuity of the inner product, conclude that \( f - Pf \perp \overline{\text{span}}(\{e_n\}) \) (exercise).

\( \square \)

**Exercise 4.16.** Use Theorem 4.15 to construct an example of a series \( \sum c_n e_n \) in a Hilbert space that converges unconditionally but not absolutely (compare Exercise 1.26).

The next theorem shows that any countable orthonormal sequence that is complete must be a Schauder basis, and vice versa. Such an orthonormal Schauder basis is usually just called an orthonormal basis (or just ONB) for \( H \).

**Theorem 4.17.** Let \( \{e_n\}_{n \in \mathbb{N}} \) be any orthonormal sequence in a Hilbert space \( H \). Then the following statements are equivalent.

(a) \( \{e_n\}_{n \in \mathbb{N}} \) is a Schauder basis for \( H \), i.e., for each \( f \in H \) there exist unique scalars \( (c_n)_{n \in \mathbb{N}} \) such that \( f = \sum_{n=1}^{\infty} c_n e_n \).

(b) For each \( f \in H \), \( f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \).

(c) \( \{e_n\}_{n \in \mathbb{N}} \) is complete (i.e, it is a Conway basis for \( H \)).

(d) Plancherel/Parseval Equality: For each \( f \in H \), \( \|f\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 \).

(e) Parseval/Plancherel Equality: For each \( f, g \in H \), \( \langle f, g \rangle = \sum_{n=1}^{\infty} \langle \langle f, e_n \rangle e_n, g \rangle \).

**Proof.** (b) \( \Rightarrow \) (e). Suppose that (b) holds, and choose any \( f, g \in H \). Then

\[
\langle f, g \rangle = \left\langle \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, g \right\rangle = \sum_{n=1}^{\infty} \langle \langle f, e_n \rangle e_n, g \rangle = \sum_{n=1}^{\infty} \langle f, e_n \rangle \langle e_n, g \rangle,
\]

where we have used Exercise 1.24 to move the infinite series outside of the inner product.

(c) \( \Rightarrow \) (b). If \( \{e_n\} \) is complete, then its closed span is all of \( H \), so by Theorem 4.15(e) we have \( f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \) for every \( f \in H \).
(d) $\Rightarrow$ (b). Suppose that $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2$ for every $f \in H$. Fix $f$, and define $s_N = \sum_{n=1}^{N} \langle f, e_n \rangle e_n$. Then, by direct calculation and the by the Pythagorean Theorem,

$$
\|f - s_N\|^2 = \|f\|^2 - \langle f, s_N \rangle - \langle s_N, f \rangle + \|s_N\|^2
$$

$$
= \|f\|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 + \sum_{n=1}^{N} |\langle f, e_n \rangle|^2
$$

$$
= \|f\|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2
$$

$$
\rightarrow 0 \text{ as } N \rightarrow \infty.
$$

Hence $f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$.

The remaining implications are exercises.

Example 4.18. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for a separable Hilbert space $H$. Define $f_n = e_1 + e_n/n$. Then $\{f_n\}_{n \in \mathbb{N}}$ is complete (exercise) but it is not a Schauder basis for $H$ (harder).

Exercise 4.19. Let $H$ be a Hilbert space. Show that $H$ is separable if and only if there exists a countable sequence $\{x_n\}_{n \in \mathbb{N}}$ that is an orthonormal basis for $H$. Hint: By Exercise 4.5 we know that there exists a complete orthonormal sequence for $H$.

We state without proof the facts about complete orthonormal systems in nonseparable Hilbert spaces.

Theorem 4.20. Let $\{e_i\}_{i \in I}$ be an orthonormal sequence in a Hilbert space $H$ (note that $I$ might be uncountable). Then the following statements hold.

(a) If $f \in H$ then $\langle f, e_i \rangle \neq 0$ for at most countably many $i$.

(b) For each $f \in H$, $\sum_{i \in I} |\langle f, e_i \rangle|^2 \leq \|f\|^2$.

(c) For each $f \in H$, $\sum_{i \in I} \langle f, e_i \rangle e_i$ converges with respect to the net of finite subsets of $I$ (see Proposition 1.27 for the meaning of net of finite subsets).

Theorem 4.21. Let $\{e_i\}_{i \in I}$ be an orthonormal sequence in a Hilbert space $H$. Then the following statements are equivalent.

(a) $\{e_i\}_{i \in I}$ is complete (i.e., is a Conway basis for $H$).

(b) For each $f \in H$, $f = \sum_{i \in I} \langle f, e_i \rangle e_i$ with respect to the net of finite subsets of $I$.

(c) For each $f \in H$, $\|f\|^2 = \sum_{i \in I} |\langle f, e_i \rangle|^2$ (only countably many terms are nonzero).
The (Hilbert space) dimension of $H$ is the cardinality of a complete orthonormal sequence. It can be shown that any two complete orthonormal sequences must have the same cardinality. Further, $H$ is separable if and only if it has a countable complete orthonormal sequence (which by Theorem 4.17 will be an orthonormal Schauder basis for $H$).

5. ISOMORPHIC HILBERT SPACES AND THE FOURIER TRANSFORM FOR THE CIRCLE

**Definition 5.1.** If $H_1$, $H_2$ are Hilbert spaces, then $L: H_1 \rightarrow H_2$ is an isomorphism if $L$ is a linear bijection that satisfies
\[
\forall f, g \in H_1, \quad \langle Lf, Lg \rangle = \langle f, g \rangle.
\]
In this case we say that $H_1$ and $H_2$ are isomorphic, and write $H_1 \cong H_2$.

If $H_1 = H_2 = H$ and $L: H \rightarrow H$ is an isomorphism, then we say that $L$ is unitary.

**Proposition 5.2.** Let $L: H_1 \rightarrow H_2$ be a linear mapping. Then the following statements are equivalent.

(a) $L$ is inner-product-preserving, i.e., $\langle Lf, Lg \rangle = \langle f, g \rangle$ for all $f, g \in H_1$.

(b) $L$ is norm-preserving (an isometry), i.e., $\|Lf\| = \|f\|$ for all $f \in H_1$.

**Proof.** We need only show that (b) implies (a). Assume that $L$ is an isometry, and fix $f, g \in H$. Then for any scalar $\lambda \in \mathbb{F}$ we have by the Polar Identity and the fact that $L$ is isometric that
\[
\|f\|^2 + 2 \text{Re} \lambda \langle f, g \rangle + |\lambda|^2 \|g\|^2 = \|f + \lambda g\|^2
\]
\[
= \|Lf + \lambda Lg\|^2
\]
\[
= \|Lf\|^2 + 2 \text{Re} \lambda \langle Lf, Lg \rangle + |\lambda|^2 \|Lg\|^2
\]
\[
= \|f\|^2 + 2 \text{Re} \lambda \langle Lf, Lg \rangle + |\lambda|^2 \|g\|^2.
\]
Thus $\text{Re} \lambda \langle Lf, Lg \rangle = \text{Re} \lambda \langle f, g \rangle$ for every $\lambda \in \mathbb{F}$. This implies (take $\lambda = 1$ and $\lambda = i$) that $\langle Lf, Lg \rangle = \langle f, g \rangle$. □

In particular, every isomorphism is an isometry and hence is automatically injective.

**Corollary 5.3.** Let $L: H_1 \rightarrow H_2$ be given. Then $L$ is an isomorphism if and only if it is a linear surjective isometry.

**Theorem 5.4.** All separable infinite-dimensional Hilbert spaces are isomorphic.

**Proof.** Let $H_1$ and $H_2$ be separable infinite-dimensional Hilbert spaces. Then $H_1$ has a countable orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$, and $H_2$ has a countable orthonormal basis $\{h_n\}_{n \in \mathbb{N}}$. If $f \in H_1$ then by Theorem 4.17 we have $f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$ and $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2$. Since this is finite and since $\{h_n\}$ is an ONB, the series $Lf = \sum_{n=1}^{\infty} \langle f, e_n \rangle h_n$ converges, and $\|Lf\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 = \|f\|^2$.

Exercise: Finish the proof by showing that $L$ is a linear surjective isometry. □
In particular, every separable Hilbert space is isomorphic to \( \ell^2(\mathbb{N}) \).

The proof can be extended to show that any two Hilbert spaces which have the same dimension are isomorphic.

**Example 5.5** (Fourier Series). Now we give one of the most important examples of an orthonormal basis.

We saw in Exercise 2.2 that if we define \( e_n(x) = e^{2\pi i nx} \), then \( \{e_n\}_{n \in \mathbb{Z}} \) is an orthonormal sequence in \( L^2[0, 1] \) (the space of square-integrable complex-valued functions on the domain \([0, 1]\), or we can consider these functions to be 1-periodic functions on the domain \( \mathbb{R} \)).

It is a fact that \( \{e_n\}_{n \in \mathbb{Z}} \) is actually complete in \( L^2[0, 1] \) and hence is an orthonormal basis for \( L^2[0, 1] \). The **Fourier coefficients** of \( f \in L^2[0, 1] \) are

\[
\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(x) e^{-2\pi i nx} \, dx, \quad n \in \mathbb{Z}.
\]

By Theorem 4.17, we have

\[
f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n \tag{5.1}
\]

(the series converges unconditionally, so it does not matter what ordering of the index set \( \mathbb{Z} \) that we use to sum with). Equation 5.1 is called the **Fourier series** of \( f \).

However, note that the series in 5.1 converges in the norm of the Hilbert space, i.e., in \( L^2 \)-norm. That is, the partial sums converge in \( L^2 \)-norm, i.e.,

\[
\lim_{N \to \infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 = 0,
\]

or,

\[
\lim_{N \to \infty} \int_0^1 \left| f(x) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i nx} \right|^2 \, dx = 0.
\]

We cannot conclude from this that the equality

\[
f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i nx}
\]

holds pointwise. In fact, one of the deepest results of Fourier series is the **Carleson–Hunt Theorem**, which proves the conjecture of Lusin that if \( f \in L^p[0, 1] \) where \( 1 < p \leq \infty \), then the Fourier series of \( f \) converges to \( f \) a.e.

**Exercise 5.6.** Show that the mapping \( \mathcal{F}: L^2[0, 1] \to \ell^2(\mathbb{Z}) \) given by \( \mathcal{F}(f) = \hat{f} = \{\hat{f}(n)\}_{n \in \mathbb{Z}} \) is exactly the isomorphism constructed by Theorem 5.4 for the case \( H_1 = L^2[0, 1] \) and \( H_2 = \ell^2(\mathbb{Z}) \). The operator \( \mathcal{F} \) is the **Fourier transform for the circle** (thinking of functions in \( L^2[0, 1] \) as being 1-periodic, the domain \([0, 1]\) is topologically a circle).
Exercise 5.7. Prove the following (easy) special case of the Riemann–Lebesgue Lemma: If $f \in L^2[0,1]$ then $\hat{f}(n) \to 0$ as $|n| \to \infty$. The full Riemann–Lebesgue Lemma for the circle states the same conclusion holds if we only assume that $f \in L^1[0,1]$.

Definition 5.8. In honor of Fourier series, if $\{e_n\}_{n \in I}$ is any orthonormal basis of a separable Hilbert space $H$, then $\{\langle f, e_n \rangle\}_{n \in I}$ is the sequence of (generalized) Fourier coefficients of $f$ and $f = \sum_{n \in I} \langle f, e_n \rangle e_n$ is the (generalized) Fourier series of $f$.

Exercise 5.9. The Plancherel formula with respect to the orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ for $L^2[0,1]$ is

$$
\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2.
$$

Use the Plancherel formula to derive a formula for $\pi$ by applying it to the function $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. 