REDUCED HOLONOMY, CONES AND BRANES

Based on results of


and some of my own explicit examples.

Slides at www.ma.ic.ac.uk/~sms
11-dimensional equations

SUGRA\(_{11}\) involves a metric \(g\) and reduction to \(\text{SO}(10, 1)\) and a 4-form \(F = dA\) satisfying

\[
R_{im} = \frac{1}{12} \left( F_{ijkl} F_{m}^{jk\ell} - \frac{1}{12} g_{im} F^2 \right)
\]

\[
dF = 0
\]

\[
d \ast F = F \wedge F
\]

Supersymmetry requires non-zero Killing spinor(s):

\[
\nabla_m \eta + \frac{1}{288} \left[ \Gamma^{ijkl}_m - 8 \delta^i_m \Gamma^{jkl} \right] F_{ijkl} \eta = 0
\]

and holonomy reduction of a suitable connection. Let \(\nu = \frac{1}{32} \text{dim(Kspinors)}\).
Special solutions

Identifying $F$ with the volume form of a 4-manifold gives an Einstein product $M^4 \times M^7$.

- With $\nu = 1$: AdS$_7 \times S^4$ or AdS$_4 \times S^7$.

- M2 brane solution with $\frac{1}{16} \leq \nu \leq \frac{1}{2}$:

\[
\left(1 + \frac{a^6}{r^6}\right)^{-2/3} g_{2,1} + \left(1 + \frac{a^6}{r^6}\right)^{1/3} (dr^2 + r^2 g_7)
\]

with $F = \text{vol}_{2,1} \wedge f(r)dr$ and $*F = 6a^6\text{vol}_7$. Interpolates between

$g_{2,1} + (dr^2 + r^2 g_7)$ on $M_{2,1} \times X^8$

as $r \to \infty$ and the near horizon limit

$\left(\frac{r^4}{a^4} g_{2,1}^2 + \frac{a^2}{r^2} dr^2\right) + a^2 g_7$ on AdS$_4 \times Y^7$

as $r \ll a$. 

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Metrics with exceptional holonomy

- To ensure that $\nu > 0$, the conical metric
  \[ dr^2 + r^2 g_7 \quad \text{on} \quad \mathbb{R}^+ \times Y^7 \]
  must have $\text{Hol} \subseteq \text{Spin}7$. This holds iff $(Y, g_7)$ has weak holonomy $G_2$ (invariant intrinsic torsion).

  **Examples** \( Y = S^7, \quad S^7_{\text{sq}} (\rightarrow S^4), \quad \frac{\text{SO}(5)}{\text{SO}(3)}, \quad \frac{\text{SU}(3)}{U(1)_{p,q}}. \)

- In turn, the conical metric
  \[ dy^2 + y^2 g_6 \quad \text{on} \quad \mathbb{R}^+ \times Z^6 \]
  has holonomy $\subseteq G_2$ iff $(Z, g_6)$ is nearly-Kähler [6].

  **Examples** The 3-symmetric spaces $Z = S^6,$
  \[ \mathbb{CP}^3, \quad F = \frac{\text{SU}(3)}{T^2}, \quad S^3 \times S^3, \]
  for which triality $\theta = \frac{1}{2}(-1 + \sqrt{3} J)$ plays a key role.
Weak $G_2$ metrics with singularities

Suppose that $(Z, g_{NK})$ is nearly-Kähler. Then

\[ g_8 = dx^2 + (dy^2 + y^2 g_{NK}) \]
\[ = (dr^2 + r^2 dt^2) + (r \sin t)^2 g_{NK} \]
\[ = dr^2 + r^2 (dt^2 + \sin^2 t g_{NK}) \]

has $\text{Hol}(g_8) \subset \text{Spin}7$.

**Corollary [4]** The ‘spherical metric’

\[ dt^2 + (\sin^2 t) g_{NK} \]

has weak holonomy $G_2$. 

![Diagram](image_url)
$S^1$ quotients of $G_2$ manifolds

If $(Y, g_7)$ has holonomy $G_2$ then $Q = Y/S^1$ has a symplectic $SU(3)$ structure with

$$g_7 = f^2(dt + \eta)^2 + \frac{1}{f} g_{\text{IIA}}$$

in which $f$ is a function on $Q$ (the dilaton).

Theorem [2] Each 3-symmetric space $Z$ admits a $S^1$ action for which $Z/S^1 \cong S^5$. Therefore, if $Y = \mathbb{R}^+ \times Z$, $Q \cong \mathbb{R}^6$ contains the $S^1$ fixed points ($f = 0$) as Lagrangian submanifolds.

Example A standard $S^1$ action on $F = \frac{U(3)}{U(1) \times U(1) \times U(1)}$ has fixed point set $S^2 \sqcup S^2 \sqcup S^2$, and $X/S^1$ has three $\mathbb{R}^3$’s intersecting at a point (generating D6 branes). A curved version is inherent in the geometry of

$$S^3 \times S^3 = \frac{SU(2) \times SU(2) \times SU(2)}{SU(2)}$$
A tri-Lagrangian example

Let $N^5 \to \mathbb{T}^2 \to T^3$ be a principal torus bundle with base 1-forms $e^1, e^3, e^5$ and connection 1-forms $e^4, e^6$ with

$$de^4 = e^{15}, \quad de^6 = e^{13}.$$ 

Then $N^5 \times S^1$ has a symplectic form

$$\omega = e^{12} + e^{34} + e^{56}.$$ 

**Theorem [10]** $N^5 \times S^1$ has a triple of mutually transverse Lagrangian submanifolds through each point.

Follows from the existence of closed simple 3-forms

$$\gamma = e^1 \wedge \theta(e^3) \wedge \theta^2(e^5)$$

$$\theta(\gamma) = \theta(e^1) \wedge \theta^2(e^3) \wedge e^5$$

$$\theta^2(\gamma) = \theta^2(e^1) \wedge e^3 \wedge \theta(e^5)$$

**Remark** $Sp(n, \mathbb{R})$ acts almost transitively on triples of transverse Lagrangian subspaces of $\mathbb{R}^{2n}$, with stabilizer $O(n)$. 

\begin{center}
\includegraphics[width=0.3\textwidth]{tri-lagrangian_example.png}
\end{center}
Half flat $SU(3)$ structures

In 6 dimensions, an $SU(3)$ structure is characterized by a 2-form $\omega$ and a $(3,0)$-form $\psi^+ + i\psi^-$. A $G_2$ structure is defined in $6+1$ dimensions by setting

$$\varphi = \omega \wedge dt + \psi^+$$

$$\ast \varphi = \psi^- \wedge dt + \frac{1}{2} \omega \wedge \omega.$$

The holonomy reduces iff

$$d\varphi = 0, \quad d \ast \varphi = 0$$

implying the half-flat conditions

$$d(\frac{1}{2} \omega \wedge \omega) = 0, \quad d\psi^+ = 0.$$

Example $N^5 \times S^1$ has a half-flat structure with

$$\psi^+(0) = e^{135} - e^{146} - e^{236} - e^{245}$$

$$\frac{1}{2} \omega(0) \wedge \omega(0) = e^{3456} + e^{5612} + e^{1234}$$

Problem: Find $\omega(t), \psi^+(t)$ satisfying the reduction.
Solution: Let $4t = 2u + \sin 2u$ and $x = \cos u$, $y = \sin u$.

$$xe^1, \quad (1+y)^{1/2}e^3, \quad (1-y)^{1/2}e^5,$$
$$x^2du, \quad (1-y)^{-1/2}e^4, \quad (1+y)^{-1/2}e^6$$

is an ON basis for a metric $g_{CY}$ on $M^6 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times N^5$ with $\text{Hol}(g_{CY}) = SU(3)$. The Kähler form is

$$\omega = x^3 e^1 du + e^{45} - e^{63}.$$ 

Proposition $M^6$ has a pencil of special Lagrangian submanifolds through each point.

Follows from the existence of closed 3-forms

$$e^{146}$$

$$x(1+y)e^{34}du + x(1-y)e^{56}du + x^2e^{135}$$

$$xe^{46}du + (1-y)e^{156} + (1+y)e^{134}$$

$$x^3e^{35}du$$

spanning the $S^1$ orbit $\{ \exp(sJ) \cdot e^{146} \}$ containing $\psi^\pm$. 

9
Kähler quotients of $G_2$ metrics

Assumption $Y^7$ has a metric with $\text{Hol} = G_2$ and the $SU(3)$ manifold $(Y/S^1, g_{IIA})$ is Kähler:

\[
\begin{array}{ccc}
Y & \downarrow & Y/S^1, g_{IIA} \\
\mu^1(t)/S^1 & \hookrightarrow & Y/S^1, g_{IIA} \\
\downarrow S^1 & \left/ \mathbb{C}^* \right. & M^4
\end{array}
\]

Theorem [5] Suppose that a 4-manifold $M$ has

- a $c\times$ structure $J_1$ and holomorphic 2-form $\omega_2 + i\omega_3$,
- a 1-parameter family of Kähler forms $\tilde{\omega} = \tilde{\omega}(t)$ s.t.

\[
\tilde{\omega}''(t) = 2i \partial \bar{\partial} f
\]

where $t\tilde{\omega} \wedge \tilde{\omega} = f\omega_2 \wedge \omega_2$. Then a rank 3 bundle over $M$ admits a Ricci-flat metric $g$ with $\text{Hol}(g) \subseteq G_2$. 
If $f$ is constant on $M$ then $\tilde{\omega}''(t) = 0$ and
\[
\tilde{\omega} = (p+qt)\omega_0 + (r+st)\omega_1,
\]
with $(\omega_1, \omega_2, \omega_3)$ a hyperkähler structure and $\omega_0$ an additional closed 2-form.

**Example** Take
\[
\begin{align*}
\omega_0 &= zdx \wedge dz - dw \wedge dy \\
\omega_1 &= zdx \wedge dy + dz \wedge dw + xdy \wedge dz \\
\omega_2 &= zdx \wedge dz + dw \wedge dy \\
\omega_3 &= dx \wedge dw - xdx \wedge dy + zdy \wedge dz,
\end{align*}
\]
corresponding to a Gibbons-Hawking (HK) metric
\[
z(dx^2 + dy^2 + dz^2) + \frac{1}{z}(dw - xdy)^2
\]
conformal to a left-invariant one on $\mathbb{CH}^2$ [8].
Solutions with $f$ non-constant are found starting from
\[ \omega_1 = \frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v}), \quad \omega_2 + i\omega_3 = du \wedge dv \]
on $T^4 = \mathbb{C}^2/\mathbb{Z}^4$ and setting $\tilde{\omega} = \omega_1 + i\partial\bar{\partial}\phi$. Then
\[ \tilde{\omega} \wedge \tilde{\omega} = \mathcal{M}(\phi)\omega_1 \wedge \omega_1, \]
and $t\mathcal{M}(\phi) = f = \frac{1}{2}\phi''$.

**Example** If $u = x + iy$, there are solutions
\[ \phi(t, x, y) = \frac{1}{3}t^3 + \ell(t)m(x, y), \]
where $\ell'' - t\ell = 0$ and $\Delta m + m = 0$ such as
\[ f(t, x) = t + \frac{1}{2}t\text{Ai}(t)\sin x. \]
\( G_2 \) quotients of hyperkähler metrics

Geometric engineering a partial deformation of \( \mathbb{C}^2/\Gamma \) relative to a Hermitian symmetric space \( \frac{G}{G' \times S^1} \):

\[
\begin{array}{ccc}
U & \xleftarrow{K'} & \mu^{-1}(0) \subset \mathbb{H}^{n+1} \\
\downarrow & & \downarrow \\
\mathbb{R}^3 & \xleftarrow{\pi} & U/S^1 \\
\downarrow & & \downarrow \\
M^4 & & \mathbb{H}\mathbb{P}^n
\end{array}
\]

- If \( K = K' \times S^1 \) Kronheimer’s construction [12] gives

\[
\mathbb{H}^{n+1} \sslash K = \pi^{-1}(x) = \begin{cases} 
\mathbb{C}^2/\Gamma, & x = 0 \\
\mathbb{C}^2/\Gamma', & x \neq 0
\end{cases}
\]

- \( U = \mathbb{H}^{n+1} \sslash K' \) is a cone over the twistor space of an Einstein self-dual (QK) orbifold \( M^4 \) [13].
Example $G = \text{SU}(6)$, $G' = \text{SU}(5)$, $K = T^5$, $K' = T^4$:

$U/S^1$ is a cone over $\mathbb{WCP}^3_{5,5,1,1}$ and corresponds to five D6 branes intersecting a D6 brane at $0 \in \mathbb{R}^6$.

Conjecture [3] This has a metric with $\text{Hol} = G_2$.

Alternative scenario:

Same definition of $U$, but with $S^1 \subset \text{Sp}(1)$. Then $S^1$ is not triholomorphic, and $U/S^1$ is a cone over the twistor space of $M$.

Example $n = 2$, $K' = U(1)$ and $M = \mathbb{WCP}^2_{p,q,r}$ [9].
Additional references


