COMPLEX STRUCTURES ON $\mathbb{R}^6$

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Based on theory of

\[
\begin{cases}
\text{Penrose (1967)} \\
\text{Atiyah-Hitchin-Singer (1978)} \\
\text{Pontecorvo (1992)} \\
\text{Slupinski (1996)}
\end{cases}
\]

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Orthogonal complex structures

Definition: An OCS on an open set \( \Omega \) of \( \mathbb{R}^{2n} \) or \( S^{2n} \) is a complex structure \( J \) compatible with the Euclidean or round metric:
\[
g(JX, JY) = g(X, Y)
\]

Equivalently, an OCS is a \( (C^1) \) map \( J: \Omega \to Z_n \) satisfying the usual integrability condition, where
\[
Z_n = \{ M \in SO(2n) : M^2 = -I \} \cong \frac{SO(2n)}{U(n)}.
\]

Problem: Given \( \Omega \subseteq \mathbb{R}^{2n} \), classify OCS’s on \( \Omega \) up to conformal equivalence.

What exceptional sets \( \Lambda = \mathbb{R}^{2n} \setminus \Omega \) occur?
\( \delta = \dim_{Hf}(\Lambda) \) plays a key role.
The twistor space of $S^{2n}$

...provides an inductive definition of the spaces of linear complex structures:

$$\frac{SO(2n+1)}{U(n)} = \frac{SO(2n+2)}{U(n+1)} = Z_{n+1}$$

$$\downarrow_{Z_n}$$

$$S^{2n} = \frac{SO(2n+1)}{SO(2n)} \subset \Omega$$

Each fibre $\pi^{-1}(x) \cong Z_n$ is a complex submanifold of the total space $Z_{n+1}$.

Proposition: An OCS on $\Omega \subset S^{2n}$ is the same as a holomorphic section $\tilde{J}: \Omega \rightarrow Z_{n+1}$.

So look at algebraic $n$-folds $X$ in $Z_{n+1}$. 

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Facts in four dimensions

\[ Z_3 = \mathbb{P}^3 \]
\[ \downarrow Z_2 = \mathbb{P}^1 \]
\[ \infty \in S^4 \]
\[ \mathbb{P}^k = \mathbb{P}(\mathbb{C}^{k+1}) \]

Any hyperplane \( \mathbb{P}^2 \) of \( Z_3 \) contains exactly one fibre \( \pi^{-1}(x) \) and defines an OCS on \( S^4 \setminus \{x\} \).

There is a \( \mathbb{P}^1 \) worth of such OCS’s containing \( \pi^{-1}(\infty) \), and these are the constant OCS’s on \( \mathbb{R}^4 \). They are in fact the only OCS’s globally on \( \mathbb{R}^4 \).

Moreover, any OCS on \( \Omega \) where \( Hf^1(\mathbb{R}^4 \setminus \Omega) = 0 \) is associated to a hyperplane \( \mathbb{P}^2 \) [VS].

A “real” quadric in \( Z_3 \) gives an OCS on \( S^4 \setminus S^1 \). A generic quadric gives an OCS on the complement of a solid torus [VS].
Examples in six dimensions

\[ Z_4^* = Q^6 = \{ x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8 = 0 \} \subset \mathbb{P}^7 \]

\[ \downarrow Z_3 = \mathbb{P}^3 \]

\[ \infty \in S^6 \]

This time, any constant OCS on \( \mathbb{R}^6 \) arises from a “horizontal” \( \mathbb{P}^3 \) that intersects \( \pi^{-1}(\infty) \) in a \( \mathbb{P}^2 \). These OCS’s are parametrized by \( (\mathbb{P}^3)^* \cong Z_3 \).

There exists a global section \( \tilde{J} : S^6 \to Q^6 \) such that \( \text{Aut}(S^6, J) \cong G_2 \) and \( \tilde{J}^\perp = Q^5 \) is holomorphic!

But there is no OCS \( J \) on \( S^6 \) because \( \tilde{J}(S^6) \) cannot be a Kähler submanifold of \( Q^6 \) [L].

There does not exist a complex structure \( J \) on \( S^6 \) for which \( \text{Aut}(S^6, J) \) has an open orbit [HKP]. A hypothetical complex structure on \( S^6 \) gives a 1-para family of exotic complex structures on \( \mathbb{P}^3 \).
"Warped product" structures

Consider an almost complex structure $J$ on

$$
\mathbb{R}^6 = \mathbb{C} \oplus \mathbb{R}^4
$$

$$
J = J_0 + K(z),
$$

where $K(z)$ is a constant OCS on $\mathbb{R}^4$ depending on $z \in \mathbb{C}$. If $K: \mathbb{C} \to \mathbb{P}^1$ is holomorphic then $J$ is integrable.

If $K$ is rational then the graph $\Gamma = \tilde{J}(\mathbb{R}^6)$ has “finite energy” in the sense that $Hf^6(\Gamma) < \infty$. In this case $\Gamma$ is an algebraic 3-fold in $Q^6$ [B].

Moreover, $\Gamma \cap \pi^{-1}(\infty) = \mathbb{P}^2$ but (unless $K = \text{const}$) this fibre contains a singular line $L \cong \mathbb{P}^1$ and

$$
\Gamma \subset Q^4_s = \{x_1x_2 + x_3x_4 = 0\} \subset \mathbb{P}^5.
$$

Theorem [BSV]: Any finite-energy OCS on $\mathbb{R}^6$ arises from a rational function $K$ as above.
Explicit coordinates

Let \([x_1, \ldots, x_8] = [x, y]\), so that
\[Q^6 = \{[x, y] \in \mathbb{P}^7 : x^\top y = 0\}\].

Suppose that \(x, y \in \mathbb{C}^4\) are both non-zero.

**Lemma:** \([x, y] \in Q^6\) if and only if \(x = My\), where
\[
M = \begin{pmatrix}
0 & -z_3 & -z_2 & -z_1 \\
z_3 & 0 & -\overline{z}_1 & \overline{z}_2 \\
z_2 & \overline{z}_1 & 0 & -\overline{z}_3 \\
z_1 & -\overline{z}_2 & \overline{z}_3 & 0
\end{pmatrix}.
\]

Moreover, \(\frac{1}{\|z\|}M \in SU(4) \cap \mathfrak{so}(4, \mathbb{C})\).

The twistor projection is given by
\([My, y] \mapsto z \in \mathbb{C}^3 \cong \mathbb{R}^6\),
with fibre parametrized by \([y] \in \mathbb{P}^3\). It is easy to identify the action on \(Q^6\) of the conformal group \(SO(7, 1)\). The latter contains the transformations
\([x, y] \mapsto [Ax, \overline{A}y], \quad A \in SU(4)\),
acting as \(SO(6)\) on \(\mathbb{R}^6\) via \(M \mapsto AMA^\top\).
Spinors and triality

Let $\Delta_\pm$ be the spin representations of $\text{Spin}(8)$. We can identify

$$ Z_4 = \{\text{pure spinor classes } [\xi] \in \mathbb{P}(\Delta_+) \}, $$

since any such $\xi$ defines a max iso subspace

$$ \Lambda^{1,0} = \{ v \in \mathbb{C}^8 : v \cdot \xi = 0 \}. $$

Now reduce to $U(4)$ by fixing $J \in Z_4$, w.r.t. which

$$ \Delta^+ = \Lambda^{0,0} \oplus \Lambda^{2,0} \oplus \Lambda^{4,0}, \quad \text{dim} = 1 + 6 + 1. $$

Then a generic pure spinor has the form

$$ e^\omega = 1 + \omega + \frac{1}{2} \omega \wedge \omega, \quad \omega \in \Lambda^{2,0}, $$

confirming that $Z_4$ is a non-singular 6-quadric $Q_+ \subset \mathbb{P}(\Delta_+)$.

Altogether we have three 6-quadrics

$$ Q_+ \subset \mathbb{P}(\Delta_+) $$
$$ Q_- \subset \mathbb{P}(\Delta_-) $$
$$ Q_0 \subset \mathbb{P}(\mathbb{C}^8). $$
Bidegree

Altogether we have three 6-quadrics
\[ Q_+ \subset \mathbb{P}(\Delta_+) \]
\[ Q_- \subset \mathbb{P}(\Delta_-) \]
\[ Q_0 \subset \mathbb{P}^0(\mathbb{C}^8). \]

\( Q_+, Q_- \) parametrize max iso subspaces of \( \mathbb{C}^8 \)
\( Q_0, Q_- \) parametrize max iso subspaces of \( \Delta_+ \),
giving rise to two families of \( \mathbb{P}^3 \)'s in the twistor space \( Q^6 = Q_+ \) of \( S^6 \):

“vertical” ones, either fibres \( \pi^{-1}(x) \) or
twistor spaces of conformal \( S^4 \)'s;

“horizontal” ones, each \( 1:1 \) outside some \( x \in S^6 \).

One from each family generates
\[ H_6(Q_+, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}. \]

Corollary: A finite-energy OCS \( J \) on \( \mathbb{R}^6 \) gives an
algebraic 3-fold \( \Gamma \) in \( Q_+ \) of bidegree \( (1, p) \).
Classification of 3-folds of order one

Theorem [BV]: An irreducible 3-fold $X$ in $Q^6$ of bidegree $(1, p)$ is one of:

(i) a horizontal $\mathbb{P}^3$ ($p = 0$),
(ii) a smooth 3-quadric $Q^3$ ($p = 1$),
(iii) the cone over a Veronese $\mathbb{P}^2 \subset Q^4$ ($p = 3$),
(iv) a Weil divisor in a rank 4 quadric $Q^4_s$ ($p \geq 1$).

Example of (iii):

$$Q^2 \subset Q^3$$

$$S^2 \downarrow \quad \downarrow$$

$$S^2 \subset S^6 \subset \mathbb{R}^3 \oplus \mathbb{R}^4 = \text{Im} \mathbb{O}$$

But we require $\pi : X \rightarrow S^6$ to be 1 : 1 except over $\infty$, and the exceptional fibre $\pi^{-1}(\infty)$ must in fact contain a $\mathbb{P}^2$. This rules out (ii) and (iii).
**Working in the singular 4-quadric**

In case (iv), take
\[
P^5 = \{ [x_1, \ldots, x_6, 0, 0] \} \subset \mathbb{P}^7,
\]
\[
Q^4_s = \{ x_1x_5 + x_2x_6 = 0 \} \subset \mathbb{P}^5,
\]
\[
L = \{ [0, 0, x_3, x_4, 0, 0] \} \subset Q^4_s.
\]

Example: Taking \( x_3x_6 + x_4x_5 = 0 \) defines
\[
\text{Segre}(\mathbb{P}^1 \times \mathbb{P}^2) \cup \mathbb{P}^3 \subset Q^4_s \subset \mathbb{P}^5.
\]

For a non-constant OCS,
\[
X = \overline{\Gamma} \subset Q^4_s, \quad L \subset X \cap \pi^{-1}(\infty) \cong \mathbb{P}^2,
\]
and we get a different subcase of (iv). Let
\[
P_\lambda = \{ [ax_1, ax_2, x_3, x_4, bx_2, -bx_1, 0, 0] \} \cong \mathbb{P}^3,
\]
with \( \lambda = b/a \in \mathbb{P}^1. \)

**Lemma:** Each \( X \cap P_\lambda \cong \mathbb{P}^2 \) defines the fibre of a projection \( X \setminus L \to C \subset \mathbb{P}^1 \times \mathbb{P}^1. \)

It follows that \( X \setminus P_0 \) is a graph \( \tilde{J} \) over \( \mathbb{R}^6 \), and \( J \) is a warped product.
Conclusions

Theorem v2 [BSV]: A finite-energy OCS on $S^6$ minus a finite set of points is a warped product arising from a rational function $K: \mathbb{C} \to \mathbb{P}^1$.

Counterexample: If $K = \wp$ is doubly-periodic then $Hf^6(\Gamma) = \infty$, but $J$ induces a non-constant OCS on the torus $T^6$.

Other examples include $S^6 \setminus S^2 \cong S^3 \times H^3$.

The generalization to $\mathbb{R}^{2n}$ with $n \geq 4$ is unclear, but an algebraic OCS $J$ on $\mathbb{R}^{2n}$ defines an $n$-fold in $\mathbb{Z}_{n+1}$ such that

$$\overline{\Gamma} \cap \pi^{-1}(\infty) \subset \mathbb{Z}_n,$$

and (if we are lucky) an OCS on $\mathbb{R}^{2n-2}$.

Example: If $J$ is “asymptotically constant” then

$$\overline{\Gamma} \cap \pi^{-1}(\infty) = \mathbb{P}^{n-1},$$

and $J$ must in fact be conformally constant.