Geometry & Topology of Wolf Spaces

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1.1 Riemannian symmetric spaces

\[ M^d = \frac{G}{H} \]

If the isometry group \( G \) acts faithfully then \( H \) is the holonomy group and \( H \subset O(d) \).

The action of \( H \) on each tangent space \( T_m M \) can give a model for more general Riemannian manifolds:

- Kähler, quaternion-Kähler, \( H \)-structure with torsion, …

Some aspects of the topology only depend on the holonomy \( H \). Others depend on \( G \); spaces with a common isometry group have a hidden affinity *
1.2 Quaternionic symmetric spaces

are analogues of the Hermitian symmetric spaces. The classical compact ones of real dimension $4n$ are

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}$$

$$\text{Gr}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))}$$

$$\text{Gr}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)}.$$

Of these, only $\text{Gr}_2(\mathbb{C}^{n+2})$ (and $\text{Gr}_4(\mathbb{R}^6)$) are Kähler.

Exceptional ones have real dimensions 8, 28, 40, 64, 112:

$$\begin{array}{cccccc}
G_2 & F_4 & E_6 & E_7 & E_8 \\
SO(4)' & Sp(3)Sp(1)' & SU(6)Sp(1)' & Spin(12)Sp(1)' & E_7Sp(1).
\end{array}$$

Recall that $SO(4) = Sp(1)Sp(1) = Sp(1) \times_{\mathbb{Z}_2} Sp(1)$ is not simple.
1.3 Wolf’s construction

Given a compact simple Lie algebra $\mathfrak{g}$, choose a Lie subalgebra $\mathfrak{su}(2) = \mathfrak{sp}(1)$ arising from a highest root. Set

$$H = KSp(1) = \{g \in G : \text{Ad}(g)(\mathfrak{su}(2)) = \mathfrak{su}(2)\}.$$ 

Then

$$M = \frac{G}{KSp(1)} = \frac{G}{H}$$

is quaternion-Kähler (QK), meaning

$$H \subseteq Sp(n)Sp(1) \subset SO(4n).$$

This means that $M$ admits a parallel 4-form $\Omega$ equivalent to

$$1234 + 5678 + \frac{1}{3}(1256 + 1278 + 3456 + 3478 + 1357 + 1386 + 4257 + 4286 + 1458 + 1467 + 2358 + 2367).$$

All compact QK homogeneous spaces arise like this (Alekseevsky). What happens if we take other $\mathfrak{su}(2)$ ’s in $\mathfrak{g}$?
1.4 The isotropy representations

of these spaces have special merit. For each Wolf space $G/K Sp(1)$, we get a symplectic representation $K \to \text{End}(\mathbb{C}^{2n})$.

**Example.** Consider $\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$, where

$$m_c = \Lambda^{3,0} \otimes \Sigma = \mathbb{C}^{40}, \quad \Sigma = \mathbb{C}^2.$$

But $E_6$ also acts on

$$\mathbb{C}^{27} = (\Lambda^{1,0} \otimes \Sigma) \oplus \Lambda^{0,2}$$

$$= 6 + 6 + 15$$

$$= \langle a_i \rangle \oplus \langle b_i \rangle \oplus \langle c_{ij} \rangle$$

giving Schl"afli’s configuration of the 27 lines on a cubic surface:
2.1 Nilpotent coadjoint orbits

can obtained [JM] by choosing $\mathfrak{su}(2) \subset \mathfrak{g}$ and setting

$$\mathcal{N} = (\text{Ad } G_c)(e) \subset \mathfrak{g}_c, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

Kronheimer proved that $Z = \mathcal{N}$ admits a hyperkähler metric, but $\mathcal{N}/\mathbb{C}^*$ is compact only if $\mathcal{N}$ is minimal. In this case, $Z = G/KU(1)$ is the so-called twistor space that fibres over $G/KSp(1)$.

**Example.** For $G_2$ there are four non-zero orbits:

$$\begin{align*}
\mathfrak{su}(2)_+ & \subset \mathfrak{so}(4) \subset \mathfrak{g}_2 \\
\mathfrak{su}(2)_- & \subset \mathfrak{so}(4) \subset \mathfrak{g}_2 \\
\mathfrak{so}(3) & \subset \mathfrak{so}(4) \subset \mathfrak{g}_2 \\
\mathfrak{so}(3)_{pr} & \subset \mathfrak{g}_2
\end{align*}$$

Then

$$Z = \frac{G_2}{U(2)_+} \longrightarrow \frac{G_2}{SO(4)} = M^8.$$ 

By contrast,

$$G_2 \frac{\mathbb{R}^7}{U(2)_-} \longrightarrow \frac{G_2}{SU(3)} = S^6,$$

in which $SU(3)$ is the fixed point set of an automorphism of order 3 on $G_2$. 
2.2 Calibrations

The fundamental 3-form

\[ F(X, Y, Z) = \langle [X, Y], Z \rangle \]

on the Lie algebra \( g \) defines a function \( f \) on \( G = \text{Gr}_3(g) \) for which

(i) \( V \in G \) is critical iff \( V \) is a subalgebra;

(ii) \( f \) achieves its maximum on the Wolf space parametrizing minimal \( su(2) \)'s;

(iii) we can easily compute \( \text{Hess}(f) \) at any \( V = su(2) \).

Example. Let \( V = so(3)_{\text{pr}} \subset su(3) \). Then \( su(3)_c \cong \Sigma^2 + \Sigma^4 \) where \( \Sigma^q = S^q(\mathbb{C}^2) \), and

\[ T_V G \cong V \otimes V^\perp \cong \Sigma^2 \otimes \Sigma^4 \cong \Sigma^2 \oplus \Sigma^4 \oplus \Sigma^6 \]

\[ + 0 - \]

Whilst the critical manifold \( C^5 = \frac{SU(3)}{\mathbb{Z}_3 SO(3)} \) has tangent space \( \Sigma^4 \), both

\[ \Sigma^2 \oplus \Sigma^4 \cong \Sigma^3 \otimes \Sigma^1 \]
\[ \Sigma^4 \oplus \Sigma^6 \cong \Sigma^5 \otimes \Sigma^1 \]

are quaternionic or \( Sp(2)Sp(1) \) modules.
2.3 Morse theory

The associated unstable manifold $U^8$ is the union of $C^5$ and the upward flow lines of the vector field $\text{grad } f$. It is diffeomorphic to a rank 5 vector bundle over $C^5$ with fibre $\Sigma^4$, and $T_U = \Sigma^2 \oplus \Sigma^4$. Moreover, [G], it is a $\mathbb{Z}_3$ quotient

$$U^8 = \frac{1}{\mathbb{Z}_3} \left( \frac{G_2}{SO(4)} \setminus \mathbb{C}P^2 \right).$$

More generally, if $G$ is any compact simple Lie group,

**Theorem** [S]. $f$ is a Morse-Bott function on $\mathbb{G}r_3(g)$. The unstable manifold determined by a critical manifold containing $su(2) \subset g$ is QK and its twistor space is $\mathcal{N}/\mathbb{C}^*$. 

A discrete version of the construction (and Nahm’s equations) gives rise to the following dynamical system. Given a subspace $V = \langle v_1, v_2, v_3 \rangle \subset g$, define

$$V' = \langle [v_2, v_3], [v_3, v_1], [v_1, v_2] \rangle.$$

For generic $V$, one expects

$$V^{(n)} \rightarrow \text{su}(2)_{\text{min}} \in \frac{G}{KSp(1)} \quad \text{as } n \rightarrow \infty.$$
3.1 The twistor space

The total space of the fibration

$$Z = \frac{G}{KU(1)} \xrightarrow{\pi} \frac{G}{KSp(1)} = M$$

is a real adjoint orbit in $\mathfrak{g}$ and a polarized variety. Wolf pointed out that $Z$ has a complex contact structure $\theta$.

**Example.** $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ has anticanonical bundle $\kappa = \mathcal{O}(2n+2)$.

In general, only $L = \mathcal{O}(2)$ is defined and $Z$ is Fano of index $n + 1$. There is a holomorphic short exact sequence

$$0 \to D \to TZ \xrightarrow{\theta} L \to 0$$

of vector bundles, in which $D$ is a horizontal distribution and $\theta \in H^0(Z, \Omega^1(L))$. The fibre

$$\pi^{-1}(m) \cong \frac{Sp(1)}{U(1)} = \mathbb{C}P^1 = S^2$$

parametrizes compatible almost complex structures on $T_mM$ and has normal bundle $2n\mathcal{O}(1)$. 

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3.2 The Penrose correspondence

between $M$ and $Z$ is much more general:

<table>
<thead>
<tr>
<th>$M$ positive QK</th>
<th>$Z$ contact Fano</th>
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<tr>
<td>point</td>
<td>rational curve</td>
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<td>complex structure</td>
<td>holomorphic section</td>
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<td>$b_2(M) + 1$</td>
<td>$= b_2(Z)$</td>
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<tr>
<td>Killing field $X$</td>
<td>$s \in H^0(Z, \mathcal{O}(2))$</td>
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<td>Dirac operator</td>
<td>$\bar{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$</td>
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</table>

The interpretation of solutions to linear field equations as elements of Čech cohomology is the essence of the Penrose programme.

**Big questions.** Is every compact QK manifold ($H \subseteq Sp(n)Sp(1)$, automatically Einstein) with scalar curvature $s > 0$ necessarily symmetric? Is every contact Fano manifold $Z^{2n+1}$ homogeneous? Open if $n \geq 3$. 

3.3 A moment mapping

Suppose that $M^{4n}$ is a QK manifold with an isometry group $G$ with $\dim G = \ell$. Consider the morphism

$$\Phi: Z \rightarrow \mathbb{P}(g_c^*) = \mathbb{P}(H^0(Z, \mathcal{O}(L))^*)$$
$$z \mapsto [s_1, \ldots, s_\ell],$$

a moment map for the contact structure $\theta$ preserved by $G_c$.

Suppose $\varphi \in S^k g^*$ is an invariant polynomial. Then either

(a) the image of $\varphi^\parallel \in S^k g_c \rightarrow H^0(Z, \mathcal{O}(L^k))$ is non-zero, or

(b) $\Phi(Z)$ lies in the zero set of $\varphi$.

In (a), the image of $\varphi^\parallel$ vanishes on $k$ local sections of $Z \rightarrow M$ each of which determines a $G$-invariant complex structure of type $aI + bJ + cK$. If these are not present, then (b) asserts that $\Phi(Z)$ lies in the nilpotent variety in $\mathbb{P}(g_c)$.

**Related question.** Does a positive QK manifold $M^{4n}$ always have isometries?

Yes, at least if $n \leq 4$. 
4.1 Witten rigidity

Let $M^{4n}$ be a Wolf space or QK manifold with isometry group $G$. Its virtual $Spin(4n)$ representation is

$$\Delta_+ - \Delta_- = \Lambda_0^n (E - \Sigma^1) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes \Sigma^q$ with $E = \mathbb{C}^{2n}$, $\Sigma^q = S^q(\mathbb{C}^2)$.

The coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \rightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $i^{p,q} = \int_M \text{ch}(R^{p,q}) \hat{A}(M)$.

**Theorem.** $(-1)^p i^{p,q} = \begin{cases} 0 & \text{if } p+q < n, \\ b_{2p-2} + b_{2p} & \text{if } p+q = n, \\ \dim G & \text{if } p=0, q=n+2. \end{cases}$

This is a $G$-equivariant statement, and if $p + q \leq n$ the associated $G$-modules are trivial.
4.2 Application to dimension 8

Index theory (and the \( \gamma \) filtration) gives a linear constraint on the Betti numbers and estimates on the isometry group, in terms of characteristic classes including the integral class \( u \in H^4(M, \mathbb{Z}) \) that represents \( \Omega \).

Example. If \( d = \dim M = 8 \) then

\[
b_2 + 1 = b_4.
\]

This suggests that \( b_2 = 0 \) or \( 1 \). Moreover

\[
\dim G = 5 + \int_M u^2.
\]

If \( b_4 = 1 \) then

\[
\dim G = \begin{cases} 
5 + 16 & = \dim Sp(3), \\
5 + 9 & = \dim G_2, \\
5 + 4 & = \dim Sp(1)^3, \\
5 + 1 & = \dim SO(4), 
\end{cases}
\]

corresponding to

\[
\mathbb{H}P^2 = \frac{Sp(3)}{Sp(2) \times Sp(1)'} \quad \frac{G_2}{SO(4)'} \quad \frac{\mathbb{H}P^2}{(\mathbb{Z}_2)^2}'.
\]

Only the first two spaces are non-singular.
4.3 Towards a classification

Let $M^{4n}$ be a compact positive QK manifold.

**Theorem** [LS,W]. If $b_2(M) > 0$ then $M$ is isometric to $\text{Gr}_2(\mathbb{C}^{n+2})$.

Proof uses Mori theory on the twistor space $Z$. If $b_2(Z) > 1$ there exists a second family of rational curves on $Z$ transverse to the fibres over $M$, and a Fano contraction

$$Z \to \mathbb{CP}^{n+1}$$

with its fibres tangent to the contact distribution $D$. This forces $Z = \mathbb{P}(T^*\mathbb{CP}^{n+1})$.

**Corollary** [GMS]. The only Wolf spaces with a (stably) almost complex structure are $\mathbb{HP}^1 = S^4$ and $\text{Gr}_2(\mathbb{C}^{n+2})$.

Proof. If $n > 1$ then

$$R^{1,n-3} \oplus R^{1,n-1} \cong (E \otimes \Sigma^1) \otimes \Sigma^{n-2} \cong (T^{1,0} \oplus T^{0,1}) \otimes \Sigma^{n-2},$$

forcing $-i^{1,n-1} = 1 + b_2$ to be even.
4.4 Spin and the $\hat{A}$ genus

Let $M^{4n}$ be compact, $H \subseteq Sp(n)Sp(1)$ and $s > 0$.

Key fact: if we ignore $\mathbb{HP}^n$ then $M$ is spin iff $n$ is even. In this case, $\hat{A}(M) = 0$ because $s > 0$. There is a dichotomy according to the parity of $n$.

**Theorem** [PS]. A positive QK manifold $M^8$ is isometric to a Wolf space.

Attempts to push this to dimension 12 relied on elliptic genera [HH], but appear to need the assumption $\hat{A}(M) = 0$. Significant progress has been made recently by Amann in higher dimensions:

**Theorem.** If $b_4 = 1$ and $3 \leq n \leq 6$ then $M \cong \mathbb{HP}^n$.

All exceptional Wolf space have $b_4 = 1$, including $\frac{F_4}{Sp(3)Sp(1)}$.

**Theorem** [A]. If $n = 5$ and $\hat{A}(M) = 0$ then $\dim G \geq 15$ and $M$ is a Wolf space if (for example) $\int_M u^5 > 384$.  


5.1 Betti numbers of symmetric spaces

Consider the Poincaré polynomial
\[ P(t) = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \]
and assume Euler characteristic \( \chi = P(-1) \neq 0 \). Then
\[ \log P(t - 1) = \log \chi - d t + \phi t^2 + \cdots \]
where \( d = \dim M \), and
\[
2\phi = \frac{P''(-1)}{2\chi} - \frac{1}{8}d^2.
\]

By construction, this coefficient is additive for products:
\[ \phi(M \times N) = \phi(M) + \phi(N). \]

**Theorems**

(i) If \( M^{4n} \) is compact hyperkähler, \( \chi = 0 \) or \( \phi = -\frac{5}{6}n \).

(ii) [FS]. If \( M^d = G/H \) is an irreducible compact SS of type ADE or a HSS,
\[
\phi = \frac{1}{12}(h(g) - 2)d,
\]
where \( h(g) \) is the Coxeter number. If \( M^{4n} \) is an ADE Wolf space then \( \phi = \frac{1}{3}n^2 \).
5.2 The case of $E_8$

The odd Betti numbers of a positive QK manifold $M^{4n}$ all vanish and the intersection form is definite: $b_{2i+1} = 0$ and $b_{2n} = b_2^+$. The signature of an ADE Wolf space space equals its rank: $b_{2n} = r$. Its Euler characteristic $\chi$ equals the number of positive roots.

$E_8/E_7Sp(1)$ has 8 primitive cohomology classes $\sigma_k \in H^{4k}(M, \mathbb{R})$;

\[ H^{56}(M, \mathbb{R}) = \langle \sigma_k \cup u^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \rangle, \]

exhibiting ‘secondary Poincaré duality’ about degree $n = 28$:

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Question [HS]. What happens over the integers? Is the quadratic form $H^{56}(M, \mathbb{Z}) \times H^{56}(M, \mathbb{Z}) \rightarrow \mathbb{Z}$ diagonalizable or the $E_8$ lattice? The quaternionic volume is

\[ \int_M u^{28} = 2^3 3^2 5^2 7 31 37 41 43 47 53 = \frac{5! 9! 57!}{19! 23! 29!} = 63468758442600. \]
## 5.2 References, see arXiv or MathSciNet

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