Tensors and representations

Let $(M, g)$ be a Riemannian manifold of dimension $d$.
Let $\phi$ be a tensor with $\{a \in SO(n) : a \cdot \phi = \phi\} = G$.

The holonomy group is a subgroup of $G$ iff $\nabla \phi \equiv 0$. Given

$$\bigwedge^2 T^* \cong so(d) = g \oplus g^\perp,$$

Lemma $\nabla \phi$ can be identified with an element of the space

$$T^* \otimes g^\perp =: \mathcal{W} = \bigoplus_{i=1}^{N} \mathcal{W}_i,$$

with say $N$ irreducible components.

Examples

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\phi$</th>
<th>$G$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2n$</td>
<td>almost complex structure $J$</td>
<td>$U(n)$</td>
<td>4</td>
</tr>
<tr>
<td>$2n$</td>
<td>non-degenerate 2-form $\omega$</td>
<td>$U(n)$</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>positive generic 3-form</td>
<td>$G_2$</td>
<td>4</td>
</tr>
<tr>
<td>$4k$</td>
<td>quaternionic 4-form $\sum_{i=1}^{3} \omega^i \wedge \omega^i$</td>
<td>$Sp(k)Sp(1)$</td>
<td>6</td>
</tr>
</tbody>
</table>

1
Sixteen classes of almost Hermitian manifolds

Given $(M^{2n}, g)$ with $\phi = J$ and $G = U(n)$,

**Proposition [GH 80]**

$$\nabla J \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$$

Two halves of equal dimension $n^2(n - 1)$ lead to a sort of duality:

- $\nabla J \in \mathcal{W}_1 \oplus \mathcal{W}_2 \iff (d\omega)^{1,2} = 0$
- $\nabla J \in \mathcal{W}_3 \oplus \mathcal{W}_4 \iff M$ is Hermitian

$M$ is Kähler iff $\nabla J \equiv 0$. 
Sixteen classes of almost Hermitian manifolds

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- \(\nabla J \in \mathcal{W}_3 \oplus \mathcal{W}_4 \Leftrightarrow M\) is **Hermitian**

If \(M\) is locally conformally Kähler then \(\nabla J \in \mathcal{W}_4\).

**Proposition [G65]** If \(M\) has \((d\omega)^{1,2} = 0\), and \(M' \subset M\) is pseudo-holomorphic, then \(M'\) is minimal.

**Definition** \(M\) is **nearly-Kähler** if \(\nabla J \in \mathcal{W}_1\), equivalently \((\nabla_X J)X = 0\) for all \(X\).

Basic model is \(S^6 = \frac{G_2}{SU(3)}\), but there is a large class of homogeneous examples.
3-symmetric spaces

$M = G/H$ is a 3-symmetric space if $H$ is the fixed point set of an automorphism $\theta$ of $G$ with $\theta^3 = 1$. Defining $J = \frac{1}{\sqrt{3}}(2\theta + 1)$ gives a canonical a.c.s. on $T_m M$.

Theorem [WG 68] Any 3-symmetric space has a nearly-Kähler metric.

Classification includes

- generalizations of $S^6$ with irreducible isotropy (e.g. $\frac{E_8}{SU(9)}$)
- $G \times G = \frac{G \times G \times G}{G}$ (e.g. $S^3 \times S^3$)
- twistor spaces over symmetric spaces (e.g. $\mathbb{CP}^3$, $\mathbb{RP}^3$)
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- generalizations of \( S^6 \) with irreducible isotropy (e.g. \( \frac{E_8}{SU(9)} \))
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- twistor spaces over symmetric spaces (e.g. \( \mathbb{CP}^3, \mathbb{RP}^3 \))

The latter are extensively used in the study of minimal surfaces and harmonic maps.

Any such \( Z \) has an integrable complex structure \( J_2 \) in addition to \( J = J_2 \).

**Proposition** If \( f: \Sigma \to (Z, J_2) \) is a pseudo-holomorphic curve then \( \pi \circ f \) is harmonic.

E.g.

\[
\begin{align*}
\frac{U(p + q + 1)}{U(p) \times U(q) \times U(1)} &= Z \\
S^2 = \mathbb{CP}^1 &\to \frac{U(n + 1)}{U(n) \times U(1)} = \mathbb{CP}^n
\end{align*}
\]
Metrics with exceptional holonomy

**Theorem** [G76] If $M^6$ is nearly-Kähler (and $\nabla J \neq 0$), it is Einstein.

E.g. $(\mathbb{CP}^3, J_2)$. In fact, $R = sR_{S^6} + R_{CY}$ with scalar curvature $s > 0$. The theory of Killing spinors implies that the cone $M \times \mathbb{R}^+$ has a Ricci-flat metric with holonomy in $G_2$.

More generally, if $X^7$ has a 3-form $\phi$ defining a $G_2$-structure, there is a vector cross product $\wedge^2 T^* \rightarrow T^*$ [G67], and

$$\nabla \phi \in T^* \otimes \mathfrak{g}_2^\perp \cong T^* \otimes T^* \cong \mathbb{R} \oplus S^2 T^* \oplus T^* \oplus \mathfrak{g}_2$$

**Corollary** [FG82]
- $X$ has holonomy in $G_2$ iff $d(*\phi) = 0$ and $d\phi = 0$
- $\nabla \phi \in \mathbb{R}$ iff $d\phi = c(*\phi)$ ($\Rightarrow d(*\phi) = 0$)
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- $X$ has holonomy in $G_2$ iff $d(*\phi) = 0$ and $d\phi = 0$
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In the second case, $X^7$ has weak holonomy $G_2$ (associative subspaces are preserved by parallel transport [G71]), and the cone $X \times \mathbb{R}^+$ has a metric with holonomy in $Spin 7$.

Standard model $S^7 = \frac{Spin 7}{G_2}$ yields the flat metric on $\mathbb{R}^8$, but $\frac{SO(5)}{SO(3)} \times \mathbb{R}^+$ has holonomy equal to $Spin 7$. 
Curvature theorems

Theorem [G77] A compact Kähler manifold with nonnegative sectional curvature and 
s constant is locally symmetric (i.e. $\nabla R \equiv 0$).

On an almost-Hermitian manifold, $R = K + K^\perp$, where
- $K$ satisfies $K(W, X, Y, Z) = K(W, X, JY, JZ)$
- $K^\perp = C(\nabla \nabla J)$ has zero holomorphic sectional curvature

$k, K^\perp$ decompose further under $GL(n, \mathbb{C})$ and $U(n)$.

Proposition [G76] If $M$ is Hermitian then


This imposes $k = \frac{1}{6}n^2(n^2 - 1)$ equations on the Weyl tensor, itself of dimension $< 8k$.

Problem A Riemannian manifold $M^{2n}$ has a \textit{finite} number $k$ of orthogonal complex structures locally. What is the maximum value of $k$?
Volume $V(r)$ of a small geodesic ball $B(r)$

- $V(1) = \pi^{d/2} / (d/2)!$ in $\mathbb{R}^d$
- If $V(r_d) = 1$ then $r_d \sim \sqrt{\frac{d}{2\pi e}}$ as $d \to \infty$
- $B(r_d) \cap (\mathbb{R}^{d-1} \times [-0.4, 0.4])$ has volume $> 0.8$

[Zoom in]
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On a Riemannian manifold,

$$V(r) = \frac{(\pi r^2)^{d/2}}{(d/2)!} \left( 1 - \frac{s}{6(d+2)} r^2 + c_4 r^4 + c_6 r^6 + c_8 r^8 + \cdots \right)$$

**Theorem** [G73] $c_4 = \frac{8\|\text{Ric}\|^2 - 3\|R\|^2 + 5s^2 - 18\Delta s}{360(d+2)(d+4)}$.

**Examples** [GV 79] There exist metrics with

- $0 = s = c_4$ and $d = 4$
- $0 = s = c_4 = c_6$ and $d = 734$

Many other asymptotic expansions, and generalizations to tubes.

**Theorem** [G88] A tube of radius $r$ surrounding a hypersurface of degree $k$ in $\mathbb{CP}^n$ has volume $\pi^n (1 - (1 - k \sin^2 r)^n)/n!$
Example Geodesic cobweb on the paraboloid $z = xy$
Plotted using GEOEQ.m [G 94]
Invariant structures on Lie groups

Any compact simple Lie group $G^{2n}$ admits a complex structure, but no symplectic one. A nilpotent Lie group $N^{2n}$ may or may not admit left-invariant complex or symplectic structures.

Compact nilmanifolds $N/\Gamma$ never admit Kähler metrics (unless $N$ is abelian). Left-invariant forms provide a minimal model for deRham cohomology with non-zero Massey products. Dolbeault cohomology is less readily computed:

Proposition [CFG91] There exist complex nilmanifolds with

- $n = 4$ and $E_2 \neq E_\infty$
- $n = 6$ and $E_3 \neq E_\infty$.

Theorem [FGM91] A compact surface $(U, \omega)$ of genus $\geq 1$ with a symplectomorphism $\varphi: U \to U$ fixing $b \in H^1(U, \mathbb{Z})$ defines a circle bundle $E \to (U \times [0, 1])/\varphi$ that is symplectic and generally non-Kähler.

E.g. For $g=1$, $E$ is a Kodaira surface with $b_1 = 3$.

A generalization of the construction accounts for all symplectic manifolds with a free $S^1$ action.
6-dimensional nilmanifolds

**Theorem** There are 34 isomorphism classes of real 6-dimensional nilpotent Lie algebras \( \mathfrak{n} \) admitting structures as shown:

For a metric on \( \mathfrak{n} \), almost-Hermitian structures define points of \( \frac{SO(6)}{U(3)} \cong \mathbb{C}P^3 \).

**Example** For the complex Heisenberg group \( N \) or Iwasawa manifold \( N/\Gamma \),

\[
\nabla J \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \Leftrightarrow J \in \mathbb{C}P^2 \cup \mathbb{C}P^2,
\]

and these two ‘faces’ contain all 15 proper Gray-Hervella classes.
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Norm of component of \( \nabla J \) in each \( \mathcal{W}_i \)
is represented by respective colour.
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- $\nabla J \in \mathcal{W}_2 \iff J \in S^3$
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  - \( \nabla J \in \mathcal{W}_2 \Leftrightarrow J \in S^3 \)
  - \( \nabla J \in \mathcal{W}_3 \Leftrightarrow J \in \{\text{pt}\} \cup \mathbb{CP}^1 \)

Final picture displays \( \nabla J \) as a function of position on the two faces. Pure blue indicates Hermitian structures, and green symplectic ones.