Index theory and special structures on 8-manifolds

An introduction

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15 July 2014
1.1 Subgroups

\[ G_2 \subset \text{Spin} 7 \subset \text{Spin} 8 \]

\[ \text{SU}(4) \cup \text{Sp}(2) \subset \text{Sp}(2)\text{Sp}(1) \]

Sp(2) fixes a HK triple \( \omega_1, \omega_2, \omega_3 \)

Sp(2)Sp(1) is the stabilizer of \( \omega_1^2 + \omega_2^2 + \omega_3^2 = \Omega \)

Spin 7 is the stabilizer of \( -\omega_1^2 + \omega_2^2 + \omega_3^2 = \Phi \) [BH]
1.2 Euler number

**Proposition [GG].** If $M^8$ (compact and oriented) has a Spin 7 or an $\text{Sp}(2)\text{Sp}(1)$ structure then

$$8 \chi = 4p_2 - p_1^2$$

**Proof.** For an $\text{SU}(4)$ structure, $TM_c = T^{1,0} \oplus T^{0,1}$ has total Chern class

$$1 - p_1 + p_2 = (1 + c_2 + c_3 + c_4)(1 + c_2 - c_3 + c_4)$$

$$= 1 + 2c_2 + (c_2^2 + 2c_4)$$

so

$$8 \varepsilon = 8c_4 = 4p_2 - p_1^2.$$  

Argument extends because $\text{SU}(4)$, Spin 7, $\text{Sp}(2)\text{Sp}(1)$ share a maximal 3-torus!
1.3 Triality

\[ \text{Spin 8 acts on } \Delta = \Delta_+ \oplus \Delta_- \]
\[ \downarrow \text{ 2: 1} \]
\[ \text{SO}(8) \text{ acts on } T = \Lambda^1 \]

Outer automorphisms of Spin 8 permute \( T, \Delta_+, \Delta_- \).

Restricting to a maximal 4-torus,

\[ \Lambda^1 \text{ has 8 weights } \pm x_1, \pm x_2, \pm x_3, \pm x_4 \]
\[ \Delta_+ \oplus \Delta_- \text{ has 16 weights } \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4) \]

(with an even number of like signs for \( \Delta_+ \)).

If \( \sum x_i = 0 \) then \( \Lambda^1, \Delta_- \) have the same weights.

In fact \( \Lambda^1 \cong \Delta_- \) as Spin 7, Sp(2)Sp(1) modules.

In general, \( \Delta \otimes \Delta = \bigoplus \Lambda^i \). Here,

\[ \Delta_+ \otimes \Delta_+ \cong \Lambda^4_+ \oplus \Lambda^2 \oplus \Lambda^0 \]
\[ \Delta_+ \otimes \Delta_- \cong \Lambda^3 \oplus \Lambda^1 \]
2.1 A hat class

The complexified tangent bundle has Chern class

\[ c(TM_c) = \prod_{1}^{4} (1 - x_i^2) \]

so \( p_1 = \sum x_i^2 \). Its Chern character is

\[ \text{ch}(TM_c) = \sum_{1}^{4} (e^{x_i} + e^{-x_i}) = 8 + 2 \sum x_i^2 + \frac{1}{12} \sum x_i^4 \]

Similarly,

\[ \text{ch}(\Delta_+ - \Delta_-) = \prod_{1}^{4} (e^{x_i/2} - e^{-x_i/2}) = \varepsilon \hat{A}(M)^{-1} \]

where \( \varepsilon = x_1 x_2 x_3 x_4 \) is the Euler class, and we define

\[ \hat{A}(M) = \prod_{1}^{4} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24} p_1 + \frac{1}{5760} (7p_1^2 - 4p_2) + \cdots \]
2.2 Dirac operator

This is defined as $\gamma \circ \nabla$ where $m$ is Clifford mult:

$$\Gamma(M, \Delta_+) \xrightarrow{\varphi} \Gamma(M, \Delta_-)$$

Given a vector bundle $V$ with connection, it extends to an elliptic operator

$$\Gamma(M, \Delta_+ \otimes V) \xrightarrow{\varphi_V} \Gamma(M, \Delta_- \otimes V).$$

The resulting index $\text{ind}(V) = \dim \ker \varphi - \dim \text{coker} \varphi$ depends only on the topology of $V$:

**Theorem [AS]**

$$\text{ind}(V) = \int_M \text{ch}(V) \hat{A}(M)$$

- $V = \Delta_+ - \Delta_-$ gives the 2-step de Rham complex

$$\bigoplus_{i=0}^{4} \Lambda^{2i} \xrightarrow{d+d^*} \bigoplus_{i=1}^{4} \Lambda^{2i-1}.$$ 

Of course, $\text{ind}(V) = \chi = \sum_{i=0}^{8} (-1)^i b_i$. 
2.3 Betti numbers

- $V = \mathbb{C}$ equates the index of $\partial$ with
  \[
  \hat{A} = \hat{A}_2 = \frac{1}{5760} (7p_1^2 - 4p_2)
  \]

- $V = \Delta_+ + \Delta_-$ gives rise to the signature operator
  \[
  \bigoplus_{i=0}^{4^+} \Lambda^i \longrightarrow \bigoplus_{i=4^-}^{8} \Lambda^i
  \]
  and so $\text{ind}(V) = b_4^+ - b_4^- = \tau$. But
  \[
  \text{ind}(V) = \int_M \left( 16 + 2p_1 + \frac{1}{24} (p_1^2 + 4p_2) \right) \hat{A}(M)
  \]
  and
  \[
  \tau = \frac{1}{45} (7p_2 - p_1^2) = L_2
  \]

**Corollary.** With a reduction to Spin 7 or Sp(2)Sp(1),

\[
48\hat{A} = 3\tau - \chi
\]

and \(24\hat{A} = -1 + b_1 - b_2 + b_3 + b^+ - 2b^-\).
### 2.4 Parallel spinors

- **$M$** is QK (holonomy $\subseteq \text{Sp}(2)\text{Sp}(1)$) with $R > 0$ so ‘nearly hyperkähler’
  
  \[ \Rightarrow \hat{A} = 0 \]

  Also $b_3 = b^- = 0$ so

  \[ b_4 = 1 + b_2 \]

- **$M$** has holonomy equal to Spin 7

  \[ \Rightarrow \hat{A} = 1 \]

  Thus

  \[ b_3 + b^+ = 25 + b_2 + 2b^- \]

- **$M$** is irreducible HK (holonomy = Sp(2))

  \[ \Rightarrow \hat{A} = 3 \]

  Then

  \[ b_3 + b_4 = 46 + 10b_2 \geq 76 \]

  Beauville’s have $(b_2, b_3, b_4) = (23, 0, 276), (7, 8, 108)$ [G].
3.1 HK constraint

Suppose that $M^{4n}$ has holonomy $Sp(n)$ with $\chi \neq 0$. Set

$$P(t) = \sum_{i=0}^{4n} b_i t^i.$$ 

Then $\chi = P(-1)$ and $P'(-1) = -2nP(-1)$. Consider

$$\log \frac{P(-1+t)}{P(-1)} = \log (1 - 2nt + \frac{P''(-1)}{2P(-1)} + \cdots) = -2nt + \frac{1}{2} \phi t^2 + \cdots$$

where $\phi + 4n^2 = \frac{P''(-1)}{P(-1)}$. By construction,

$$\phi(M \times N) = \phi(M) + \phi(N)$$

is additive.

**Theorem** [S]. Any cpt HK manifold $M^{4n}$ has $\phi = -5n/3$. Equivalently

$$n\chi = 6 \sum_{i=0}^{2n-1} (-1)^i (2n - i)^2 b_i$$

and as a corollary, $24 \mid (n\chi)$.

$n = 1 \Rightarrow 4b_1 + b_2 = 22$

$n = 2 \Rightarrow 25b_1 - 10b_2 + b_3 + b_4 = 46.$

$\phi$ plays a role in the theory of symmetric holonomy.
3.2 QK topology

By analogy to $\text{Spin } 8/\text{Spin } 7 \cong S^7$,

$$\frac{\text{Spin } 8}{\text{Sp}(2)\text{Sp}(1)} \cong \frac{\text{SO}(8)}{\text{SO}(5) \times \text{SO}(3)} = \text{Gr}_3(\mathbb{R}^8)$$

Given an $\text{Sp}(2)\text{Sp}(1)$ structure,

$$TM_c \cong E \otimes H \cong \Delta_-$$

$$\Delta_+ \cong \Lambda^2_0 E \oplus S^2H$$

where $S^2H \cong \langle I, J, K \rangle_c$. We have

$$\Delta_+ - \Delta_- = \Lambda^2_0 E - E \otimes H + S^2H = \Lambda^2_0(E - H).$$

**Proposition.** Relative to $\text{Sp}(n)\text{Sp}(1)$,

$$V = \Delta_+ - \Delta_- \cong \Lambda^n_0(E - H).$$

This explains why $\text{ch}(V) \in H^{4n}(M, \mathbb{R})$. Similar techniques can be used in other situations to prove, e.g. $\mathcal{M}_g$ inside $\mathcal{F}_g \to \text{Gr}_4(2g+2)$ has $TM_g = Q \otimes W - \psi^2Q$ and $p^1_g = 0$ [K].
3.3 Isometry groups

Over $M = \mathbb{HP}^2$, $H$ is the tautological line bundle. In general,

$$h = -4c_2(H) \in H^4(M, \mathbb{Z})$$

represents the class generated by $\Omega$, and $h^2 \in \mathbb{N}$.

**Proposition.** Suppose $M^8$ is QK with $R > 0$. Then

$$\text{ind}(S^2 H) = 1, \quad \text{ind}(TM) = -1 - b_2, \quad \text{ind}(\Lambda^2_0 E) = 2b_2 + 1$$

The corresponding modules are trivial representations of the isometry group $G$. On the other hand,

$$\text{ind}(S^4 H) = \dim G = 5 + h^2 \geq 6.$$

By twistor/Mori theory, one knows that $b_2 > 0$ implies $M \cong \text{Gr}_2(\mathbb{C}^4)$. So we can assume $b_2 = 0$ and $b_4 = 1$. Then

$$\dim G = 21, 14, 9, 6.$$  

The first two cases are $\mathbb{HP}^2$ and $G_2/\text{SO}(4)$, the other two can be eliminated [PS].
### 3.4 Rigid operators

When an isometry group $S^1$ or $G$ acts on $M$, the indices become virtual $G$ modules. Operators like $\hat{\phi} \otimes \Delta_+$ that involve Betti numbers will be rigid, meaning that the indices are sums of trivial modules.

**Theorem** [AH]. If a compact spin manifold admits a non-trivial $S^1$ action then $\hat{A} = 0$. (cf. Spin 7)

Define a sequence of virtual vector bundles $R'_i$ by

$$R'(q) = \sum_{i=0}^{\infty} q^k R'_k = \bigotimes_{i=1}^{\infty} \Lambda(q^{2i-1})/\Lambda(-q^{2i}).$$

Explicitly, $R'_0 = \mathbb{C}$

$R'_1 = TM = \Lambda^1$ (Rarita-Schwinger)

$R'_2 = \Lambda^2 \oplus \Lambda^1$

$R'_3 = \Lambda^3 + \Lambda^2 + S^2 + \Lambda^1$

$R'_4 = 2\Lambda^1 + \Lambda^2 + \Lambda^3 + \Lambda^4 + 2S^2 + V_{sw}$

**Theorem** [W, BT]. If $M^{2n}$ is a compact spin manifold then $\text{ind}(R'_k)$ is rigid for each $k$.

**Strategy:** $\text{ind}(R'(q))$ is a mero function on $\mathbb{C}/\langle 1, e^{2\pi i q} \rangle$. 
4.1 Fernández example

**Theorem [CF].** There are 12 nilpotent Lie algebras \( \mathfrak{g} \) that admit left-invariant \( G_2 \) structures with \( d\varphi = 0 \).

They all give rise to compact nilmanifolds \( N = \Gamma \backslash G \) but with \( \pi_1 = \Gamma \) infinite and \( p_1 = 0 \)!

An easy one is

\[
\mathfrak{g}^* = \langle e^1, \ldots, e^7 \rangle = \mathfrak{g}_5 \oplus \mathbb{R}^2
\]

with \( de^4 = e^{12} \) and \( de^5 = e^{13} \). The closed 3-form is

\[
(45 - 67)1 + (46 - 75)2 + (47 - 56)3 + 123.
\]

The cohomology ring of \( N \) is isomorphic to that of the DGA \( (\bigoplus \Lambda^i \mathfrak{g}^*, d) [N] \), which:

(i) is freely generated by \( e^1, \ldots, e^7 \),

(ii) has a nilpotent property \( de^k \in \bigwedge^2 \langle e^1, \ldots, e^{k-1} \rangle \).

This means that it is a *minimal model* for the de Rham algebra (over \( \mathbb{R} \)). It has a non-zero Massey product

\[
\langle [e^2], [e^1], [e^3] \rangle = [-e^{43} + e^{25}] \in H^2(N, \mathbb{R})/\langle [e^3], [e^2] \rangle.
\]
4.2 Formality

If \( M \) is simply connected or ‘nilpotent’, its minimal model determines \( \pi_\ast(M) \otimes \mathbb{Q} \).

A manifold is formal if there is a morphism of its minimal model to \( (H^\ast(M, \mathbb{R}), d = 0) \) inducing an isomorphism on cohomology (so the latter determines rational homotopy). All Massey products must vanish.

- Spheres are formal: e.g. the minimal model for \( S^{2n} \) is \( S(x_{2n}) \otimes \Lambda(y_{4n-1}) \) with \( dy = x^2 \). Indeed, \( \pi_i(S^{2n}) \otimes \mathbb{Q} \) is non-trivial iff \( i = 0, 2n, 4n - 1 \).

- Compact symmetric spaces are formal: cohomology is represented by parallel forms, so Massey products vanish.

- Compact Kähler manifolds are formal, thanks to the \( \partial \bar{\partial} \) lemma and Chern’s theorem [DGMS].

- A nilmanifold is formal only if it is a torus [H], so nilmanifolds can’t be Kähler even though many admit both complex and symplectic structures.

- Positive QK manifolds are formal, because they have Kähler twistor spaces [A].
4.3 Low dimensions

- Any simply-connected (compact oriented) 6-manifold is formal. Any $k$-connected manifold $M^n$ with $n \leq 4k + 2$ is formal [M].
- Any $M^7$ or $M^8$ with $b_2 \leq 1$ is formal [C].
- There exist simply-connected symplectic manifolds $M^{2n}$ that are not formal for all $n \geq 4$.
- Which of the known manifolds with holonomy $G_2$ or Spin 7 are formal? Many have vanishing Massey products since $[\alpha] \cup [\beta] = 0$ implies that $\alpha \wedge \beta = 0$ as forms.

Why is the cohomology ring of a manifold with special holonomy compatible with index theory constraints, like $b_3 + b^+ = 25 + b_2 + 2b^-$ for Spin 7?

What about the topology of compact 8-manifolds with holonomy $Sp(2)Sp(1)$ and $R < 0$?
5.1 References

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