The local isomorphism between the special orthogonal group $SO(4)$ and the product $SO(3) \times SO(3)$ manifests itself in the conformally invariant decomposition of the bundle of 2-forms

$$\Lambda^2 T^* M = \Lambda^2_+ T^* M \oplus \Lambda^2_- T^* M$$

over an oriented Riemannian 4-manifold $M$. There is a corresponding decomposition of the Weyl curvature tensor $W = W_+ + W_-$, and $M$ is said to be self-dual if $W_+ = 0$. If $M$ is compact, its signature is given by

$$\tau = \frac{1}{3} p_1 = \frac{1}{12\pi^2} \int_M (|W_+|^2 - |W_-|^2) v,$$

where $v$ is the volume form. Consequently, if $M$ is self-dual but not conformally flat, then $\tau > 0$.

Self-duality is the integrability condition for a natural almost complex structure on the 6-dimensional sphere bundle of $\Lambda^2 T^* M$ [1]. Motivated in part by this result, we study the 7-dimensional total space $X$ of $\Lambda^2 T^* M$, and characterize curvature conditions on $M$ by means of differential relations between invariant forms on $X$. First though, we define the exceptional Lie group $G_2$ using the inclusion $SO(4) \subset G_2$, corresponding to a splitting of dimensions $7 = 3 + 4$. This enables us to construct a family of $G_2$-structures on $X$, which amounts to assigning a metric and vector cross product on each tangent space.

There are only two exceptions in the list of holonomy groups of irreducible non-symmetric Riemannian manifolds, namely $G_2$ and $Spin(7)$ [2,3,5,11]. This explains the importance of $G_2$-structures, which, in the light of [7], seem to be a little richer than their $Spin(7)$ counterparts. An examination of the structure on $X$ leads us to exhibit there a Riemannian metric with holonomy group $G_2$, when $M$ is the self-dual Einstein manifold $S^4$ or $CP^2$. No such complete metrics were previously known. This, and analogous examples with holonomy $G_2$ and $Spin(7)$, are the subject of a forthcoming joint paper with R. L. Bryant.
1. Definition of G₂

Let \( V \) denote an oriented \( n \)-dimensional vector space with a positive definite inner product \(<,>\). The inner product extends to one on \( \Lambda^k V^* \), and together with the orientation defines a unit volume form \( \nu \in \Lambda^n V^* \) and an isomorphism \(*: \Lambda^k V^* \rightarrow \Lambda^{n-k} V^*\), where

\[
\sigma (\tau \nu) = <\sigma, \tau > \nu, \quad \sigma, \tau \in \Lambda^k V^*.
\]  

(1)

Here and in the sequel, an exterior product of differential forms is denoted by their juxtaposition.

Now take \( n = 4 \) and \( k = 2 \). Then \( * \) is an involution on \( \Lambda^2 V^* \), and we consider the 7-dimensional space

\[ A = \Lambda^2 V^* \oplus V^*, \]

where \( \Lambda^2 V^* \) is the \(-1\)-eigenspace of \( * \). If \( \{e^1, e^5, e^6, e^7\} \) is an oriented orthonormal basis of \( V^* \), then \( \Lambda^2 V^* \) is the span of

\[ e^1 = e^4 e^5 - e^6 e^7, \quad e^2 = e^4 e^6 - e^7 e^5, \quad e^3 = e^4 e^7 - e^5 e^6. \]

(2)

Regarding now \( e^1, \ldots, e^7 \) as all elements of \( A \), rather than \( \Lambda^2 A \), we set

\[ \varphi' = e^1 e^2 e^3, \quad \varphi'' = e^1 (e^4 e^5 - e^6 e^7) + e^2 (e^4 e^6 - e^7 e^5) + e^3 (e^4 e^7 - e^5 e^6). \]

Then \( \varphi = \varphi' + \varphi'' \) is the sum of 7 simple 3-forms on a 7-dimensional vector space, and has the following well-known property (see [5]).

**Proposition 1** \( G_2 = \{g \in GL(V) : g^* \varphi = \varphi\} \) is a compact Lie group of dimension \( 14 \).

**Proof.** \( G_2 \) is defined above as a closed subgroup of \( GL(V) \) containing \( SO(4) \). Decreeing \( \{e^1, \ldots, e^7\} \) to be an oriented orthonormal basis of \( A \) defines an action of \( SO(7) \) with Lie algebra

\[
\mathfrak{so}(7) \cong \Lambda^2 A \cong \Lambda^2 (\Lambda^2 V^*) \oplus (\Lambda^2 V^* \otimes V^*) \oplus \Lambda^2 V^* \\
\cong \Lambda^2 V^* \oplus (V^* \oplus K) \oplus (\Lambda^1_+ V^* \oplus \Lambda^2_+ V^*). \]

(3)

2
Here $K$ denotes the 8-dimensional subspace of $\Lambda^2 V^* \otimes V^*$ of elements with zero contraction; for example $K$ contains $e^1 \otimes e^4 + e^2 \otimes e^7$ which defines a skew-symmetric endomorphism of $V$ annihilating $\varphi$. Hence the Lie algebra $G_2$ contains $K$, not to mention $\Lambda^2_+ V^*$ and one copy of $\Lambda^2_- V^*$. Now $S^2 A \cong R \oplus S^2_0 A$, where

$$S^2_0 A \cong S^2_0 (\Lambda^2_- V^*) \oplus R \oplus V^* \oplus K \oplus S^2_0 V^*$$

is the space of traceless symmetric endomorphisms of $A$, decomposed into $SO(4)$-modules. Consideration of the action of $K \subset G_2$ shows that $S^2_0 A$ is $G_2$-irreducible. Thus

$$G_2 = \text{so}(4) \oplus K,$$

and it is not hard to check that $G_2 \subset SO(7)$. Q.E.D.

The form $\varphi$ defines by contraction a two-fold vector cross product

$$m : \Lambda^2 A \rightarrow A,$$  \hspace{1cm} (4)

of the sort that exists only on a space of dimension 3 or 7 [4]. Using $m$, $\mathbf{O} = R \oplus A$ can be identified with the alternative algebra of Cayley numbers, to give the description of $G_2$ as the group of automorphisms of $\mathbf{O}$. The subspace $H = R \oplus \Lambda^2_- V^*$ corresponds to a quaternionic subalgebra, and $K$ may be identified with the tangent space of the quaternionic symmetric space $G_2/SO(4)$, parametrizing all quaternionic subalgebras in $\mathbf{O}$ [9].

Like $S^2_0 A$, the $G_2$-modules $A$ and $G_2$ are irreducible, and from (4), the orthogonal complement $G_2^\perp$ of $G_2$ in $\text{so}(7)$ must be isomorphic to $A$. The derivative

$$\delta : \text{End}(A) \cong A \otimes A \rightarrow \Lambda^3 A$$

of the action of $GL(V)$ on $\varphi$ has kernel $G_2$. It follows that the orbit $GL(V)/G_2$ containing $\varphi$ is open in $\Lambda^3 A$; in fact there is just one other open orbit, containing the form $\varphi' - \varphi''$, with stabilizer the non-compact form $G^*$ [5].

Anyway, the above remarks establish

**Proposition 2**  \hspace{1cm} $\Lambda^2 A \cong G_2 \oplus A$, \hspace{1cm} $\Lambda^3 A \cong R \oplus S^2_0 A \oplus A$. 

3
2. Four-dimensional Riemannian Geometry

Let $M$ be an oriented Riemannian 4-manifold. We shall now use the symbols $e^4, e^5, e^6, e^7$ to denote elements of an oriented orthonormal basis of 1-forms on an open set $U$ of $M$. Accordingly $e^1, e^2, e^3$ defined by (2) form a basis of sections over $U$ of $\Lambda^2 T^*M$. The Levi Civita connection on $M$ induces a covariant derivative $\nabla$ on this vector bundle, and we set

$$\nabla e^i = \Sigma \omega_j^i \otimes e^j, \quad \Omega_j^i = d\omega_j^i - \Sigma \omega_k^i \omega_j^k.$$ 

Summations here and below are exclusively over the range of indices 1,2,3.

Let $X$ denote the total space of $\Lambda^2 T^*M$; its cotangent space at $x$ admits a splitting $T^*_x X = V^o \oplus H^o$, (5)

where $H^o$ is the annihilator of the horizontal subspaces defined by $\nabla$, and $V^o = \pi^* T^*_m M$, $m = \pi(x)$. A local section $\Sigma a^i e^i$ of $\Lambda^2 T^*M$ is covariant constant iff $\Sigma (da^i + \Sigma a^j \omega_j^i) \otimes e^i = 0$, so $H^o$ is spanned by 1-forms

$$f^i = da^i + \Sigma a^j \pi^* \omega_j^i,$n

where $a^1, a^2, a^3$ are now interpreted as fibre coordinate functions on $X$. Of course $V^o$ is spanned by $\pi^* e^4, \pi^* e^5, \pi^* e^6, \pi^* e^7$.

Omitting the symbol $\pi^*$, consider the following invariant forms, defined globally on $X$, independently of the choice of basis:

$$r = \Sigma (a^i)^2$$
$$dr = 2\Sigma a^i f^i$$
$$\alpha = \Sigma a^i e^i$$
$$da = \Sigma e^i f^i, \quad \beta = f^1 f^2 f^3$$
$$\gamma = e^1 f^2 f^3 + e^2 f^3 f^1 + e^3 f^1 f^2, \quad v = -\frac{1}{6} \Sigma e^i e^i$$

For example $r$ is simply the radius squared, $\alpha$ is the tautological 2-form on $X$, and $v = e^4 e^5 e^6 e^7$ is the pullback of the volume form on $M$.

**Proposition 3** (i) $M$ is self-dual if and only if $d\gamma = 2 tv dr$ for (the pullback of) some scalar function $t$ on $M$; (ii) $M$ is self-dual and Einstein if and only if $d\beta = \frac{1}{2} t d a, t d r$, for some constant $t$. If $t$ exists in either case, it equals $\frac{1}{12}$ of the scalar curvature of $M$.  

4
Proof. We refer the reader to [1] for basic properties of the curvature tensor of a Riemannian 4-manifold. The curvature of the induced connection on the bundle $\Lambda^2 T^*M$ is determined by the Ricci tensor, and the half $W_-$ of the Weyl tensor which may be regarded as a section of $\Lambda_+^2 T^*M \otimes \Lambda_+^2 T^*M$. Moreover $M$ is self-dual and Einstein if

$$
\Omega_2^1 = te^3, \quad \Omega_3^2 = te^1, \quad \Omega_3^3 = te^2, \quad (6)
$$

where $t = \frac{1}{12}$ (scalar curvature). Since the trace-free Ricci tensor essentially belongs to $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$, $M$ is self-dual iff (6) holds modulo elements of $\Lambda^2_+ T^*M$. The proposition is now the result of a computation involving the formulae

$$
de^i = \Sigma \omega^i_j e^j, \quad df^i = \Sigma (f^j \omega^i_j + a^j \Omega_1^j).
$$

Q.E.D.

Motivated by section 1, we next consider the 3-form

$$
\varphi = \lambda^i \beta + \lambda \mu^2 d\alpha, \quad (7)
$$

where $\lambda$ and $\mu$ are scalar functions on $X$. Observe that

$$
\varphi = E^1 E^2 E^3 + E^1 E^4 E^5 - E^1 E^6 E^7 + E^2 E^4 E^6 - E^2 E^7 E^5 + E^3 E^4 E^7 - E^3 E^5 E^6,
$$

where $E^i$ equals $\lambda f^i$ for $i = 1, 2, 3$ and $\mu \pi^* e^i$ for $i = 4, 5, 6, 7$, and forms an oriented orthonormal basis of 1-forms for the underlying $SO(7)$-structure on $X$. In view of (1), we also have

$$
*\varphi = E^4 E^5 E^6 E^7 + E^2 E^3 E^6 E^7 - E^2 E^3 E^4 E^5 + E^3 E^1 E^7 E^5
\quad - E^3 E^1 E^4 E^6 + E^1 E^2 E^6 E^6 - E^1 E^2 E^4 E^7
\quad = \mu^4 \nu - \lambda^2 \mu^2 \gamma. \quad (8)
$$

Proposition 1 implies

**Proposition 4** If $\lambda$ and $\mu$ are strictly positive everywhere, (7) determines a $G_2$-structure on $X$, i.e. a $G_2$-subbundle $P$ of the principal frame bundle of $X$, whose underlying Riemannian metric has the form $\lambda^2 g^V + \mu^2 g^H$ in terms of the splitting (5).
3. Torsion considerations

If $D$ denotes the Levi Civita connection of the Riemannian metric in Proposition 4, the quantity $D\phi$ measures the failure of the holonomy group to reduce to $G_2$, i.e. the extent to which parallel transport does not preserve the principal subbundle $P$. Its properties were studied by Fernández and Gray in [7], and we first summarize their approach.

Choose any connection $\tilde{D}$ that reduces to $P$, so that $\tilde{D}\phi = 0$. Fix a frame $p \in P$ at the point $x = \pi(p) \in X$, and a vector $v \in T_xX$. The difference $D_v - \tilde{D}_v$ defines, relative to $p$, an element of the Lie algebra $\mathfrak{so}(7)$. The same is true of $D_v\phi = (D_v - \tilde{D}_v)\phi$, but since this is independent of the choice of $\tilde{D}$, it actually belongs to the subspace $G_2^\perp$. Therefore $(D\phi)_x$ may be regarded as an element of

$$T_x^*X \otimes G_2^\perp \cong A \otimes A \cong \mathbb{R} \oplus G_2 \oplus S_0^2A \oplus A.$$  \hspace{1cm} (9)

Let $W_1X \cong X \times \mathbb{R}$, $W_2X$, $W_3X$, $W_4X \cong TX \cong T^*X$ denote the vector bundles associated to $P$ with fibre $\mathbb{R}$, $G_2$, $S_0^2A$, $A$ respectively. Corresponding to (9), there is a decomposition

$$D\phi = w_1 + w_2 + w_3 + w_4,$$

in which $w_i$ is a section of $W_iX$. Now $D$ is torsion-free, and there exist surjective homomorphisms

$$a : T^*X \otimes \Lambda^3T^*X \longrightarrow \Lambda^4T^*X \cong W_1X \oplus W_3X \oplus W_4X$$

$$a^* : T^*X \otimes \Lambda^3T^*X \longrightarrow \Lambda^5T^*X \cong W_2X \oplus W_4X,$$

such that $d\phi = a(D\phi)$ and $d^*\phi = a^*(D\phi)$ (cf. Proposition 2). Thus

**Proposition 5** [7] **With the above identifications,** $d\phi = (w_1, w_3, w_4)$, and $d^*\phi = (w_2, w_4)$, so $D\phi = 0$ if and only if $d\phi = 0 = d^*\phi$.

Call a differential form on $X$ of type $(p, q)$ if, at each point, it is built up from forms on the base of degree $p$ and forms of degree $q$ involving $f^i$. Endow $X$ with the $G_2$-structure of Proposition 4, with $\lambda$ and $\mu$ arbitrary positive scalar functions on $X$. Then $d\phi$, unlike $\ast\phi$, has no component of type $(4, 0)$. Moreover $\phi d\phi = 0$, whence $d\phi$ has no component in the subbundle
$W_1X \subset \Lambda^4 T^* X$, and we always have $w_1 = 0$. Further components of $D\varphi$ can be eliminated by a suitable choice of $\lambda$ and $\mu$.

**Theorem** (i) If $M$ is self-dual, an open set of $X$ admits a $G_2$-structure with $D\varphi = w_3$; (ii) if $M$ is self-dual and Einstein, an open set of $X$ admits a $G_2$-structure with $D\varphi = 0$.

**Proof.** We apply Proposition 3. If $M$ is self-dual, we seek $\lambda, \mu$ such that

$$d\ast \varphi = d(\mu^4) \nu - d(\lambda^2 \mu^2) \gamma - \lambda^2 \mu^2 2t \nu dr,$$

vanishes. Taking $\lambda \mu = c = \text{constant}$, we obtain a solution

$$\mu = (2c^2 tr + d)^{\frac{4}{5}}, \quad \lambda = c(2c^2 tr + d)^{-\frac{4}{5}}, \quad (10)$$

where $d$ is another constant. If $M$ is also Einstein, then $dt = 0$ and

$$d\varphi = d(\lambda^3) \beta + \lambda^3 \frac{1}{2} t d\alpha dr + d(\lambda \mu^2) d\alpha = 0.$$ 

Note that $\lambda, \mu$ can only be strictly positive on all of $X$ if $t$ is everywhere non-negative. Q.E.D.

In [7] it is shown that any minimally embedded hypersurface of $\mathbb{R}^8$ also has a $G_2$-structure with $D\varphi = w_3$. A contrasting example with $D\varphi = w_2 \neq 0$ has been found in [6]. We remark that in general $w_2$ is the obstruction to the existence of a short elliptic complex

$$0 \to C^\infty(X) \xrightarrow{\text{grad}} C^\infty(X, TX) \xrightarrow{\text{curl}} C^\infty(X, TX) \xrightarrow{\text{div}} C^\infty(X) \to 0,$$

on $X$ whose operators are manufactured using $D$ and (4) in analogy with the 3-dimensional case. Indeed, if $f \in C^\infty(X)$ is a function, and $v \in C^\infty(X, TX)$ is a vector field, $\text{curl}(\text{grad} f) = m(D \wedge (\text{grad} f))$ vanishes identically, but $\text{div}(\text{curl} v)$ equals the contraction of $Dv$ with $w_2$. We conjecture that a complex of this sort can be defined on $X$, using only the self-dual conformal structure of $M$. Topological consequences of the existence of a self-dual metric with $t$ non-negative have been given by LeBrun [10].

Self-dual Einstein metrics have been generated by quaternionic Kähler reduction [8]. However a theorem of Hitchin states that a complete Riemannian 4-manifold which is self-dual, Einstein and of positive scalar curvature is
necessarily isometric to the sphere $S^4$, or the complex projective plane $CP^2$ [3, 13.30]. In either of these two cases, the Riemannian metric
\[(2tr + 1)^{-\frac{1}{2}}g^V + (2tr + 1)^{\frac{1}{2}}g^H\]
on $X$ corresponding to the solution (10) with $c = d = 1$ is complete, essentially because \(f_0^\infty(2tr + 1)^{-\frac{1}{2}}d(r^\frac{1}{2})\) diverges. Because $D\varphi = 0$, the holonomy group $H$ is contained in $G_2$, which in turn implies that the Ricci tensor is zero [3]. Furthermore, the respective groups $SO(5), SU(3)$ act as isometries on $X$ with generic orbits of codimension 1. Consideration of the induced action on a hypothetical space of covariant constant 1-forms shows that $X$ is locally irreducible, and it follows that $H = G_2$ [5]. In conclusion:

**Corollary** The total space of $\Lambda^2 T^*S^4$ and $\Lambda^2 T^*CP^2$ admits a complete Ricci-flat Riemannian metric with holonomy equal to $G_2$.

**References**


