Milan j. math. 72 (2002) 1–0
1424-9286/02/040001-0 $ 1.50+0.20/0

Milan Journal of Mathematics

A tour of exceptional geometry

Simon Salamon

Abstract. A discussion of G_2 and its manifestations is followed by the definition of various groups acting on $\mathbb{R}^8$. Calculation of exterior and covariant derivatives is carried out for a specific metric on a 7-manifold, as a means to illustrate their dependence on the underlying Lie algebra. This example is used to construct an explicit metric with holonomy $Spin\,7$, which is reduced so as to obtain both G_2 and $SU(3)$ structures. Special categories of such structures are investigated and related to metrics with holonomy G_2. A final section describes the orbifold construction leading to a known hyperkähler 8-manifold.

Mathematics Subject Classification (2000). 53C25; 53C26.

Keywords. holonomy.

Intoduction

This article is concerned with examples of differential geometric structures that exist on manifolds of real dimensions $6, 7$ and 8. From the mathematical point of view, these examples are based on the structure groups $SU(3), G_2, Spin\,7$, all of which occur as holonomy groups of Ricci-flat metrics. Surprising though it might have seemed twenty years ago, it now turns out that local examples of all such metrics can be found quite explicitly. To some extent, this has come about from a clearer understanding of 4-dimensional Riemannian geometry. The explicit examples enable one to explore more general properties of these Ricci-flat metrics. Following the pioneering work of Yau and later Joyce (all described in [33]), many compact manifolds are now known to carry metrics with the respective holonomy groups. However, the emphasis in the present article will be on explicit local constructions.

Compact Riemannian manifolds with holonomy group equal $SU(n)$ are the so-called Calabi-Yau spaces and amongst compact Kähler manifolds are characterized by the presence of closed n-forms. The case $n = 2$ is rather special as the underlying diffeomorphism class is unique (that of a K3 surface) and the complex manifold need not be projective. Whilst the study of Calabi-Yau spaces has much in common for all values of $n \geqslant 3$, the local theory of $SU(3)$ structures has characteristics that favour its study in relation with other geometries. A complex structure on a 6-manifold is in fact determined by a 3-form that lies in an open orbit under the action of the group $GL(6, \mathbb{R})$ of all invertible linear transformations.

Similar phenomena occur in dimensions 7 and 8, since open orbits also exist in the spaces of 3-forms. In 8 dimensions, the canonical 3-form on the simple Lie algebra $\mathfrak{su}(3)$ spans an open orbit under $GL(8, \mathbb{R})$. The corresponding 3-form on the group $SU(3)$ is parallel relative to the biinvariant metric that it determines, and the holonomy reduces to $SU(3)/\mathbb{Z}_2$. But this is not a holonomy group in Berger's list [7], and a more general study of manifolds modelled on this subgroup proceeds by requiring the 3-form to be harmonic (closed and coclosed) but not parallel [29]. This theory has little direct contact with $SU(3)$ structures on 6-manifolds, which are instead linked to other structures on 8-manifolds, namely those defined by the groups *Spin* 7 and (to a lesser extent) $Sp(2)Sp(1)$. But it does help us to understand these other structures and the important role played by Lie algebras.

In the 'intermediate' dimension 7, a 3-form φ that is generic and positive at each point completely determines a G_2 structure and thereby a Riemannian metric. If φ is closed and coclosed, the holonomy does reduce to G_2 and the metric has zero Ricci tensor. Examples are given in §7. First however, we study more general G_2 structures defined by a form φ that is not parallel, and use this to describe a metric g with holonomy *Spin* 7 that was discovered in [24]. Whilst this example can be constructed from a 4-torus, it allows us to study the relationship between quotienting out by an S^1 action on the one hand, and restricting g to associated hypersurfaces on the other. Performing both operations produces a metric in 6 dimensions, and this type of reduction is common to many situations involving special geometrical structures.

Metrics with holonomy G_2 are relevant in the compactifications of M-theory with unbroken symmetry in 4 dimensions [3, 24]. The standard examples have singularities that are conical in nature, and are constructed using nearly-Kähler manifolds. They have deformations to complete metrics that are asymptotically conical [12, 25]. New examples have so-called asymptotically locally conical (ACL) behaviour [9].

A final section returns to the theme of quaternionic structures on $\mathbb{R}^8$ introduced in §2. The aim is to discuss the topology underlying an 8-dimensional orbifold and hyperkähler manifold, as a means of indicating some of the problems associated to the construction of compact manifolds with reduced holonomy. Our tour is an incomplete one, in more than one sense of the word.

Acknowledgments. This article was written in belated association with a talk the author gave in Milan in 2002, aimed at providing a brief introduction to metrics with exceptional holonomy. I am grateful to Bernard Ruf and others in this connection.

1. A seven times table

Let V denote the vector space $\mathbb{R}^7$ endowed with a standard inner product and a choice of orientation. The automorphism group of this structure is of course $SO(7)$. The double-covering $\mathit{Spin}\,7$ of $SO(7)$ can be viewed (via its so-called spin representation) as a subgroup of $SO(8)$ that acts transitively on the sphere S^7. The reader is referred to [31] for a discussion of the groups $\mathit{Spin}(n)$ in an informal geometrical setting.

The Lie group G_2 then arises as the isotropy group of $\mathit{Spin}\,7$ at a point $x \in S^7$, and acts faithfully on $T_xS^7 \cong V$. Given that $\mathit{Spin}\,6 \cong SU(4)$ also acts transitively on S^7, we may deduce that

$$\frac{SO(6)}{SU(3)} \cong \frac{SO(7)}{G_2} \cong \frac{SO(8)}{\mathit{Spin}\,7}$$

are all isomorphic to the real projective space $RP^7 = S^7/\mathbb{Z}_2$. In fact, $SO(7)$ also acts transitively on the Grassmannian of oriented 2-dimensional subspaces of $\mathbb{R}^8$, via the isomorphisms of homogeneous spaces

$$\frac{SO(7)}{U(3)} \cong \frac{SO(8)}{U(4)} \cong \frac{SO(8)}{SO(2)\times SO(6)}.$$

Related to this is the fact that G_2 acts transitively on the sphere S^6 of unit vectors in V, with isotropy subgroup isomorphic to $SU(3)$.

The importance of G_2 representations is illustrated by Table 1.

n	vector space or manifold of dimension n
7	fundamental representation V of G_2 and $SO(7)$
14	the Lie group G_2 itself
21	$\bigwedge^2 V$, and the groups $SO(7), \mathit{Spin}\,7$
28	$\mathsf{S}^2 V$, and the group $SO(8)$
35	space $\bigwedge^3 V$ of 3-forms on a 7-manifold
42	intrinsic torsion of an $SU(3)$ metric
49	intrinsic torsion of a G_2 metric, and $GL(7,\mathbb{R})$
56	intrinsic torsion of a $\mathit{Spin}\,7$ metric
63	the group $SL(8,\mathbb{R})$
70	space $\bigwedge^4\mathbb{R}^8$ of 4-forms on an 8-manifold
77	curvature of a metric with holonomy G_2
84	curvature of a metric with holonomy $SU(4)$
168	curvature of a metric with holonomy $\mathit{Spin}\,7$

Table 1

As mentioned in the Introduction, there exist Riemannian manifolds of dimensions 6, 7, 8 whose holonomy group is equal to $SU(3), G_2, \mathit{Spin}\,7$ respectively. Due to the work described in [33], one can also stipulate that the manifolds be compact, at the expense of finding an explicit expression for the metric. But it is a much more elementary matter to find manifolds that admit structures defined by these groups without satisfying the much more stringent holonomy condition. The failure of the holonomy group to reduce to G is measured by the so-called *intrinsic torsion* τ. The numerology in Table 1 is then no great mystery, as τ is a tensor which, for each tangent vector $X \in V$, takes values in the quotient $\mathfrak{so}(n)/\mathfrak{g}$, which is isomorphic to V for each $n = 6, 7, 8$ [38, 17].

Observations 1.1. (i) If one is honest, 27 is more important than 28 as it is the dimension of the irreducible the space $\mathsf{S}^2_0 V$ of traceless symmetric tensors, a representation of $SO(7)$ that stays irreducible when restricted to G_2. It is also the dimension of the space of curvature tensors of a metric with holonomy $SU(3)$.

(ii) 35 is also the dimension of the space of traceless symmetric tensors in 8 dimensions. Indeed, there is an $\mathit{Spin}\,7$ equivariant isomorphism

$$\bigwedge^3\mathbb{R}^7 \cong \mathsf{S}^2_0\mathbb{R}^8 \tag{1.1}$$

relevant to the geometry of S^7. Here, $\mathbb{R}^8$ denotes the representation of *Spin* 7 induced from the appropriate Clifford algebra.

(iii) The entry for $n = 63$ (not one's favourite multiple of 7) is a feeble choice, but emphasizes the relevance of volume forms. Actually, 64 has more significance than 63, being the dimension of the kernel of the composition

$$\mathfrak{g}_2 \otimes V \subset \bigwedge^2 V \otimes V \to \bigwedge^3 V;$$

the extra 1 dimension compensates for the missing trace in the cokernel $\mathsf{S}^2_0 V \oplus V$.

(iv) There are in fact two non-isomorphic 77-dimensional irreducible representations R_1, R_2 of G_2, both of which are summands of $\mathfrak{g}_2 \otimes \mathfrak{g}_2$. The space R_1 of curvature tensors of a metric with holonomy G_2 corresponds to the highest weight submodule of

$$\mathsf{S}^2 \mathfrak{g}_2 \cong R_1 \oplus \mathsf{S}^2_0 V \oplus \mathbb{R}.$$

Elements in the other two summands fail to satisfy the first Bianchi identity, and this explains why a manifold with holonomy G_2 is necessarily Ricci-flat. The anti-symmetric part of the tensor product admits a decomposition

$$\bigwedge^2 \mathfrak{g}_2 \cong R_2 \oplus \mathfrak{g}_2$$

that allows one to regard R_2 as the tangent space of the isotropy irreducible space $SO(14)/G_2$ [42].

We now turn attention to G_2 and its subgroups. The equation $49 - 14 = 35$ implies that $GL(7,\mathbb{R})/G_2$ is an open subset $\mathcal{O}$ of $\bigwedge^3 V$. Thus, G_2 can be defined as the subgroup of $SO(7)$ leaving invariant a 3-form φ in the open orbit $\mathcal{O}$ (one of two). In fact, we now define

$$\varphi = e^{127} + e^{347} - e^{567} + e^{135} - e^{245} + e^{146} + e^{236}, \qquad (1.2)$$

explaining this particular choice of canonical form later. Here e^{ijk} stands for $e^i \wedge e^j \wedge e^k$. Thus,

$$G_2 = \{g \in GL(7,\mathbb{R}) : g\varphi = \varphi\}.$$

For example, each of the seven simple 3-forms determines a 3-dimensional subspace $V \subset \mathbb{R}^7$. Then multiplication by -1 on $V^\perp$ is an element of G_2.

Changing the sign of just one term in (1.2) would result in a 3-form with stabilizer the non-compact Lie group ${G_2}^*$ contained in $SO(3,4)$. The 3-form φ determines an inner product and orientation. Exactly how is explained in [10], but the details are immaterial for our purposes. Suffice it to say that once one has expressed φ in the form (1.2) one already has the required oriented orthonormal basis. One may then consider the 'dual form'

$$*\varphi = e^{3456} + e^{1256} - e^{1234} - e^{2467} + e^{1367} + e^{2375} + e^{1475}.$$

One can almost recover φ from $*\varphi$, but there is a slight hitch. The stabilizer of $*\varphi$ in $GL(7,\mathbb{R})$ is $\mathbb{Z}_2 \times G_2$ since -1 preserves $*\varphi$, so the latter fails to determine the overall orientation.

subgroup	$\mathbb{C}^7$	3-forms
$SU(3)$	$\mathbb{C} \oplus (\mathbb{C}^3 \oplus \overline{\mathbb{C}^3})$	3
$SO(4)$	$\mathbb{C}^4 \oplus \bigwedge^2_+\mathbb{C}^4$	2
$SO(3)$	$\mathsf{S}^6\mathbb{C}^2$	1

Table 2

The maximal Lie subgroups of G_2 are listed in the above table, that shows the corresponding representation and the dimension of the space of G-invariant 3-forms. Each choice of subgroup gives rise to an associated homogenous space, a way of defining G_2 and associated geometrical constructions:

i. As already remarked, $G_2/SU(3)$ is the sphere S^6. The relationship between $SU(3)$ and G_2 is a very intimate one that leads to an active interaction between geometrical structures described by the two groups (the author likes to call this 'SUG' theory) [17].
ii. $G_2/SO(4)$ is an 8-dimensional symmetric space with a quaternion-Kähler structure. Starting from $SO(4)$ leads to explicit metrics with holonomy groups equal to G_2 on the total spaces of vector bundles over manifolds of dimension 3 and 4 [12, 25].
iii. $G_2/SO(3)$ is an 11-dimensional isotropy irreducible space [8, 42], though the represention $\mathsf{S}^6\mathbb{C}^2$ is realized as the tangent space of $SO(5)/SO(3)$. This homogeneous space has a natural G_2 structure that was used to construct a metric with holonomy *Spin* 7 [10, 38]. Since the space of holomorphic sections of a sextic curve in the plane $\mathbb{CP}^2$ is isomorphic to $\mathsf{S}^6\mathbb{C}^2$, one can define a G_2 structure on a real subvariety of such curves.

There are three conjugacy classes of 3-dimensional subgroups of $SO(4)$. The principal or generic one corresponds to (iii) above. Using the isomorphism

$$Spin\,4 = SU(2)_+ \times SU(2)_- \tag{1.3}$$

(see §3) exhibits two more, namely $SU(2)_\pm$. A third is formed from the diagonal $SO(3)$. Each of the four such subgroups gives rise to a nilpotent coadjoint orbit of $\mathfrak{g}_2^{\mathbb{C}}$ [35]. Indeed, $G_2/U(2)_+$ is the projectivization $\mathcal{O}/\mathbb{C}^*$ of the minimal nilpotent orbit $\mathcal{O}$. Remarkably, it is also locally isomorphic to the principal nilpotent $SL(3,\mathbb{C})$ orbit in $\mathfrak{sl}(3,\mathbb{C})$. On the other hand,

$$G_2/U(2)_- \cong \frac{SO(5)}{SO(2)\times SO(3)} \tag{1.4}$$

can be identified with the complex quadric $\mathbb{Q}^5$ in $\mathbb{CP}^6$, a fact exploited in [13].

2. Groups acting on $\mathbb{R}^8$

We begin this section by reviewing the well-known isomorphism $Sp(2) \cong Spin\,5$.

The groups $Sp(n)$ consists of quaternionic matrices $Q \in \mathbb{H}^{n,n}$ for which $Q^*Q = I$ where $Q^* = \overline{Q}^T$. The mapping

$$Q = A + Bj \mapsto \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} = M \tag{2.1}$$

is a homomorphism for matrix multiplication that also commutes with the operation $*$ (with conjugation applied in the quaternionic and complex sense respectively). In particular if $Q \in Sp(n)$ then $Q^* \in Sp(n)$ and we may write the defining equation unambiguously as $Q^{-1} = Q^*$. The group $Sp(1)$ consists of the unit quaternions and (2.1) identifies it with $SU(2)$. It is alsways true that $\det M = 1$, so $Sp(n) \subset SU(2n)$ for $n \geqslant 2$.

Let us pass to the the case $n = 2$. If $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$ then $|a| = |d|$ and $|b| = |c|$. Whilst the Lie algebra $\mathfrak{sp}(2)$ may be identified with the space of 'anti-Hermitian' matrices P for which $P^* + P = 0$, consider instead the vector space

$$V = \left\{ \begin{pmatrix} \lambda & q \\ \overline{q} & -\lambda \end{pmatrix} : \lambda \in \mathbb{R},\ q \in \mathbb{H} \right\}$$

consisting of trace-free matrices P for which $P^* = P$. Observe that V can be identified with $\mathbb{R}^5$ and that

$$P^2 = (\lambda^2 + |q|^2)I = |P|^2 I, \tag{2.2}$$

where $|P|$ indicates the Euclidean norm of (λ, q) in $\mathbb{R}^5$. The equation (2.2) tells us that, for each $Q \in Sp(2)$, the endomorphism f_Q defined by

$$f_Q(P) = QPQ^* = QPQ^{-1} \tag{2.3}$$

is a linear isometry. Since $Q = -I$ is the only non-identity element that commutes with all $P \in \mathbb{R}^5$, the image of f must be a connected component; thus

Proposition 2.1. *The resulting homomorphism* $f\colon Sp(2) \to O(5)$ *has kernel* $\{I, -I\} \cong \mathbb{Z}_2$ *and image* $SO(5)$.

Let $SO(4)$ denote the subgroup of $SO(5)$ preserving the 4-dimensional subspace of V for which $\lambda = 0$, and acting as $+1$ on the 1-dimensional subspace for which $q = 0$. It is easy to check that

$$f^{-1}(SO(4)) = \left\{ \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} : p_i \in \mathbb{H},\ |p_i| = 1 \right\} \cong Sp(1) \times Sp(1) \tag{2.4}$$

(though if one drops the '+1' hypothesis one obtains a second copy of $Sp(1) \times Sp(1)$ represented by off-diagonal matrices). The restriction of (2.3) to (2.4) corresponds to

$$q \mapsto p_1 q \overline{p}_2 = p_1 q {p_2}^{-1}, \tag{2.5}$$

whence (1.3). These group actions lead to the concept of self-duality [41, 4], that we discuss next.

Let x^1, x^2, x^3, x^4 be coordinates on $\mathbb{R}^4$ and set

$$q = x^1 + x^2 i + x^3 j + x^3 k.$$

Under the action (2.5), the quaternion-valued 1-form $\overline{q}dq$ transforms as

$$\overline{q}dq \mapsto \overline{p_1 q \overline{p}_2}\, d(p_1 q \overline{p}_2) = p_2(\overline{q}dq)\overline{p}_2,$$

and is thus unaffected by $Sp(1)_+$. If we set $e^i = dx^i$ then the quaternion-valued 2-form

$$\begin{aligned} d(\overline{q}dq) &= d\overline{q} \wedge dq \\ &= (e^1 + ie^2 + je^3 + ke^4) \wedge (e^1 - ie^2 - je^3 - ke^4) \\ &= i\varpi^1 + j\varpi^2 + k\varpi^3, \end{aligned}$$

where

$$\left\{\begin{array}{l} \varpi^1 = e^{12} - e^{34}, \\ \varpi^2 = e^{13} - e^{42}, \\ \varpi^3 = e^{14} - e^{23}. \end{array}\right. \tag{2.6}$$

The triple (ϖ^i) forms a basis of an $SO(4)$-invariant subspace Λ^2_- of $\bigwedge^2(\mathbb{R}^4)^*$. The corresponding triple (ω^i) with a plus sign is a basis of a complementary $SO(4)$-invariant subspace Λ^2_+. The two subspaces $\Lambda^2_\pm$ are mutual annihilators for the wedge product $\bigwedge^2(\mathbb{R}^4)^* \times \bigwedge^2(\mathbb{R}^4)^* \to \bigwedge^4(\mathbb{R}^4)^*$, and the mapping

$$(e^i) \mapsto ((\omega^i), (\varpi^i))$$

is a double covering $SO(4) \mapsto SO(3)_+ \times SO(3)_-$.

Returning $\mathbb{R}^8 = \mathbb{H}^2$, consider the 2-form

$$\begin{aligned} \Sigma \;&=\; d\overline{q}^1 \wedge dq_1 + d\overline{q}^2 \wedge d\overline{q}_2 \;=\; \begin{pmatrix} dq^1 \\ dq^2 \end{pmatrix}^* \wedge \begin{pmatrix} dq^1 \\ dq^2 \end{pmatrix} \\ &=\; i\sigma^1 + j\sigma^2 + k\sigma^3, \end{aligned} \tag{2.7}$$

with values in $\operatorname{Im}\mathbb{H}$. Its three components σ^i are non-degenerate 2-forms on $\mathbb{R}^8$ extending (2.6); for example we may write

$$\sigma^1 = e^{12} - e^{34} + e^{56} - e^{78}.$$

The group $Sp(2)$ acts by left multiplication on the column vector in (2.7) and leaves each σ^i unchanged. Whereas the subgroup of $GL(8,\mathbb{R})$ preserving any one of σ^i is the symplectic group $Sp(4,\mathbb{R})$ (a non-compact group with the same dimension of $Sp(4)$), the stabilizer of all three is precisely $Sp(2)$.

A manifold of dimension $4n$ with an $Sp(n)$ structure for which the corresponding invariant 2-forms are all closed is called hyperkähler. The closure condition ensures that the 2-forms are all parallel relative to the induced Riemannian metric [32]. A 4-dimensional hyperkähler manifold therefore has a triple of closed 2-forms like $\varpi^1, \varpi^2, \varpi^3$ or $\omega^1, \omega^2, \omega^3$, depending on the orientation. An 8-dimensional hyperkähler manifold has a triple of closed 2-forms linearly equivalent to $\sigma^1, \sigma^2, \sigma^3$. The real-valued 4-form

$$\Omega = \Sigma \wedge \Sigma = \sum_{i=1}^{3} \sigma^i \wedge \sigma^i \tag{2.8}$$

invariant by the larger group $Sp(2)Sp(1)$ of transformations of the type

$$\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \mapsto Q \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \overline{p}$$

with $p \in Sp(1)$. This leads to the theory of quaternion-Kähler manifolds [38].

Example 1. The isotropy action of the space $\mathbb{Q}^5$ (see (1.4)) identifies its holomorphic cotangent space with the tensor product $L \otimes K$ where $L \cong \mathbb{C}$ and $K \cong \mathbb{R}^3$ are the respective representations of $SO(2) = U(1)$ and $SO(3)$. The total space M of the standard $\mathbb{R}^2$ bundle over (1.4) (whose fibre has complexification $L \oplus \overline{L}$) carries an $SO(5)$-invariant 4-form linearly equivalent to (2.8), defined as follows.

The space of $(2,1)$-forms on $\mathbb{Q}^5$ is

$$\begin{aligned} \Lambda^{2,1}\mathbb{Q}^5 = \textstyle\bigwedge^2(L \otimes K) \otimes (\overline{L} \otimes K) \ &\cong\ L \otimes K \otimes K \\ &\cong\ L \oplus (L \otimes K) \oplus (L \otimes \mathsf{S}^2_0 K), \end{aligned}$$

and it follows that the 1-dimensional subspace L belongs to the subspace $\Lambda^{2,1}_0$ of primitive forms. In this way, M has a local basis of sections α, β consisting of 3-forms on $\mathbb{Q}^5$, and these can be paired naturally with 1-forms e^7, e^8 transverse to the base. If τ is the Kähler form on $\mathbb{Q}^5$ then we set $\sigma^1 = \tau + e^{78}$ and

$$\sigma^2 \wedge \sigma^2 + \sigma^3 \wedge \sigma^3 = \alpha \wedge e^7 + \beta \wedge e^8.$$

The resulting metric is locally symmetric. However, similar techniques can be used to construct 8-manifolds with a closed but non-parallel form Ω [39].

The stabilizer of

$$\sigma^1 \wedge \sigma^1 + \sigma^2 \wedge \sigma^2 - \sigma^3 \wedge \sigma^3 \tag{2.9}$$

is a different group, isomorphic to the double-covering *Spin* 7 of $SO(7)$ [11]. One can therefore use (2.9) to define *Spin* 7 as a subgroup of $SO(8)$.

3. The Levi-Civita connection on a 7-manifold

One purpose of this section is to explain how in certain simple situations it is possible to calculate the Levi-Civita connection ∇ directly from a knowledge of the exterior derivative d.

We first define a 7-dimensional Lie algebra to model the representation in Table 2 defining the inclusion $SO(4) \subset G_2$. Extend the basis (e^1, e^2, e^3, e^4) of the previous section to $(\mathbb{R}^7)^*$ by adding e^5, e^6, e^7, and decree that

$$de^5 = \omega^2, \quad de^6 = \omega^3, \quad de^7 = \omega^1.$$

Then d defines an isomorphism $\langle e^5, e^6, e^7\rangle \to \bigwedge^2_+\mathbb{R}^4$ so as to couple these two subspaces (one of $(\mathbb{R}^7)^*$ and the other of $\bigwedge^2(\mathbb{R}^4)^*$). The unusual choice of the identification is tailored to fit the conventions that will intervene below.

Since $d^2 = 0$, we obtain a Lie algebra $\mathfrak{n}$ whose automorphism group contains $SO(4)$. In terms of a dual basis (e_i) the brackets are given by

$$[e_1, e_2] = [e_3, e_4] = e_7, \quad -[e_1, e_3] = [e_2, e_4] = e_6, \quad [e_1, e_4] = [e_2, e_3] = e_5.$$

The associated simply-connected nilpotent Lie group can be realized as

$$N = \{(\mathbf{p}, \mathbf{q}) \in \mathbb{H}^2 : \mathrm{Re}(\mathbf{p}^2 - \mathbf{q}) = 0\},$$

with multiplication

$$(\mathbf{p}, \mathbf{q})(\mathbf{p}', \mathbf{q}') = (\mathbf{p} + \mathbf{p}',\ \mathbf{q} + \mathbf{q}' + \mathbf{p}\mathbf{p}')$$

that arises from the matrix representastion

$$(\mathbf{p}, \mathbf{q}) \leftrightarrow \begin{pmatrix} 1 & \mathbf{p} & \frac{1}{2}\mathbf{q} \\ 0 & 1 & \mathbf{p} \\ 0 & 0 & 1 \end{pmatrix}.$$

See [18]. The forms e^1, e^2, e^3, e^4 are the components of $d\mathbf{p}$ whilst e^5, e^6, e^7 are the components of $d\mathbf{q} - 2\mathbf{p}\,d\mathbf{p}$ (which is purely imaginary).

Let Γ denote the subgroup of N consisting of pairs $(\mathbf{p}, \mathbf{q})$ for which the real and imaginary components of $\mathbf{p}, \mathbf{q}$ are integers and $\mathrm{Re}(\mathbf{q}) = \mathrm{Re}(\mathbf{p}^2)$. The quotient $M = \Gamma\backslash N$ of N by left translation by Γ is a compact smooth manifold. Its points consist of right cosets of Γ, and there is a smooth surjection $\pi\colon M \to T^4$ defined by

$$\Gamma(\mathbf{p}, \mathbf{q}) \mapsto \mathbb{Z}^4 + \mathbf{p}.$$

This realizes M as a T^3 bundle over T^4, where T^n denotes the standard torus $\mathbb{Z}^n\backslash\mathbb{R}^n$. Being left-invariant (by N and so certainly Γ), the forms e^i pass to the quotient. In symbols, each e^i equals the $\pi^*\check{e}^i$ for some 1-form $\check{e}^i$ (which from now on we can also denote e^i without undue confusion). This allows us to carry out all the following computations on the compact manifold M.

Consider the metric g on M for which $(\check{e}^i = e^i)$ is an orthonormal basis of 1-forms, and let ∇ denote the Levi-Civita connection. Then

$$\nabla e^j = \sum \sigma^j_i \otimes e^i,$$

with $\sigma_i^j + \sigma_j^i = 0$. Using the natural isomorphism $TM \cong T^*M$ induced by g, we may identify the Levi-Civita connection with the tensor

$$\vec{\nabla} = \sum \sigma_i^j \otimes e^i \otimes e^j \in \mathfrak{n} \otimes \bigwedge^2 \mathfrak{n}.$$

Since $de^i = \sum \sigma_j^i \wedge e^j$, exterior differentiation likewise corresponds to the tensor

$$\vec{d} = \sum (\sigma_i^j \wedge e^i) \otimes e^j \in \bigwedge^2 \mathfrak{n} \otimes \mathfrak{n}.$$

Moreover, $\vec{d}$ is the image of $\vec{\nabla}$ under the composition

$$f \colon \mathfrak{n} \otimes \bigwedge^2 \mathfrak{n} \subset \mathfrak{n} \otimes \mathfrak{n} \otimes \mathfrak{n} \to \bigwedge^2 \mathfrak{n} \otimes \mathfrak{n}$$

induced from the inclusion $\bigwedge^2 \mathfrak{n} \subset \mathfrak{n} \otimes \mathfrak{n}$ with wedging. The linear mapping f is an isomorphism, as its inverse can be computed explicitly from the formula

$$2f^{-1}(e^{jk} \otimes e^i) = e^i \otimes e^{jk} - e^{(ki)\wedge e^j} + e^{(ij)} \wedge e^k, \tag{3.1}$$

with the conventions

$$\begin{aligned} e^{(ij)} &= e^i \odot e^j = \tfrac{1}{2}(e^i \otimes e^j + e^j \otimes e^i) \\ e^{ij} &= e^i \wedge e^j = \tfrac{1}{2}(e^i \otimes e^j - e^j \otimes e^i). \end{aligned}$$

(We write skew forms so frequently that we omit the customary square brackets for anti-symmetrization.) The equation (3.1) is valid because its right-hand side belongs to $\mathfrak{n} \otimes \bigwedge^2 \mathfrak{n}$.

In the above terms,

Lemma 3.1. *$\vec{\nabla} = f^{-1}(\vec{d})$.*

We may now write down the covariant derivatives ∇e^i without further ado. The last two terms on the right-hand side of (3.1) tell us what we have to add on (the 'symmetric part') to obtain ∇ from d. The result is displayed below.

k	∇e^k
1	$e^{(27)} + e^{(35)} + e^{(46)}$
2	$-e^{(17)} + e^{(36)} - e^{(45)}$
3	$-e^{(15)} - e^{(26)} + e^{(47)}$
4	$-e^{(16)} + e^{(25)} - e^{(37)}$
5	$e^{13} + e^{42}$
6	$e^{14} + e^{23}$
7	$e^{12} + e^{34}$

Table 3

The fact that ∇e^k are totally symmetric for $1 \leqslant k \leqslant 4$ is simply the assertion that the corresponding e^k are closed. On the other hand, the fact that ∇e^k are skew for $5 \leqslant k \leqslant 7$ implies that each of e^5, e^6, e^7 is dual to a Killing vector field. They generate the T^3 action that makes π a principal fibration and allows us to write $M/T^3 = T^4$.

Now

$$\textstyle\bigwedge^2 \mathfrak{n} \otimes \mathfrak{n} \cong \mathfrak{n} \oplus \bigwedge^3 \mathfrak{n} \oplus W,$$

where W is an irreducible representation of dimension 105. We can decompose ∇ into these three components. There is no $\mathfrak{n}$ component; this corresponds to the fact that $d^* e^i = 0$ for all i, a consequence of nilpotency. The component of ∇ in $\bigwedge^3 \mathfrak{n}$ can be identified with the 3-form

$$\delta = \sum_{i=1}^{7} e^i \wedge de^i = e^5 \wedge \omega^2 + e^6 \wedge \omega^3 + e^7 \wedge \omega^1. \tag{3.2}$$

Using (1.1), we may regard δ as a trace-free symmetric tensor on the 8-dimensional spin representation Δ of *Spin* 7. It follows that δ can be identified with the Dirac operator

$$\Gamma(M, \Delta) \to \Gamma(M, \Delta) \tag{3.3}$$

restricted to the finite-dimensional space of invariant spinors. An introduction to spinors can be found in [31].

4. G_2 and *Spin* 7 structures

A G_2 structure on a manifold is determined by a 3-form linearly equivalent at each point to (1.2), and will define $*\varphi$. If ∇ denotes the Levi Civita connection of the metric determined by φ then $\nabla\varphi$ is completely determined by the pair $(d\varphi, d * \varphi)$ in a way first prescribed in [20]. If φ is closed and coclosed then the holonomy reduces to G_2 or a subgroup thereof.

We endow N with a G_2 structure based on the form δ. More precisely, we set

$$\begin{aligned} \varphi_0 &= \delta - e^{567} \\ &= e^{135} - e^{245} + e^{146} + e^{236} + e^{127} + e^{347} - e^{567}. \end{aligned} \tag{4.1}$$

A Hodge operator $*$ is now defined by decreeing (e^i) to be an oriented orthonormal basis of $\mathbb{R}^7$. The dual 4-form is then

$$\begin{aligned} *\varphi_0 &= \varepsilon - e^{1234} \\ &= -e^{2467} + e^{1367} + e^{2375} + e^{1375} + e^{3456} + e^{1256} - e^{1234}, \end{aligned}$$

where

$$\varepsilon = \omega^2 \wedge e^{67} + \omega^3 \wedge e^{75} + \omega^1 \wedge e^{56}. \tag{4.2}$$

Observe that $d * \varphi_0 = 0$ whereas

$$d\varphi_0 = -\varepsilon + 6e^{1234}.$$

This example illustrates the difficulty in finding a G_2 structure for which both $\varphi, *\varphi$ are closed. In fact, no compact 7-manifold with first Pontrjagin class p_1 equal to zero (in $H^4(N, \mathbb{R})$) can have a metric with holonomy equal to G_2.

Now let $p = p(t)$ and $q = q(t)$ be functions of $t \in \mathbb{R}^+$ to be determined. The plan is to weight the two subspaces of $\mathbb{R}^7$ by p, q respectively, and to modify the above definitions so as to give

$$\varphi = p^2 q\delta - q^3 e^{567}, \qquad *\varphi = p^2 q^2 \varepsilon - p^4 e^{1234}.$$

On $N \times \mathbb{R}^+$, consider the 4-form

$$\Omega = \varphi \wedge dt + *\varphi. \tag{4.3}$$

This is known to be linearly equivalent to the form (2.9) and therefore has stabilizer *Spin* 7.

Given that $d * \varphi = 0$,

$$d\Omega = d\varphi \wedge dt + dt \wedge (*\varphi)'.$$

The 4-form Ω will be closed if and only if $d\varphi = -(*\varphi)'$, or equivalently

$$6p^2 q = 4p^3 p', \qquad q^3 = (p^2 q^2)'.$$

The first equation gives $q = \frac{2}{3} pp'$ whence $(p^{5/3} p')' = t$ and

$$p(t) = (at + b)^{3/8}, \quad q(t) = \tfrac{1}{4}(at + b)^{-1/4}.$$

Taking $a = 1$ and $b = 0$ for simplicity,

$$\varphi = \tfrac{1}{4} t^{1/2} \delta - \tfrac{1}{64} t^{-3/4} e^{567} \qquad * \varphi = \tfrac{1}{16} t^{1/4} \varepsilon - t^{3/2} e^{1234}.$$

One can eliminate the fractional powers by means of the substitution $t = 2^{-16/5} u^4$, and neglecting an overall factor of $2^{-14/5}$, we may write

$$\Omega = u^5 \delta \wedge du - e^{567} \wedge du + u\varepsilon - u^6 e^{1234}.$$

To sum up,

Proposition 4.1. *The metric*

$$g = u^3 \sum_{i=1}^{4} e^i \otimes e^i + u^{-2} \sum_{i=5}^{7} e^i \otimes e^i + u^6 (du)^2 \tag{4.4}$$

on $N \times \mathbb{R}^+$ determined by (4.3) has holonomy equal to $Spin\,7$.

Strictly speaking, the closure of Ω tells us only that the holonomy is *contained* in $Spin\,7$, though the abence of other closed forms can be used to show that the holonomy does not reduce further.

In this relatively simple situation, we can integrate the coefficients so as to exhibit Ω as an exact form on $N \times \mathbb{R}^+$. Indeed, in analogy to (4.3), we may write

$$\Omega = -\Phi' \wedge du + d_N \Phi,$$

where $'$ denotes $\partial/\partial u$, and the right-hand side is the full exterior derivative of

$$\Phi = -\tfrac{1}{6} u^6 \delta + u e^{567} \tag{4.5}$$

on the 7-manifold $N \times \mathbb{R}^+$. This 3-form Φ acts as a 'potential' for the $Spin\,7$ structure.

A different type of G_2 structure can be obtained by factoring out by one of the S^1 actions generated by the vector fields dual to e^5, e^6, e^7. For definiteness, let X be the vector field for which $g(\,\cdot\,, u^2 X) = e^7$, so that $e^7(X) = 1$. Using

$$X \lrcorner \Phi = -\tfrac{1}{6} u^6 \omega^1 + u e^{56},$$

it is easy to verify that

$$\mathcal{L}_X \Phi = X \lrcorner d\Phi + d(X \lrcorner \Phi) = X \lrcorner \Omega + d(X \lrcorner \Phi) = 0.$$

The exact 3-form

$$\phi = -X \lrcorner \Omega = (-u^5 \omega^1 + e^{56}) \wedge du - u(\omega^3 \wedge e^5 - \omega^2 \wedge e^6) \tag{4.6}$$

has stabilizer G_2 and determines a metric on the quotient. Since e^7 has norm u relative to (4.4), ϕ is in effect weighted by u and we may write

$$\Omega = \phi \wedge e^7 + u^{4/3} * \phi,$$

where $*\phi$ is computed relative to the Riemannian metric induced from ϕ. It follows that

$$*\phi = -u^{14/3} e^{1234} + u^{-1/3} \omega^1 \wedge e^{56} + u^{11/3} (\omega^2 \wedge e^5 + \omega^3 \wedge e^6) \wedge du,$$

and

$$d * \phi = d(u^{-1/3}) \wedge (2u^5 e^{1234} + \omega^1 \wedge e^{56}) = \phi \wedge \tilde{\Phi},$$

where $\tilde{\Phi} = \frac{1}{3} u^{-19/3} (u^5 \omega^1 + 2e^{56})$ is a variant of Φ.

Whilst the G_2 structure (4.1) satisfied $d{*}\varphi = 0$, the one defined by (4.6) satisfies $d\phi = 0$. The former type is called *cocalibrated* and the latter type *calibrated*. It is much easier to find cocalibrated structures; for example, any hypersurface of R^8 has one. The cocalibrated condition indicates a partial integrability of the G_2 structure in the sense that it is possible to define various subcomplexes of the de Rham complex [21]. A very special class of cocalibrated structures are those for which $*\phi$ is a constant times $d\phi$; this is the 'weak holonomy condition' discussed in [26, 22, 14].

Let W denote the hypersurface, isomorphic to N/S^1, formed by taking $u = 1$. There is the following diagram.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & N \times \mathbb{R}^+ \arrow[d] \\ W \arrow[r, hook] & W \times \mathbb{R}^+ \end{tikzcd}$$

The restriction of ϕ to W is then the form

$$\psi^+ = \omega^2 \wedge e^6 - \omega^3 \wedge e^5 = d(e^{56}),$$

which on W is 'dual' (in a sense to be explained in §5) to the 3-form

$$\psi^- = \omega^2 \wedge e^5 + \omega^3 \wedge e^6.$$

The restriction of $*\phi$ is

$$-e^{1234} + e^{1256} + e^{3456} = \tfrac{1}{2}\sigma^2, \tag{4.7}$$

where $\sigma = -\omega^1 + e^{56}$. The restriction of the mapping $\sigma \mapsto \sigma \wedge \sigma$ to the set of non-degenerate 2-forms is 2:1, so σ is determined by (4.7) and the orientation. Observe that

$$d\sigma = \psi^+, \qquad d\psi^- = e^{1234}.$$

The passage from 8 to 6 dimensions in exceptional geometry was described in [13] in a different context. The interaction of low dimensional structures discussed in [37]. A study of the induced structure on W motivates the following section.

5. $SU(3)$ structures

Consider the group $GL(2n, \mathbb{R})$ of invertible real matrices of order $2n$. Whilst the standard symmetric bilinear form on $\mathbb{R}^{2n}$ is represented by the identity matrix I_{2n} of order $2n$, the standard antisymmetric form is represented by the matrix

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{5.1}$$

The group $U(n)$ is isomorphic to the intersection of any two of the following subgroups of $GL(2n,\mathbb{R})$:

i. the orthogonal group $O(n)$ of matrices X that satisfy $X^T X = I_{2n}$;
ii. the symplectic group $Sp(n,\mathbb{R})$ consisting of matrices X satisfying $X^T J_n X = J_n$;
iii. the subgroup, isomorphic to $GL(n,\mathbb{C})$, of matrices X for which $J_n X = X J_n$ or equivalently $X^{-1} J_n X = J_n$.

The last two correspond to thinking of J_n representing either a bilinear form or a linear transformation.

A manifold of real dimension $2n$ is called almost-Hermitian if its principal frame bundle contains a $U(n)$ subbundle. Such a manifold possesses respectively (i) a Riemannian metric g, (ii) a 2-form σ, and (iii) an orthogonal almost-complex structure J, related by the formula

$$\sigma(X,Y) = g(JX,Y).$$

In accordance with (5.1), one typically chooses an orthonormal basis (e^i) 1-forms at each point for which

$$\begin{gathered} Je^k = -e^{n+k}, \quad Je^{n+k} = e^k, \qquad 1 \leqslant k \leqslant n, \\ \sigma = \sum_{k=1}^{n} e^{k,n+k}. \end{gathered}$$

With this convention, $\alpha = e^1 + ie^{n+1}$ is a form satisfying $J\alpha = i\alpha$ and has type $(1,0)$.

Let us pass to the case $n = 3$. For the particular $U(3)$ structure considered in the previous section, we actually chose an orthonormal basis (e^i) of 1-forms such that

$$\begin{gathered} Je^1 = e^2, \quad Je^3 = e^4, \quad Je^5 = -e^6, \\ \sigma = -e^{12} - e^{34} + e^{56}. \end{gathered} \tag{5.2}$$

Although this differs from the above, w^3 is still a positive multiple of e^{123456} so that J has a positive orientation.

The action of $U(3)$ on the space $\Lambda^{3,0}$ of $(3,0)$-forms relative to J corresponds to the determinant representation, and the structure of M is reduced further to $SU(3)$ by the assignment of a non-zero $(3,0)$-form Ψ. Consistent with the choice (5.2), we may for example take

$$\Psi = -i(e^1 - ie^2) \wedge (e^3 - ie^4) \wedge (e^5 + ie^6),$$

and we define

$$\begin{aligned} \psi^+ &= \operatorname{Im}\Psi = e^{136} - e^{145} - e^{235} - e^{246}, \\ \psi^- &= \operatorname{Re}\Psi = e^{135} + e^{146} + e^{236} - e^{245}. \end{aligned}$$

These real 3-forms satisfy the compatibility relations

$$\sigma \wedge \psi^{\pm} = 0 \tag{5.3}$$

and

$$\psi^+ \wedge \psi^- = \tfrac{2}{3}\sigma^3. \tag{5.4}$$

There are two complementary ways of viewing (5.3):

i. If we fix J (we shall explain below that this is determined by ψ^+) then it asserts that σ belongs to the space $\Lambda^{1,1}$ of $(1,1)$ forms at each point. This follows because in general $\sigma^{2,0}$ is fully detected by $(\sigma \wedge \psi^+)^{2,3}$.

ii. If we fix σ then (5.3) asserts that ψ^+ belongs to the space Λ^3_0 of primitive 3-forms (isomorphic to quotient of Λ^3 by the image of Λ^1 under wedging with σ).

   In the second point of view, $Sp(3,\mathbb{R})$ acts transitively on forms ψ^+ satisfying (5.3) and (5.4), and has orbits of codimension 1 on Λ^3_0.

The relationship between ψ^+ and ψ^- on a 6-manifold is analogous to that between φ and $*\varphi$ on a 7-manifold. It is known that ψ^+ actually determines ψ^- in a pointwise algebraic sense [29]. More precisely, any 3-form ψ^+ arising from an $SU(3)$ structure belongs to an open $GL(6,\mathbb{R})$ orbit $\mathcal{O}$ in $\bigwedge^3\mathbb{R}^6$ and has isotropy subgroup conjugate to a standard $SL(3,\mathbb{C})$. By means of this subgroup, ψ^+ determines the almost complex structure J, and therefore the 3-form $\psi^- = J\psi^+$.

To make this construction more explicit, suppose that

$$\psi^+ = \sum_{i<j<k} a_{ijk} e^{ijk}.$$

We extend the definition of the coefficients a_{ijk} so that they are antisymmetric in the indices i,j,k. The associated tensors J and ψ^- can in theory be determined via

Lemma 5.1. *At a given point, Je^1 is proportional to the 1-form*

$$\begin{aligned}
a_{123}a_{456} - a_{124}a_{356} + a_{125}a_{346} - a_{126}a_{345} + a_{134}a_{256}& \\
-a_{135}a_{246} + a_{136}a_{245} + a_{145}a_{236} - a_{146}a_{235} + a_{156}a_{234})e^1& \\
+2(a_{234}a_{256} + a_{235}a_{264} + a_{236}a_{245})e^2& \\
+2(a_{345}a_{362} + a_{346}a_{325} + a_{342}a_{356})e^3& \\
+2(a_{456}a_{423} + a_{452}a_{436} + a_{453}a_{462})e^4& \\
+2(a_{562}a_{534} + a_{563}a_{542} + a_{564}a_{523})e^5& \\
+2(a_{623}a_{645} + a_{624}a_{653} + a_{625}a_{634})e^6&
\end{aligned}$$

Example 2. Suppose that ψ^+ is a 3-form for which $a_{i56} = 0$ for $1 \leqslant i \leqslant 4$. There are examples in which this condition is a consequence of assuming that $d\psi^+ = 0$. The lemma implies that $Je^1 \in \langle e^1, e^2, e^3, e^4 \rangle$. By symmetry the same is true of Je^i for $i = 2, 3, 4$, so that the space

$$\langle e_5, e_6 \rangle = \langle e^1, e^2, e^3, e^4 \rangle^o$$

is J-invariant, and ψ^- also lacks coefficients with indices ${}_{i56}$.

6. Kähler and nearly-Kähler metrics

Let M be an almost-Hermitian manifold of real dimension $2n$. Then M is Kähler if the following hold:

i. J is integrable, in the sense that (M, J) is locally equivalent to $\mathbb{C}^n$ with its standard complex coordinates,
ii. σ is closed.

It is well known that these condition are sufficient to guarantee that $\nabla J = 0$ and $\nabla\sigma = 0$ so that parallel transport preserves not only g but J and σ. In the case the holonomy group is conjugate to $U(n)$ or a subgroup thereof.

Definition 6.1. A Calabi-Yau manifold is a compact Kähler manifold with holonomy group equal to $SU(n)$.

Since it is precisely the determinant of $U(n)$ that acts on the space $\Lambda^{n,0}$, the extra reduction is characterized by having a parallel $(n, 0)$-form. On the other hand, suppose that M is a compact Kähler manifold of real dimension $2n$ with a nowhere-zero closed $(n, 0)$ form Φ. Then the canonical bundle $\Lambda^{n,0}$ is trivial, $c_1(M) = -c_1(\Lambda^{n,0})$ vanishes in $H^2(M, \mathbb{R})$. Yau's Theorem implies that M has a Ricci-flat Kähler metric, and it follows (non-trivially) that $\nabla\Phi = 0$. If $n \geqslant 3$, such an M is projective, i.e. a submanifold of some

$\mathbb{CP}^m$. This relies on fact that the cone of Kähler forms on (M,J) is open in $H^2(M,\mathbb{R}) = H^{1,1}$ and so intersects $H^2(M,\mathbb{Z}) \cong H^1(M,\mathcal{O}^*)$; the result then follows from Kodaira embedding theorem.

We return again to the 6-dimensional case $n=3$.

Example 3. Consider the intersection of two hypersurfaces S_1, S_2 in $\mathbb{CP}^5$ defined by polynomials f_1, f_2 of degrees d_1, d_2. If $df_1 \wedge df_2 \neq 0$ at all points of $M = S_1 \cap S_2$ then M is a complex manifold. Since

$$T\mathbb{CP}^5|_M \cong TM \oplus \mathcal{O}(d_1) \oplus \mathcal{O}(d_2),$$

we have $6 = c_1(TM) + d_1 + d_2$. Thus, $c_1 = 0$, (d_1, d_2) is one of $(1,5), (2,4), (3,3)$. The first case gives $S \subset \mathbb{CP}^4$, and $\chi = c_3 = -200, -176, -144$ respectively.

An example more akin to a K3 surface is the following. Let

$$\varepsilon = e^{2i\pi/3} = \tfrac{1}{2}(-1+\sqrt{3})/2, \tag{6.1}$$

and set

$$\Gamma = \{(z^1, z^2, z^3) : z^r = a^r + \varepsilon b^r \in \mathbb{Z}[\varepsilon]\}.$$

Then $\Gamma \backslash (\mathbb{C}^3, +)$ is diffeomorphic to T^6. Multiplication by ε on $\mathbb{C}^3$ induces a mapping $\theta \colon T^6 \to T^6$ with $\theta^3 = 1$ that preserves the canonical 3-form $dz^1 \wedge dz^2 \wedge dz^3$. Then θ has 27 fixed points, and $O = T^6/\langle 1, \theta, \theta^2 \rangle$ has 27 singular points locally resembling $\mathbb{C}^3/\mathbb{Z}_3$. Each of these is resolved by considering the total space of $\Lambda^{2,0} = \mathcal{O}(-3) \to \mathbb{CP}^2$ that has a canonical $(3,0)$-form. This gives an overall resolution $\widetilde{O}$ of O with a nowhere-zero $(3,0)$-form. It is Calabi-Yau manifold with $b_2 = 9 + 27$ and $b_3 = 2$, so $h^{1,1} = 0$ and $\chi = 76$. We remark that the total space $\mathcal{O}(-3)$ admits a metric with holonomy $SU(3)$ of the form

$$(r+1)^{1/4} g_{FS} + (r+1)^{-3/4} g_{\text{fibre}}$$

[15].

Let M be a manifold of real dimension 6, with an $SU(3)$ structure determined by a $\Psi = \psi^+ + i\psi^-$ of type $(3,0)$. Since exterior differentiation maps forms of type $(3,0)$ into those of type $(3,1)$, we may write

$$d\Psi = \Psi \wedge \xi^{0,1}, \tag{6.2}$$

for some *real* 1-form ξ. It follows that the 4-forms $d\psi^+, d\psi^-$ have no component of type $(2,2)$. Conversely, the Nijenhuis tensor of J (the obstruction to J defining a complex structure) can be identified with $(d\psi^+)^{2,2} + (d\psi^-)^{2,2}$. The structure is therefore Kähler if and only if

$$d\sigma = 0, \quad (d\psi^+)^{2,2} = 0, \ (d\psi^-)^{2,2} = 0. \tag{6.3}$$

Since the norm of Ψ is constant, $\nabla\Psi$ is completely determined by its skew-symmetric part $d\Psi$, and the vanishing of this characterizes a fur'ther reduction of the holonomy group to $SU(3)$. The holonomy of M therefore reduces $SU(3)$ if, in addition to (6.3), the remaining component (**??**) vanishes.

In the general case of an $SU(3)$ structure with forms satisfying (5.3), we always have

$$d\sigma \wedge \psi^{\pm} = \sigma \wedge d\psi^{\pm},$$

so that there is some redundancy in (6.3). The form ξ represents the so-called W_5 component of the intrinsic torsion, in the terminology of [17] that extends [28]. When M is complex, two special situations are worth emphasizing:

i. If ξ is exact, we may write $\xi = df$ for some real-valued function f, and

$$d(e^f\Psi) = e^f(\overline{\partial} f \wedge \Psi + d\Psi) = 0.$$

Then $e^f/2g$ is a Hermitian metric with a closed $(3,0)$-form.

ii. If $J\xi = dg$ is exact then

$$d(e^{ig}\Psi) = e^{ig}(i\overline{\partial} g \wedge \Psi + d\Psi) = 0,$$

and g already possesses a closed $(3,0)$-form.

We now turn attention to situations in which J is typically non-integrable.

Definition 6.2. We shall say that an $SU(3)$ structure is *coupled* if $d\sigma$ is proportional to ψ^+ at each point.

Such structures are necessarily non-symplectic as $d\sigma \neq 0$. On the other hand, a coupled $SU(3)$ structure satisfies

i. the 'co-symplectic' condition whereby

$$d(*\sigma) = d(\tfrac{1}{6}\sigma^2) = \tfrac{1}{3}\sigma \wedge d\sigma = 0,$$

and

ii. $d\psi^+ = 0$.

These two conditions, taken together, correspond to the vanishing of half of the torsion components of the $SU(3)$ structure, so we may speak of the structure being 'half-flat' or 'half-integrable'.

A subclass of coupled $SU(3)$ structures are the nearly-Kähler ones. In general an almost Hermitian metric is said to be nearly-Kähler if

$$(\nabla_X J)(X) = 0$$

for all X [27]. This is equivalent to asserting that the torsion of the $U(n)$ structure is completely determined by $(d\sigma)^{3,0}$, and lies in the real space underlying $\Lambda^{3,0}$. For example, if G is a compact Lie group then it is known that $G\times G=(G\times G\times G)/G$ is a 3-symmetric space and admits a nearly-Kähler metric [43].

Example 4. (i) The sphere $S^6=G_2/SU(3)$ has an $SU(3)$ structure and corresponding differential forms σ,ψ^+,ψ^- compatible with its standard metric g. The flat metric on $\mathbb{R}^7-\{0\}=S^6\times\mathbb{R}^+$ has the conical form t^2g+dt^2. Set $e^7=dt$ and consider

$$\begin{aligned}\varphi &= t^2\sigma\wedge dt+t^3\psi^+,\\ *\varphi &= t^3\psi^-\wedge dt+\tfrac{1}{2}t^4\sigma^2.\end{aligned}$$

Then

$$\begin{gathered}0=d\varphi=t^2d\sigma\wedge dt+3t^2dt\wedge\psi^+\\ 0=d*\varphi=t^3d\psi^-\wedge dt+2t^3dt\wedge\sigma^2.\end{gathered}\tag{6.4}$$

This implies that

$$d\sigma=3\psi^+,\qquad d\psi^-=-2\sigma^2,$$

equations which characterize the nearly-Kähler condition in 6 dimensions.
(ii) Take $G=SU(2)$ and consider $M=G\times G\times G)/G\cong S^3\times S^3$. It has a global basis of 1-forms $(e^1,\dots,e^6)$ such that

$$\begin{array}{lll} de^1=e^{35}, & de^3=e^{51}, & de^5=e^{13},\\ de^2=e^{46}, & de^4=e^{62}, & de^6=e^{24}.\end{array}$$

We have chosen the odd indices to refer to the first factor, and the even ones to the second. If we choose represent a point of $S^3\times S^3)$ by a triple (g_1,g_2,g_3) with g_1 the identity, then the isotropy represention makes G acts diagonally on $\langle e^1,e^3,e^5\rangle\times\langle e^2,e^4,e^6\rangle$. The 2-form

$$\sigma=e^{12}+e^{34}+e^{56}$$

is then invariant relative to this action.

The automorphism θ of order 3 that is cyclic on the factors induces the action

$$(v_1,v_2)\mapsto(-v_2,v_1-v_2)$$

on T_xM and has eigenvalues $1,\varepsilon,\varepsilon^2$ (notation as in (6.1)). One can interpret the ε eigenspace

$$\Lambda^{1,0}=\langle e^1+\varepsilon e^2,\ e^3+\varepsilon e^4,\ e^5+\varepsilon e^6\rangle$$

of θ as the space of $(1,0)$ forms of almost complex structure J on M. An $SU(3)$ structure is then defined by setting

$$\psi^+ + i\psi^- = i(e^1 + \varepsilon e^2) \wedge (e^3 + \varepsilon e^4) \wedge (e^5 + \varepsilon e^6).$$

On now verifies that

$$d\sigma = -\tfrac{2}{\sqrt{3}}\psi^+, \qquad d\psi = \tfrac{1}{2}\sigma^2,$$

equations that are equivalent to (6.4).

Nearly-Kähler metrics also exist on the complex projective space $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}^3 = SU(3)/T^2$.

The complex Hesienberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}$$

is a T^2 bundle over T^4. On H there are left-invariant 1-forms

$$\begin{aligned} dx &= e^1 + ie^2, \\ dy &= e^3 + ie^4, \\ dz - xdy &= e^6 + ie^5 \end{aligned}$$

that satisfy

$$de^i = \begin{cases} 0, & i = 1, 2, 3, 4, \\ -e^{14} - e^{23}, & i = 5 \\ -e^{13} + e^{24}, & i = 6. \end{cases}$$

These pass to the compact quotient $M^6 = \mathbb{Z}^6 \backslash H$ [1]. Complex structures are discussed in [34].

7. Metrics with holonomy G_2

We now describe some metrics with holonomy G_2 based on various previous examples. One of the first realizations of a metric with holonomy G_2 came about from nearly-Kähler metrics [12, 25].

Proposition 7.1. *If M^6 has a strict nearly-Kähler metric then $M \times \mathbb{R}^+$ has a metric g_0 with holonomy G_2.*

The word ‘strict’ means that the structure is not Kähler, and in particular not flat. This result is part of a more comprehensive theory linking manifolds with Killing spinots to ones with excpetional holonomy [5, 22]. A lot more is true – if M is one of the known nearly-Kähler 6-manifolds $S^3 \times S^3$, $\mathbb{CP}^3$, $\mathbb{F} = U(3)/U(1)^3$ then g_0 can be deformed to a complete metric g_λ with holonomy G_2 and asymptotic to G_0. This is indicated as follows.

$$\begin{tikzcd}
S^3 \times S^3 \arrow[r, phantom, "\subset"] & V \arrow[d, "\mathbb{R}^4"] & \mathbb{CP}^3 \arrow[r, phantom, "\subset"] & \Lambda^+ \arrow[d, "\mathbb{R}^3"] & \mathbb{F} \arrow[r, phantom, "\subset"] & \Lambda^+ \arrow[d, "\mathbb{R}^3"] \\
 & S^3 & & S^4 & & \mathbb{CP}^2
\end{tikzcd}$$

These metrics have isometry groups $SU(2)^3$, $SO(5)$ and $SU(3)$.

For example, in the second case,

$$g_\lambda = (r+\lambda)^{-1/2} g_{\mathbb{R}^3} + (r+\lambda)^{1/2}\pi^* g_{S^4}, \quad \lambda > 0,$$

and $g_\lambda \sim g_0$ at infinity. The three models admit S^1 quotients with M^7/S^1 homeomorphic to $\mathbb{R}^6$.

The methods of §3 show that if M has an $SU(3)$ structure depending on a real parameter t then the metric compatible with either (and both) of the forms

$$\begin{array}{rcl} \varphi & = & \sigma \wedge dt + \psi^+ \\ *\varphi & = & \psi^- \wedge dt + \frac{1}{2}\sigma^2 \end{array}$$

has holonomy contained in G_2 if and only if

i. it is half-flat (see above),
ii. it satisfies

$$d\sigma = (\psi_+)', \qquad d\psi^- = -(\tfrac{1}{2}\sigma^2)'. \tag{7.1}$$

See [30].

If the structure is coupled, we can expect to solve (7.1) with $\psi^+ = f(t)\psi_0^+$. This implies that the induced almost-complex structure is constant.

The simplest examples that fall into the above category begin from a hyperkähler 4-manifold. Let M be such a manifold, with closed 2-forms $\varpi^1, \varpi^2, \varpi^3$ defined as in (2.6). A bundle W over M with fibre T^2 can be defined from connection 1-forms e^5, e^6.

Theorem 7.2. *Let* $\tilde{\omega}_1 = (a+bt)\omega_1$ *and* $u = t(a+bt)^2$. *Then the forms*

$$\begin{aligned} \varphi &= -\tilde{\omega}_1 \wedge e^5 + dt \wedge e^{56} + t(\omega_2 \wedge e^6 + u\omega_3 \wedge dt) \\ *\varphi &= \omega_3 \wedge e^{56} + t^3\omega_2 \wedge dt \wedge e^5 + t^3\omega_1 \wedge dt \wedge e^6 + \tfrac{1}{2}t^4\omega_1 \wedge \omega_1. \end{aligned} \tag{7.2}$$

determines a metric on $W \times \mathbb{R}^+$ *with holonomy in* G_2.

Example 5. A more complicated example is associated to the 6-dimensional Lie algebra with

$$de^i = \begin{cases} 0, & i = 1,2,3,4, \\ -e^{14} - e^{23}, & i = 5, \\ e^{34}, & i = 6. \end{cases}$$

This can be regarded as a degeneration of (**??**). For simplicity we take $a = 0, b = 1$. Then the associated metric is

$$t^2 \sum_1^2 e^i \otimes e^i + t^4 \sum_1^2 e^i \otimes e^i + t^{-4} e^5 \otimes e^5 + t^{-2} e^6 \otimes e^6 + 4t^8 dt^2.$$

Consider the $SU(3)$-structure for which

$$\begin{aligned} \omega &= t^3(e^{13} + e^{42}) + t^{-3}e^{56}, \\ \psi^+ + i\psi^- &= (e^1 + ite^3) \wedge (e^2 - ite^4) \wedge (e^5 + ite^6), \end{aligned}$$

by analogy to (**??**). This yields a G_2-structure with closed forms

$$\begin{aligned} \varphi &= 2t^7(e^{13} + e^{42}) \wedge dt + 2te^{56} \wedge dt - e^{125} - t^2(e^{146} + e^{236} + e^{345}), \\ *\varphi &= -2t^7 e^{346} \wedge dt + 2t^5(e^{145} - e^{126} + e^{235}) \wedge dt + e^{1256} - e^{2456} + t^6 e^{1234}. \end{aligned}$$

There are many generalizations in which M^6 is replaced by a nilmanifold. [24]

On $S^3 \times S^3$, to satisfy $\sigma \wedge d\sigma = 0$ define

$$\sigma = x'e^{12} + y'e^{34} + z'e^{56}$$

To satisfy $d\psi^+ = 0$ we take ψ^+ to have a similar form to $d\sigma$, but more precisely

$$\psi^+ = x(e^{352} - e^{146}) + y(e^{514} - e^{362}) + z(e^{136} - e^{524})$$

so that the forms extend to $N = M \times (a,b)$ with

$$\begin{aligned} d\varphi &= (d\sigma - (\psi^+)') \wedge dt = 0 \\ d*\varphi &= (d\psi^- - \tfrac{1}{2}(\sigma^2)') \wedge dt \end{aligned}$$

Proposition 7.3. *$d\psi^- \in \langle e^{3456}, e^{5612}, e^{1234}\rangle$ and $d*\varphi = 0$ iff*

$$(y'z')' = \frac{2x(x^2-y^2-z^2)}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}}$$

cyclically.

Example 6. If $x = y = z$ then $\sqrt{3}x'x'' = -x$ and $x = -\frac{1}{18\sqrt{3}}t^3$ gives the nearly-Kähler metric on M. Other variants of the construction give rise to complete metrics with holonomy G_2 on 7-manifolds foliated by M_t [9].

Reductions of G_2 meytrics are discussed in [2, 3].

8. A compact example

A serious discussion of the known examples of compact manifolds with exceptional holonomy [33, 36] is beyond the scope of this article. Nevertheless, this final section contains some topological observations that are relevant to the constructions.

We return to the quaternionic formalism of §4 and the description of $\mathbb{R}^8$ as $\mathbb{H}^2$. There are two groups acting on $\mathbb{H}^2$ that potentially give rise to compact Riemannian manifolds which have reduced holonomy but are not locally symmetric. They are $Sp(2)$ and $Sp(2)Sp(1)$, and the resulting structures are characterized by the forms (2.7) and (2.8). To the author's knowledge there are as yet no known compact examples with holonomy equal to $Sp(2)Sp(1)$ that other than symmetric spaces and their finite quotients. Any new examples will necessarily have negative Ricci tensor. On the other hand, there are two known examples of compact manifolds with holonomy equal to $Sp(2)$. Below, we shall describe the first example, found by Fujiki [23], albeit in a setting more adapted to [33]. It was generalized by Beauville [6].

Let $\mathbb{Z}^8$ denote the standard lattice consisting of points (q^1, q^2) all of whose real components are integral, so that $T^8 = \mathbb{H}^2/\mathbb{Z}^8$ is a torus. The elements

$$\alpha : (q^1, q^2) \mapsto (-q^2, q^1)$$
$$\sigma : (q^1, q^2) \mapsto (q^2, q^1)$$

generate the standard action of the dihedral group D on the plane (in this case quaternionic). Whilst α is 'rotation by 90°', σ is 'diagonal reflection'. Observe that this action on $\mathbb{H}^2$ preserves the 2-form (2.7) and its real components $\sigma^1, \sigma^2, \sigma^3$. It also preserves $\mathbb{Z}^8$ and passes to T^8.

For each $g \in D$ we may consider

i. the centralizer $C(g)$ of g in D,
ii. the fixed point set F^g of g acting on T^8,
iii. the topological space $F(g) = F^g/C(g)$,
iv. the Poincaré polynomial $P(F(g)) = \sum\limits_{k=0}^{8} b_k(F(g))t^k$,

where $b_k = \dim H^k(F(g), \mathbb{R})$ is the kth Betti number. If g_1, g_2 are conjugate in D then $F(g_1)$ and $F(g_2)$ are obviously homeomorphic.

Table 4 lists a representative g of each conjugacy class in D. In each case the codimension c of F^g is either 4 or 8. The product of $P(F(g))$ with $t^{c/2}$ will thereofre be a polynomial that satisfies Poincaré duality $b_k = b_{8-k}$. Moreover, it is then possible to subdivide the new middle Betti number in the form $\genfrac{}{}{0pt}{}{m}{n}$ to reflect the decomposition $H^4 = H_+ \oplus H_-$ arising from the Hodge $*$ mapping.

g	$g(q^1, q^2)$	F^g	$C(g)$	$t^{c/2}P(F(g))$
e	(q^1, q^2)	T^8	D	$1 + 6t^2 + \genfrac{}{}{0pt}{}{13}{9}t^4 + 6t^6 + t^8$
α	$(-q^2, q^1)$	16 points	$\mathbb{Z}_4$	$\genfrac{}{}{0pt}{}{16}{0}t^4$
α^2	$(-q^1, -q^2)$	256 points	D	$\genfrac{}{}{0pt}{}{136}{0}t^4$
σ	(q^2, q^1)	T^4	$\mathbb{Z}_2^2$	$t^2 + \genfrac{}{}{0pt}{}{3}{3}t^4 + t^6$
$\sigma\alpha$	$(-q^1, q^2)$	$16T^4$	$\mathbb{Z}_2^2$	$16(t^2 + \genfrac{}{}{0pt}{}{3}{3}t^4 + t^6)$
				$1 + 23t^2 + \genfrac{}{}{0pt}{}{216}{60}t^4 + 23t^6 + t^8$

Table 4

The last row of the table is then merely the sum of the various terms $t^{c/2}P(F(g))$. It represents the so-called *orbifold cohomology* of the singular space T^8/D [16, 19].

Theorem 8.1. *There is a resolution $S \to T^8/D$ with Betti numbers*

$$b_2 = 23, \quad b_4^+ = 216, \quad b_4^- = 60.$$

The smooth 8-manifold S possesses a metric with holonomy $Sp(2)$.

The idea is to write

$$T^8/D = \frac{(T^4/\pm 1) \times (T^4/\pm 1)}{\langle\sigma\rangle},$$

and use the fact that the resolution of $T^4/\pm$ is a Kummer surface, and the resolution of $(K \times K)/\langle\sigma\rangle$ is the Hilbert scheme discussed in [6]. In practice, the hyperkähler structure is detected by the presence of a complex symplectic form on S.

Let M be a compact 8-manifold with holonomy group H contained in *Spin* 7. Then

$$\hat{A} = \tfrac{1}{24}(-1 - b^2 + b^3 + b^4_+ - 2b^4_-)$$

is an integer. It is the index of the Dirac operator (3.3) and counts the number of harmonic (necessarily parallel) spinors. In particular, it equals $1, 2, 3$ if H *equals Spin* $7, SU(4), Sp(2)$ respectively. Moreover, if $H \subseteq Sp(2)$ then

$$b_3 + b_4 = 46 + 10b_2,$$

so that $H = Sp(2)$ implies

$$b_3 + b_4^+ = 75 + 7b_2$$

[40]. These relations are satisfied by S with $\hat{A} = 3$.

One can modify the action of D by adding translations. By way of a simple illustration, let p be a quaternion satisfying $2p \in \mathbb{Z}^4$ (such as $p = \frac{1}{2}$), and re-define

$$\alpha(q^1, q^2) = (p - q^2, p + q^1).$$

Whilst α^2 and σ remain invariant, $\sigma\alpha(q^1, q^2) = (p - q^1, p + q^2)$. The induced action on the cohomology of T^8, and so the polynomial $P(F^e) = P(T^8/D)$, remains the same. But the table is affected in the following way.

i. For $g = \alpha^2$, the action of $D/\lambda\alpha\rangle$ on F^g has 16 orbits of size 2 (8 for which α acts trivially and 8 for which σ acts trivially) and 56 of size 4. The entry ${}^{136}_{\;0}t^4$ is therefore replaced by ${}^{72}_{\;0}t^4$.

ii. $\sigma\langle$ no longer has fixed points.

iii. As a consequence, the cash total becomes $1 + 7t^2 + {}^{104}_{12} + 7t^6 + t^8$.

This provides another solution of the above relations with $\hat{A} = 3$. This time however, there is no resolution of the orbifold T^8/D carrying a metric with holonomy $Sp(2)$. The problem is that, for the new action, there are isolated points, but it is well known however that the singularity $\mathbb{C}^4/\mathbb{Z}_2$ has no hyperkähler resolution. For the original action, the fixed point set $F^{\sigma\langle} = 16T^4$ acts to 'mask' or 'censure' such isolated points.

As an extreme case, one can replace D by $\mathbb{Z}_2$ so as to obtain the orbifold Poincaré polynomial

$$P(T^8/\mathbb{Z}_2) + {}^{256}_{\;\;0}t^4 = 1 + 28t^2 + {}^{291}_{\;35}t^4 + 28t^6 + t^8,$$

and $\hat{A} = 8$. This is completely consistent in this sense that any even spinor is $\mathbb{Z}_2$-invariant and parallel, so 8 is just the dimension of the spin bundle. In theory, one might expect to realize smooth hyperkähler 8-manifolds as resolutions of the form T^8/Γ where Γ is a more complicated finite group acting on T^8 preserving the $Sp(2)$. In practice though one comes up against the problem of unresolvable singularities. On the other hand, the method has been spectacularly successful in constrcting manifolds with holonomy 6 and G_2 [33].

References

[1] E. Abbena, S. Garbiero, and S. Salamon. Almost Hermitian geometry of 6-dimensional nilmanifolds. *Ann. Sc. Norm. Sup.*, 30:147–170, 2001.

[2] V. Apostolov and S. Salamon. Kähler reduction of metrics with holonomy g_2. DG/0303197.

[3] M. Atiyah and E. Witten. M-theory dynamics on a manifold of G_2 holonomy. hep-th/0107177.

[4] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional riemannian geometry. *Proc. Roy. Soc. London*, A 362:425–461, 1978.

[5] C. Bär. Real Killing spinors and holonomy. *Comm. Math. Phys.*, 154:509–521, 1993.

[6] A. Beauville. Variétés Kählériennes dont la première class de Chern est nulle. *J. Differ. Geom.*, 18:755–782, 1983.

[7] M. Berger. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[8] M. Berger. Les variétés riemanniennes homogène normales simplement conexes à courbure strictemente positive. *Ann. Sc. Norm. Sup. Pisa*, 15:179–246, 1961.

[9] A. Brandhuber, J. Gomis, S. S. Gubser, and S. Gukov. Gauge theory at large n and new g_2 holonomy metrics. hep-th/0106034.

[10] R. Bryant. Metrics with exceptional holonomy. *Annals of Math.*, 126:525–576, 1987.

[11] R. Bryant and R. Harvey. Submanifolds in hyper-Kähler geometry. *J. Amer. Math. Soc.*, 2:1–31, 1989.

[12] R. Bryant and S. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58:829–850, 1989.

[13] R. L. Bryant. Submanifolds and special structures on the octonians. *J. Differ. Geom.*, 17:185–232, 1982.

[14] F. M. Cabrera, M. D. Monar, and A. F. Swann. Classification of G_2-structures. *London Math. Soc.*, 53:407–416, 1996.

[15] E. Calabi. Métriques kählériennes et fibrés holomorphes. *Ann. Ec. Norm. Sup.*, 12:269–294, 1979.

[16] W. Chen and Y. Ruan. A new cohomology theory of orbifold. math.AG/0004129.

[17] S. Chiossi and S. Salamon. The intrinsic torsion of SU(3) and G_2 structures. In *Differential Geometry, Valencia 2001.* World Scientific, 2002.

[18] I. G. Dotti and A. Fino. Abelian hypercomplex 8-dimensional nilmanifolds. *Ann. Global Anal. Geom.*, 18:47–59, 2000.

[19] B. Fantechi and L. Göttsche. Orbifold cohomology for global quotients. math.AG/0104207.

[20] M. Fernández and A. Gray. Riemannian manifolds with structure group G_2. *Ann. Mat. Pura Appl.*, 32:19–45, 1982.

[21] M. Fernández and L. Ugarte. Dolbeault cohomology for G_2-structures. *Geom. Dedicata*, 70:57–86, 1998.

[22] T. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann. On nearly parallel G_2 structures. *J. Geom. Phys.*, 27:155–177, 1998.

[23] Fujiki. ??? *?*, ?:?, ?

[24] G. W. Gibbons, H. Lü, C. N. Pope, and K. S. Stelle. Supersymmetric domain walls from metrics of special holonomy. hep-th/0108191.

[25] G. W. Gibbons, D. N. Page, and C. N. Pope. Einstein metrics on S^3, R^3, and R^4 bundles. *Commun. Math. Phys.*, 127:529–553, 1990.

[26] A. Gray. Weak holonomy groups. *Math. Z.*, 123:290–300, 1971.

[27] A. Gray. The structure of nearly Kähler manifolds. *Math. Ann.*, 223:233–248, 1976.

[28] A. Gray and L. Hervella. The sixteen classes of almost Hermitian manifolds. *Ann. Mat. Pura Appl.*, 282:1–21, 1980.

[29] N. Hitchin. The geometry of three-forms in six dimensions. *J. Differ. Geom.*, 55:547–576, 2000.

[30] N. Hitchin. Stable forms and special metrics. In *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*, volume 288 of *Contemp. Math.*, pages 70–89. American Math. Soc., 2001.

[31] N. Hitchin. The Dirac operator. In *Invitations to Geometry and Topology*, pages ??–?? Oxford University Press, 2002.

[32] N. J. Hitchin. The self-duality equations on a riemann surface. *Proc. London Math. Soc.*, 55:59–126, 1989.

[33] D. D. Joyce. *Compact manifolds with special holonomy.* Oxford University Press, 2000.

[34] G. Ketsetzis and S. Salamon. Complex structures on the iwasawa manifold. DG/0112295, to appear in Advances in Geometry.

[35] B. Kostant. The principal three-dimensional subgroup and the betti numbers of a complex simple lie group. *Amer. J. Math.*, 81:973–1032, 1959.

[36] Kovalov. ??? *?*, ?:?, ?

[37] R. Reyes-Carrion. A generalization of the notion of instanton. *Differ. Geom. Appl.*, 8:1–20, 1998.

[38] S. Salamon. *Riemannian geometry and holonomy groups.* Pitman Research Notes in Mathematics 201. Longman, 1989.

[39] S. Salamon. Almost product structures. In *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*, volume 288 of *Contemp. Math.*, pages 162–181. American Math. Soc., 2001.

[40] S. M. Salamon. Spinors and cohomology. *Rend. Mat. Univ. Torino*, 50:393–410, 1992.

[41] I. M. Singer and J. A. Thorpe. The curvature of 4-dimensional einstein spaces. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 355–365. Univ. Tokyo Press, 1969.

[42] J. A. Wolf. The geometry and structure of isotropy irreducible homogeneous spaces. *Acta Math*, 120:59–148, 1968.

[43] J. A. Wolf and A. Gray. Homogeneous spaces defined by lie group automorphisms I. *J. Differ. Geom.*, 2:77–114, 1968.

Simon Salamon
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino
Italy

E-mail: `salamon@calvino.polito.it`