Notes 24 – Conics and Quadrics

In the last lecture, we discussed conics defined by setting a quadratic form equal to a constant. After defining a general conic, we list the eight different types and investigate when a conic has a centre of symmetry. We then move on to discuss quadrics from a similar point of view.

L24.1 Definition of a conic. A conic \( C \) is the set of points \((x, y)\) in \( \mathbb{R}^2 \) determined by an equation of the form

\[
Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \tag{1}
\]

where \( A, \ldots, F \) are real constants, and \( A, B, C \) are not all zero so that the left-hand side is a polynomial of degree 2 in the two variables \( x, y \). (The 2’s are for convenience later.)

Often we shall refer to an equation like (1) as a conic, though strictly speaking the latter is a set of points. The conics we discussed before were those for which \( D = E = 0 \). In this case, by diagonalizing the symmetric matrix

\[
S = \begin{pmatrix} A & B \\ B & C \end{pmatrix},
\]

we can always rotate the coordinate system so that the equation of \( C \) becomes

\[
\lambda_1 x^2 + \lambda_2 y^2 = \mu, \quad \mu = -f.
\]

It follows that \( C \) is one of

(i) an ellipse (if \( \lambda_1, \lambda_2, \mu \) all have the same sign); a circle is the special case in which \( \lambda_1 = \lambda_2 \);

(ii) a hyperbola (if \( \lambda_1, \lambda_2 \) have opposite signs and \( \mu \neq 0 \)); the corresponding special case \( \lambda_1 = -\lambda_2 \) gives rise to a rectangular hyperbola whose asymptotes are perpendicular;

(iii) two straight lines intersecting in one point (if \( \lambda_1, \lambda_2 \) have opposite signs but \( \mu = 0 \));

(iv) two parallel lines (if one of \( \lambda_1, \lambda_2 \) is zero and the other has the same sign as \( \mu \));

(v) a single line (if one of \( \lambda_1, \lambda_2 \) is zero and \( \mu = 0 \), for then the equation actually defines two coincident lines, though only one is visible to the naked eye);

(vi) a point (if \( \lambda_1, \lambda_2 \) have the same sign but \( \mu = 0 \));

(vii) in all other cases, the set of points satisfying (1) is empty.

Allowing \( D, E \) to be nonzero produces only one other type, namely

(viii) a parabola (such as \( x^2 + y = 0 \), or less obviously \( 4x^2 + 6xy + 9y^2 + x = 0 \)).

Given the general equation (1), we can try first to eliminate the term \( 2Dx + 2Ey \) of degree 1 by a change of coordinates of type

\[
\begin{align*}
&\begin{cases} 
  x = X + u, \\
  y = Y + v.
\end{cases}
\end{align*} \tag{3}
\]

This corresponds to a translation in which the new system OXY has its origin O at the old point \((x, y) = (u, v)\). Substituting (3) into (1), we see that the new term of degree 1 is

\[
2AuX + 2B(Xv + uY) + 2CvY + 2DX + 2EY = 2(Au + Bv + D)X + 2(Bu + Cv + E)Y.
\]
The conic (1) can only be a parabola if \( C \) and \( A \) the form

\[
A'X^2 + 2B'XY + C'Y^2 + F' = 0.
\]

In this case, \((X, Y) \in \mathcal{C} \iff (-X,-Y) \in \mathcal{C} \), and the centre of symmetry is the point \((X, Y) = (0, 0)\) or \((x, y) = (u, v)\).

From the analysis above, we know that there is only one case in which (4) is incompatible and \( \mathcal{C} \) is not central, namely (viii).

**Corollary.** The conic (1) can only be a parabola if \( B^2 = 4AC \).

**Example.** Given the conic \( x^2 + 4y^2 - 6x + 8y = 3 \), we can locate a centre by completing the squares:

\[
(x - 3)^2 - 9 + 4(y + 1)^2 - 4 = 3.
\]

Thus \( u = 3, v = -1 \), and the equation becomes

\[
X^2 + 4Y^2 = 16, \quad \text{or} \quad \frac{X^2}{4^2} + \frac{Y^2}{2^2} = 1,
\]

which is an ellipse with width twice its height. In general, if the orginal equation has a term in \( xy \), one needs to find the centre by solving (4).

**L24.2 Central quadrics.** One can carry out a parallel discussion in space by adding a third variable.

**Definition.** A quadric \( \mathcal{Q} \) is the locus of points \((x, y, z)\) in \( \mathbb{R}^3 \) satisfying an equation of the form

\[
Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy + 2Gx + 2Hy + 2Iz + J = 0.
\]

The word ‘quadric’ implies that (5) has order 2, so not all of \( A, B, C, D, E, F \) are zero.

Just as we did for conics in L23.1, one can list all the different types of quadrics; whilst there were 8 types of conics there are 15 types of quadrics. However, we shall only consider the more interesting cases in this course.

Let us start with an obvious example. The equation

\[
x^2 + y^2 + z^2 = r^2
\]

fits the definition (with \( A = B = C = 1 \), \( J = -r^2 \), and all other coefficients zero). It is of course a sphere of radius \( r \) with centre the origin. Indeed if \( \mathbf{v} = (x, y, z)^T \), then the equation becomes \( |\mathbf{v}|^2 = r^2 \) or \( |\mathbf{v}| = r \), and asserts that the distance of \((x, y, z)\) from the origin is \( r \) (see L9.1).

In the light of the discussion of ellipses, it should now come as no surprise that the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

represents an ellipsoid that fits snugly into a box centred at the origin of dimension \( 2a \times 2b \times 2c \).
Definition. A central quadric is the locus of points \((x, y, z)\) satisfying an equation
\[
Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy + J = 0,
\]
or equivalently
\[
(x \ y \ z) \begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -J.
\]

The \(3 \times 3\) matrix here is symmetric, and we can rewrite the equation as
\[
v^T S v = -J,
\]
where \(\mu = -J\).

We know from L21.1 that there exists a \(3 \times 3\) orthogonal matrix \(P\) so that \(P^{-1}SP = P^T SP\) is diagonal. We may also suppose that \(\det P = 1\) (for if not, \(\det P = -1\) and we merely replace \(P\) by \(-P\) and note that \(\det(-P) = 1\)). It follows from remarks in L21.3 that \(P\) represents a rotation; thus we have the

**Theorem.** Given a central quadric (6), it is possible to rotate the coordinate system about the origin in space so that in the new system the equation becomes
\[
\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = \mu.
\]
The numbers \(\lambda_1, \lambda_2, \lambda_3\) are (in no particular order) the eigenvalues of \(S\).

Here are some examples of central quadrics in which the eigenvalues are all nonzero:
(i) an ellipsoid (if \(\lambda_1, \lambda_2, \lambda_3, \mu\) all have the same sign);
(ii) a hyperboloid of one sheet (if for example \(\lambda_1, \lambda_2, \mu\) are positive and \(\lambda_3 < 0\));
(iii) a hyperboloid of two sheets (if for example \(\lambda_1, \lambda_2\) are negative and \(\lambda_3, \mu\) are positive),
(iv) a cone (if not all \(\lambda_1, \lambda_2, \lambda_3\) have the same sign and if \(\mu = 0\)).

In the last case, the cone is **circular** if two of the eigenvalues are equal, otherwise it is called **elliptic**. We shall explain this case further in the next lecture.

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L24.3 *Paraboloids.* In some ways the simplest equation in \(x, y, z\) of second order is
\[
z = xy.
\]

This is the equation (5) of a quadric for which all the coefficients are zero except \(F = -I\). If we perform a rotation of \(\pi/4\) of the \(xy\) plane corresponding to the matrix
\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},
\]
then we can replace \(x\) by \(\frac{1}{\sqrt{2}}(X - Y)\) and \(y\) by \(\frac{1}{\sqrt{2}}(X + Y)\), and leave \(z = Z\) alone. Our quadric \(\mathcal{Q}\) becomes
\[
z = \frac{1}{2}(X - Y)(X + Y), \quad \text{or} \quad 2Z = X^2 - Y^2.
\]

This is an example of a **hyperbolic paraboloid** that resembles a ‘saddle’ for a horse, or (on a bigger scale) a ‘mountain pass’. Any plane \(X = c\) or \(Y = c\) intersects \(\mathcal{Q}\) in a parabola, whereas a plane \(Z = c\) intersects \(\mathcal{Q}\) in a hyperbola or (if \(c = 0\)) a pair of lines.
Paraboloids are quadrics that cannot be put into the form (6) or (7), and therefore possess no central point of symmetry. The standard form of a paraboloid is the equation

\[ Z = aX^2 + bY^2. \]

If \( a, b \) have opposite signs, it is again a hyperbolic paraboloid. If \( a, b \) have the same sign, the quadric is easier to draw and is called an elliptic hyperboloid (circular if \( a = b \)). Its intersection with the plane \( Z = a \) is an ellipse (circle).

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**L24.4 Further exercises.**

1. For each of the following conics, find the centre \((u, v)\) and the equations that results by setting \( x=X+u, \ y=Y+v \): (i) \( x^2 + y^2 + x = 3 \), (ii) \( 3y^2 - 4xy - 4x = 0 \), (ii) \( 3x^2 - xy + 2y = 9 \).

2. Find the centre \((u, v)\) of the conic \( C : x^2 + xy + y^2 - 2x - y = 0 \), and a symmetric matrix \( S \) so that the equation becomes \( (X, Y)S \begin{pmatrix} X \\ Y \end{pmatrix} = 1 \) with \( x=X+u, \ y=Y+v \). Diagonalize \( S \), and sketch \( C \) relative to the original axes \((x, y)\).

3. Let \( \mathcal{S} \) be the sphere with centre \((3, 1, 1)\) passing through \((3, 4, 5)\). Find the radius of \( \mathcal{S} \), and write down its equation.

4. Let \( \pi \) be the plane \( x-2y+2z=0 \) and let \( O \) denote the origin \((0,0,0)\). Find
   (i) the line \( \ell \) orthogonal to \( \pi \) that passes through \( O \),
   (ii) the point \( P \) on \( \ell \) a distance 6 from \( O \) with \( z > 0 \);
   (iii) a sphere \( \mathcal{S} \) of radius 6 tangent to \( \pi \) at \( O \).

5. Match up, in the correct order, the quadrics

\[
\begin{align*}
x^2 &= 3y^2 + z^2 + 1, \\
z^2 &= xy, \\
x^2 + 2y^2 - z^2 &= 1, \\
-x^2 - y^2 + 2x + 1 &= 0
\end{align*}
\]

with (i) a hyperboloid of 1 sheet, (ii) a hyperboloid of 2 sheets, (iii) a cone, (iv) a cylinder.

6. Show that the line \( \ell \) with parametric equation \((x, y, z) = (1, -t, t)\) is contained in the quadric \( D : x^2 + y^2 - z^2 = 1 \). Draw \( D \) and \( \ell \) in the same coordinate system. Find a second line \( \ell' \) that lies in \( D \).

7. Decide which of the following equations describes the circular cone that is obtained when one rotates the line \( \{(x, y, z) : x = 0, \ z = 2y\} \) around the \( z \)-axis:

\[
\begin{align*}
x^2 + 4y^2 &= z^2, \\
4x^2 + 4y^2 - z^2 &= 0, \\
2(x^2 + y^2) - z^2 &= 0, \\
z &= 4x^2 + 4y^2.
\end{align*}
\]

8. The quadrics \( D_1 : z = x^2 + y^2 \) and \( D_2 : z = x^2 - y^2 \) are both examples of paraboloids. Write down the equations of planes \( \pi_1, \pi_2, \pi_3, \pi_4 \) parallel to the coordinate planes but such that
   (i) \( D_1 \cap \pi_1 \) is a parabola,
   (ii) \( D_1 \cap \pi_2 \) is a circle,
   (iii) \( D_2 \cap \pi_3 \) is a hyperbola,
   (iv) \( D_2 \cap \pi_4 \) is a pair of lines.