SOBOLEV AND STRICT CONTINUITY
OF GENERAL HYSTERESIS OPERATORS

VINCENZO RECUPERO

Abstract. The most natural and important topologies connected with hysteresis operators are those induced by uniform convergence, $W^{1,1}$-convergence, and strict convergence. Indeed the supremum norm and the variation are invariant under reparametrization. We prove a general result which implies that if a hysteresis operator is continuous with respect to the topology of $W^{1,1}$, then it is continuous with respect to the strict topology.

1. Introduction

Usually the term hysteresis is used in connection with a relation $R : u \mapsto R(u)$ between functions of time $u(t)$ and $R(u)(t)$ that can be represented by using loops in the plane like in the following figure:

![Diagram of hysteresis loop]

These diagrams appear in several areas of physics and engineering such as ferromagnetism, ferroelectricity, elastoplasticity, superconductivity, thermostats. The suggestive terms input and output are often used for $u$ and $R(u)$ respectively. Such diagrams implicitly show that the operatorial relation $R$ is in some way independent of the velocity of the system. More precisely, if we reparametrize the function $u$ by a change $\phi(t)$ of the time scale, then $R(u)$ has to be reparametrized by the same change of variable $\phi(t)$. This suggests the following definition. Let $F([0, T])$ be a set of real continuous functions defined on the interval $[0, T]$, with $T > 0$. An operator $R : F([0, T]) \rightarrow F([0, T])$ is called rate independent if the equality

$$R(u \circ \phi) = R(u) \circ \phi$$

(1.1)

2000 Mathematics Subject Classification. 34C55, 74N30, 74C05, 47H30.

Key words and phrases. hysteresis operators, rate independence, continuity of hysteresis operators.
holds for every $u \in F([0,T])$ and every increasing surjective $\phi : [0,T] \rightarrow [0,T]$ such that $u \circ \phi \in F([0,T])$. Another feature which characterizes hysteresis phenomena is memory, that is $R(u)(t)$ is not determined by the value $u(t)$, but by the entire evolution of $u$ on $[0,t]$. Hence the second formal definition: the operator $R : F([0,T]) \rightarrow F([0,T])$ is called causal if the following implication holds for every $t \in [0,T]$:

\[ u = v \text{ on } [0,t] \implies R(u) = R(v) \text{ on } [0,t]. \] (1.2)

Finally we follow the convention that $R : F([0,T]) \rightarrow F([0,T])$ is called a hysteresis operator if it is rate independent and causal. In the next section we will give a slightly more general definition. Let us also recall that the most useful class of hysteresis operators is that of locally monotone ones, i.e. those operators $R$ such that

\[ u \text{ increasing (resp. decreasing) on } [c,d] \implies R(u) \text{ increasing (resp. decreasing) on } [c,d] \]

for every $c,d \in [0,T], c < d$. The local monotonicity means exactly that the branches of hysteresis loops are graphs of increasing functions.

The theory of hysteresis operators has been the object of intensive study in the last twenty years, starting from the mathematical investigation summarized in the monograph by M. A. Krasnosel’skii and A. V. Pokrovskii [5]. This fundamental book has been followed by a large number of research papers on this subject and by the important monographs of I. D. Mayergoyz [7], A. Visintin [11], M. Brokate and J. Sprekels [3], and P. Krejčí [6]. In this books one can find the mathematical analysis of many relevant concrete hysteresis operators appearing in the applications. Typically this analysis starts by defining such hysteresis operators on the space of piecewise monotone (or piecewise linear) functions: this is in fact very natural due to the nature of the phenomena involved. Then one has to face the problem of extending the definition to more general classes of inputs. Such extension has to be constructed in such a way the resulting operator is continuous with respect to suitable and natural topologies on the involved space of functions. For example one can check that a given hysteresis operator $R$ is uniformly continuous on the space of piecewise linear functions $C_{pl}([0,T])$ when endowed with the topology of uniform convergence. Then, since $C_{pl}([0,T])$ is dense in the complete space $C([0,T])$ of continuous functions, it can be automatically continuously extended to $C([0,T])$.

On the other hand in the applications one often has to deal with absolutely continuous functions, therefore it is natural to verify if $R$ is uniformly continuous on $C_{pl}([0,T])$ with respect to the topology induced by the usual norm

\[ \|u\|_{W^{1,1}} := \int_0^T |u(t)| \, dt + \int_0^T |u'(t)| \, dt. \] (1.3)

In this case one can extend the operator from $C_{pl}([0,T])$ to the Sobolev space $W^{1,1}(I)$. The next natural step is to define $R$ on $BV([0,T]) \cap C([0,T])$ or even on
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In order to deal with discontinuous inputs, the $W^{1,1}$-norm defined in (1.3) naturally generalizes to

$$\|u\|_{BV} := \int_0^T |u(t)| \, dt + V(u, [0, T]),$$  \hspace{1cm} (1.4)

where $V(u, [0, T])$ is the variation of $u$. But it is well known that this topology is too strong for applications, and the more natural strict topology is considered, that is the topology induced by the strict distance

$$d_s(u, v) := \int_0^T |u(t) - v(t)| \, dt + |V(u, [0, T]) - V(v, [0, T])|.$$  \hspace{1cm} (1.5)

Therefore in the analysis of hysteresis operators, it is usually checked if the continuity with respect to the strict metric is satisfied. This is in particular important in order to investigate possible extensions to the space of functions of bounded variation, indeed in [8, 9] it is proved that every $R : W^{1,1}(I) \rightarrow W^{1,1}(I)$ which is continuous with respect to the strict metric, can be uniquely extended to $BV([0, T]) \cap C([0, T])$ in a continuous manner. Moreover if $R$ is locally monotone it extends to the whole $BV([0, T])$.

As one can see in the monographs cited above, the most important hysteresis operators are continuous on $C_{pl}([0, T])$ and $W^{1,1}(I)$ when they are endowed with the topologies defined by (1.3) and (1.5). Usually the proof of the continuity with respect to the strict metric is more involved and less trivial.

The aim of this paper is to show that in order to prove that a hysteresis operator is continuous with respect to the Sobolev topology and to the strict topology, it is sufficient to check only that it is continuous with respect to the Sobolev topology. For example if an operator $R : W^{1,1}(I) \rightarrow W^{1,1}(I)$ is continuous with respect to the Sobolev topology, then we show that the property of rate independence ensures that $R$ is necessarily continuous with respect to the strict topology, an implication which is in general false. In fact we prove a more general continuity result where the causality plays no role and larger spaces than $W^{1,1}(I)$ are allowed. A consequence of this result is that every continuous $R : W^{1,1}(I) \rightarrow W^{1,1}(I)$ can be uniquely extended to $BV([0, T]) \cap C([0, T])$ in a continuous manner with respect to the metric (1.5). If $R$ is also locally monotone, it can be further extended to $BV([0, T])$.

Let us conclude this introduction with a brief plan of the paper. In the next section we recall some definitions and technical tools. In Section 3 we state precisely the main results and in Section 4 we present all the proofs. Finally in the last section we apply the results to several particular hysteresis operators.

2. Preliminaries

In this section we recall the technical tools needed in the paper. We fix $I := ]a, b[$, the open interval with endpoints $a, b \in \mathbb{R}$, $a < b$, and we deal with functions in $\mathbb{R}^I$. 
the set of real functions defined on $I$. In the sequel $\mathbb{N}$ is the set of strictly positive integers $\{1, 2, \ldots\}$. If $f \in \mathbb{R}^I$ is given, $\text{Lip}(f)$ denotes its Lipschitz constant, $\text{Cont}(f)$ represents the set where $f$ is continuous and $\text{Discont}(f) := I \setminus \text{Cont}(f)$. Moreover $\|f\|_\infty := \sup\{|f(t)| : t \in I\}$. We say that $f$ is increasing if $(f(t_1) - f(t_2))(t_1 - t_2) \geq 0$ for every $t_1, t_2 \in I$. Same convention for the terms decreasing and monotone. Finally we use the symbol $\chi_S$ to denote the characteristic function of the set $S$, i.e. $\chi_S(t) := 1$ if $t \in S$ and $\chi_S(t) := 0$ if $t \notin S$.

2.1. Signed measures. In order to fix the notation we recall some definitions about signed measures. Details can be found, e.g., in [10, Chapter 6]. Given a locally compact topological space $X$, the symbol $\mathcal{B}(X)$ denotes the family of Borel subsets of $X$, i.e. the smallest $\sigma$-algebra of subsets of $X$ containing all the open sets. Let us recall that a signed Borel measure on $X$ is a function $\nu : \mathcal{B}(X) \rightarrow \mathbb{R}$ which is countably additive, i.e. $\nu(\bigcup_{j=1}^\infty B_j) = \sum_{j=1}^\infty \nu(B_j)$ whenever $(B_j)$ is a family of mutually disjoint Borel sets. If the range of $\nu$ is $[0, +\infty[$, the measure $\nu$ is also called a positive finite Borel measure. In the sequel we will omit the term “Borel”. The adjective “finite” is used to stress the fact that the measure does not attain the value $+\infty$, at variance with positive measures $\mu : \mathcal{B}(X) \rightarrow [0, +\infty]$ allowing for sets of infinite measure. Theory of positive measures and integration with respect to positive measures is standard and can be found in [10, Chapters 1, 2]. The ordinary Lebesgue measure on $\mathbb{R}$ will be denoted by $\mathcal{L}^1$. Many properties of a finite signed measure $\nu : \mathcal{B}(X) \rightarrow \mathbb{R}$ can be obtained from the theory of positive measures by means of the total variation of $\nu$, which is the smallest positive measure dominating the map $B \mapsto |\nu(B)|$ and will be denoted by $|\nu|$. It turns out that $|\nu|$ is finite and setting $||\nu|| := ||\nu||(X)$ we define a norm on the vector space of finite signed measures on $X$. Setting $\nu_+ := (|\nu| + \nu)/2$ and $\nu_- := (|\nu| - \nu)/2$ we have defined two finite positive measures (the positive part and the negative part of $\nu$) such that $\nu = \nu_+ - \nu_-$ and $|\nu| = \nu_+ + \nu_-$. If $f$ is a $|\nu|$-integrable function we define $\int f \, d\nu := \int f \, d\nu_+ - \int f \, d\nu_-$ and $\int_B f \, d\nu := \int_I \chi_B f \, d\nu$ for $B \in \mathcal{B}(X)$.

2.2. Functions of bounded variation. Now let us summarize the main definitions and properties of functions of bounded variation. We refer to [1, Section 3.2] for the omitted proofs. Let us recall that $I = [a, b]$ with $a, b \in \mathbb{R}$, $a < b$.

Definition 2.1. Let $u \in \mathbb{R}^I$. We say that $u$ is of (essentially) bounded variation on $I$ if $u \in \mathcal{L}^1(I)$ and its distributional derivative is a finite signed Borel measure, i.e. there exists a (unique) measure $\nu =: \text{D}u : \mathcal{B}(I) \rightarrow \mathbb{R}$ such that
\[
\int_I \varphi \, d\text{D}u = -\int_a^b \varphi'(t)u(t) \, dt \quad \forall \varphi \in C^1_c(I).
\]
The vector space of all functions of bounded variation on $I$ is denoted by $\text{BV}(I)$. Let $J$ be an interval contained in $I$. We recall that a subdivision of $J$ is a finite
family of points \((t_j)_{j=0}^m, m \in \mathbb{N}\), with the property that \(t_0 < t_1 < \cdots < t_m\) and \(t_j \in J\) for \(j = 0, 1, \ldots, m\). The pointwise variation \(V_p(u, J)\) of \(u\) on \(J\) is defined as

\[
\sup \left\{ \sum_{j=1}^m |u(t_j) - u(t_{j-1})| : m \in \mathbb{N}, (t_j)_{j=0}^m \text{ subdivision of } J \right\}.
\]

The essential variation \(V_e(u, J)\) of \(u\) on \(J\) is defined by

\[
V_e(u, J) := \inf \left\{ V_p(w, J) : u = w \text{ } L^1\text{-a.e. in } J \right\}.
\]

If \(v \in \mathbb{R}^J\) is such that \(V_p(v, I) = V_e(u, I) < +\infty\), then \(v\) is called a good representative of the \(L^1\)-class of \(u\).

If \(V_p(u, I) < +\infty\), it is well known that \(u\) is bounded, \(\text{Discont}(u)\) is at most countable, and \(u\) admits left and/or right limits \(u(t-), u(t+)\) at every limit point \(t\) of \(\overline{I}\) (\(\overline{I}\) is the closure of \(I\)). If \(u \in BV(I)\) we will adopt the convention \(u(a) := v(a+)\) and \(u(b) := v(b-)\), where \(v\) is any good representative of \(u\).

**Proposition 2.1.** Let \(u \in L^1(I)\). Then \(u \in BV(I)\) if and only if \(V_e(u, I) < +\infty\). In this case there exists a good representative \(v \in \mathbb{R}^J\) of \(u\), therefore for every open interval \(A \subseteq I\) we have \(V_p(v, A) = V_e(u, A) = |Dv|(A)\) and in particular \(V_e(u, I) = |Dv|\). There exists a unique constant \(c \in \mathbb{R}\) such that the functions \(w^v, w^l \in \mathbb{R}^J\) defined by

\[
\begin{align*}
 w^v(t) &= c + Dv([a, t]), \\
 w^l(t) &= c + Du([a, t]),
\end{align*}
\]

are good representatives; \(w^v\) is right-continuous, \(w^l\) is left-continuous. Any other good representative \(v\) of \(u\) is characterized by the condition

\[
V_e(u, I) = \int_I |w^v(t)|\,dt = \|w^v\|_{L^1(I)} \quad \forall u \in W^{1,1}(I).
\]

For a proof of the previous proposition we refer to [1, Theorems 3.27, 3.28]. We will use the Sobolev space \(W^{1,p}(I)\), \(p \in [1, +\infty]\), of functions in \(u \in L^p(I)\) having distributional derivative \(w^v\) in \(L^p(I)\). Since \(I\) has finite measure we have \(W^{1,p}(I) \subseteq W^{1,q}(I)\) for \(p > q\). It is well-known that \(W^{1,1}(I) \subseteq BV(I)\) and

\[
V_e(u, I) = \int_I |w^v(t)|\,dt = \|w^v\|_{L^1(I)} \quad \forall u \in W^{1,1}(I).
\]

We also recall that \(u \in W^{1,1}(I)\) if and only if \(u\) has a \(L^1\)-representative \(v \in AC(I)\), the space of absolutely continuous functions on \(I\). Moreover \(u \in W^{1,\infty}(I)\) if and only if \(u\) has a \(L^1\)-representative \(v \in Lip(I)\), where \(Lip(I)\) is the space of functions on \(I\) with finite Lipschitz constant (see e.g. [1, 2] for these facts). A very special class of Lipschitz continuous functions is \(C_{\rho}(\overline{I})\), the space of piecewise linear continuous functions, that is functions \(v \in C(\overline{I})\) such that there exists a subdivision \((t_j)_{j=1}^m\) of
such that \( v \) is affine on each interval \([t_{j-1}, t_j] \), \( j = 1, \ldots, m \), \( a = t_0 \), \( b = t_m \). It is well known that \( C(\mathcal{T}) \) is dense in \( W^{1,1}(I) \) for the topology induced by \( \| \cdot \|_{W^{1,1}} \).

The space \( BV(I) \) can be endowed with the complete seminorm
\[
\|u\|_{BV(I)} := \|u\|_{L^1(I)} + \|D u\|, \quad u \in BV(I). \tag{2.3}
\]
The resulting norm-topology is too strong for applications, indeed, e.g., smooth functions are not dense in \( BV(I) \). It is more natural to consider the topology induced by the strict semimetric \( d_s \) defined by
\[
d_s(u, v) := \|u - v\|_{L^1(I)} + \|D u\| - \|D v\|, \quad u, v \in BV(I). \tag{2.4}
\]
Indeed we have the following approximation result, which is proved using regularization by convolution (cf., e.g., [1, Theorem 3.9]).

**Proposition 2.2.** If \( u \in BV(I) \) then there exists a sequence \( u_n \in C^\infty(\mathcal{T}) \cap BV(I) \) which strictly converges to \( u \).

It is worth noting that the semimetric \( d_s \) is not complete and that it induces a topology that is not linear. It is clear that the norm-topology is stronger than the strict topology, indeed we have \( \|D u\| - \|D v\| \leq \|D(u - v)\| \) for every \( u, v \in BV(I) \). With abuse of notation the norm-topology of \( BV \) is often called \( W^{1,1} \)-topology, because on \( W^{1,1}(I) \) it coincides with the usual Sobolev topology:
\[
\|u\|_{BV(I)} = \|u\|_{W^{1,1}(I)} \quad \forall u \in W^{1,1}(I). \tag{2.5}
\]
where
\[
\|u\|_{W^{1,p}(I)} := \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}, \quad u \in W^{1,p}(I), \quad p \in [1, +\infty]. \tag{2.6}
\]
We will use weak convergence in the Lebesgue spaces \( L^p(I), p \in [1, +\infty] \): a sequence \( u_n \) weakly converges to \( u \) in \( L^p(I) \) (\( u_n \rightharpoonup u \)) if \( \int_I u_n v \, dL^1 \to \int_I u v \, dL^1 \) for every \( v \in L^q(I) \) with \( p^{-1} + q^{-1} = 1 \). Properties of weak convergences can be found in [2, Chapter 3].

### 2.3. Rate independent and hysteresis operators

Now we recall the notions of rate independent operator and of hysteresis operator. In the last decades, operators of this kind have been extensively studied in several research articles and in the monographs [5, 7, 11, 3, 6]. We refer to these books for concrete examples. However in Section 5 we recall the definitions of several hysteresis operators and we apply to them our results.

**Definition 2.2.** Let \( F(I) \subseteq \mathbb{R}^\mathcal{T} \), \( F(I) \neq \emptyset \), and let \( R : F(I) \to \mathbb{R}^\mathcal{T} \).

a) The operator \( R \) is called rate independent if
\[
R(u \circ \phi) = R(u) \circ \phi
\]
for every \( u \in F(I) \) and for every \( \phi : \mathcal{T} \to \mathcal{T} \) increasing, surjective such that \( u \circ \phi \in F(I) \).
b) We say that $R$ is \textit{causal} if

$$u = v \text{ on } [a, t] \implies R(u)(t) = R(v)(t).$$

for every $u, v \in F(I)$ and for every $t \in I$.

c) If $R$ is rate independent and causal, then it is called \textit{hysteresis operator}.

Notice that in defining $\phi$ from $\overline{T}$ into itself, we allow, e.g., time rescalings that are equal to $b \in \mathbb{R}$ on an interval $[t_0, b[ \text{ for a certain time } t_0 \in ]a, b[.$

\textbf{Definition 2.3.} Let $F(I) \subseteq \mathbb{R}^I$, $F(I) \neq \emptyset$, and let $R : F(I) \rightarrow \mathbb{R}^I$.

a) We say that $R$ is \textit{locally monotone} if for every $c, d \in I$, $c < d$,

$$u \text{ increasing (resp. decreasing) on } [c, d] \implies R(u) \text{ increasing (resp. decreasing) on } [c, d].$$

b) We say that $R$ is \textit{locally isotone} if for every $c, d \in I$, $c < d$,

$$u \text{ monotone on } [c, d] \implies R(u) \text{ monotone on } [c, d].$$

The class of locally monotone operators contains the most significant hysteresis operators: in fact under natural assumptions all the most common hysteresis operators are locally monotone.

\section{Main results}

In order to state the main results let us recall the notion of reparametrization by arc-length of a function of bounded variation. The description we give follows essentially [4, §2.5.16, p. 109] even if the situation here is slightly different because we want two functions which are equal $L^1$-a.e. to have the same arc length; moreover a normalization factor is needed. Recall that $I = ]a, b[ \text{ and that } a, b \text{ are finite. If } u \in BV(I) \text{ we define the normalized arc length function } \ell_u : \overline{T} \rightarrow \overline{T} \text{ by setting}$

$$
\ell_u(t) := \begin{cases}
  a + \frac{b - a}{\| Du \|} |Du| (]a, t[) & \text{if } \| Du \| \neq 0 \\
  a & \text{if } \| Du \| = 0
\end{cases}, \quad t \in \overline{T}. \tag{3.1}
$$

Observe that $\ell_u(a) = a$ and, if $\| Du \| \neq 0$, $\ell_u(b) = b$. By elementary properties of the pointwise variation we have $|Du| (]a, t[) = V_e(u, ]a, t[) = V_p(u^l, ]a, t[) = V_p(u^l, ]a, t[)$ for every $t \in \overline{T}$, therefore the function $\ell_u$ is increasing and left continuous, $\text{Discont}(\ell_u) = \text{Discont}(u^l)$, and

$$
\ell_u(\overline{T}) = \overline{T} \setminus \bigcup_{t \in \text{Discont}(u)} [\ell_u(t), \ell_u(t+)].
$$

If $t_1 < t_2$ we have $|u^l(t_1) - u^l(t_2)| \leq V_p(u^l, [t_1, t_2]) = V_p(u^l, ]a, t_2[) - V_p(u^l, ]a, t_1[)$ therefore

$$
|u^l(t_1) - u^l(t_2)| \leq \frac{\| Du \|}{b - a} |\ell_u(t_1) - \ell_u(t_2)| \quad \forall t_1, t_2 \in \overline{T}. \tag{3.2}
$$
This inequality yields that $u^l(\ell_u^{-1}(\sigma))$ is a singleton for every $\sigma \in \ell_u(I)$, therefore there is a unique function $U : \ell_u(I) \rightarrow \mathbb{R}$ such that $U \circ \ell_u = u^l$. From (3.2) it also follows that $U$ is the unique Lipschitz function such that $U \circ \ell_u = u$ $\mathcal{L}^1$-a.e. and that $\text{Lip}(U) \leq \|D\ u\|/(b-a)$. In order to extend $U$ to $I$ we define $\tilde{u} : I \rightarrow \mathbb{R}$ by setting

$$\tilde{u}(\sigma) := (1-\lambda)u^l(t) + \lambda u^l(t+) \quad \text{if } \sigma = (1-\lambda)\ell_u(t) + \lambda \ell_u(t+)$$

$t \in I, \lambda \in [0,1]$.

It is clear that $\tilde{u}$ extends $U$ and that $\text{Lip}(\tilde{u}) = \text{Lip}(U)$. The function $\tilde{u}$ may be regarded as a kind of reparametrization of $u$ by the normalized arc length. In the case $\|D\ u\| = 0$, i.e. $u$ is $\mathcal{L}^1$-a.e. equal to a constant $u_0 \in \mathbb{R}$, then $\tilde{u} \equiv u_0$. For convenience we summarize the previous discussions in the following

**Proposition 3.1.** Assume $u \in BV(I)$. Let $\ell_u : I \rightarrow \overline{I}$ be its “normalized” arc length defined by (3.1). Then there exists a unique function $\tilde{u} \in \text{Lip}(I)$ such that

$$u = \tilde{u} \circ \ell_u \quad \mathcal{L}^1 \text{-a.e. in } I,$$

$\tilde{u}$ is affine on $[\ell_u(t-),\ell_u(t+)]$ $\forall t \in \text{Discont}(u^l)$. (3.4)

The main theorem of this paper is the following.

**Theorem 3.1.** Let $F(I)$ be such that $\text{Lip}(I) \subseteq F(I) \subseteq BV(I) \cap C(I)$. Assume that $R : F(I) \rightarrow BV(I) \cap C(I)$ is rate independent and has the following continuity property:

$$v, v_n \in \text{Lip}(I), \quad \|v_n - v\|_{W^{1,1}(I)} \rightarrow 0 \implies R(v_n) \rightarrow R(v) \quad \text{strictly on } I$$

as $n \rightarrow \infty$. Then $R$ is continuous with respect to the strict topology, i.e.

$$u_n \rightarrow u \quad \text{strictly on } I \implies R(u_n) \rightarrow R(u) \quad \text{strictly on } I$$

as $n \rightarrow \infty$. (3.6)

An immediate consequence is the continuity property of rate independent operators stated in the Introduction.

**Corollary 3.1.** Assume that $R : W^{1,1}(I) \rightarrow W^{1,1}(I)$ is rate independent and continuous with respect to $W^{1,1}$-topology. Then $R$ is continuous with respect to the strict topology, i.e.

$$u_n \rightarrow u \quad \text{strictly on } I \implies R(u_n) \rightarrow R(u) \quad \text{strictly on } I$$

as $n \rightarrow \infty$. (3.7)

Theorem 3.1 also applies to the problem of extending hysteresis operators to spaces of functions with bounded variation. Indeed we have
**Theorem 3.2.** Assume that $R : C_p(I) \rightarrow W^{1,1}(I)$ is rate independent and continuous with respect to the $W^{1,1}$-topology. Then $R$ is continuous with respect to the strict topology and can be extended to a continuous operator $\overline{R} : BV(I) \cap C(I) \rightarrow BV(I) \cap C(I)$. Moreover $\overline{R}$ can be further continuously extended to an operator $\overline{R} : BV(I) \rightarrow BV(I)$ if and only if $R$ is locally isotone on $\text{Lip}(I)$. If we do not distinguish functions which are equal $L^1$-almost everywhere, then the extension $\overline{R}$ is unique and is given by the formula

$$\overline{R}(u) = R(\tilde{u}) \circ \ell_u \quad \forall u \in BV(I).$$

(3.8)

Finally we conclude with a version of Theorem 3.1 for operators defined on the whole $BV(I)$. The same conclusion holds true provided we ask $R$ to be locally isotone.

**Theorem 3.3.** Let $R : BV(I) \rightarrow BV(I)$ be a locally isotone, rate independent operator. Moreover let us assume that $R(\text{Lip}(I)) \subseteq BV(I) \cap C(I)$ and that $R$ has the following continuity property:

$$v, v_n \in \text{Lip}(I), \quad \|v_n - v\|_{W^{1,1}(I)} \rightarrow 0 \implies R(v_n) \rightarrow R(v) \text{ strictly on } I$$

(3.9)

as $n \rightarrow \infty$. Then $R$ is continuous with respect to the strict topology, i.e.

$$u_n \rightarrow u \text{ strictly on } I \implies R(u_n) \rightarrow R(u) \text{ strictly on } I$$

(3.10)

as $n \rightarrow \infty$.

Other applications will be presented in Section 5.

4. Proofs

Let us start by recalling some results we proved in [8]. The first one concerns with the variation of the composition of functions. For the proof see [8, Lemma 4.3].

**Lemma 4.1.** Let $c, d \in \mathbb{R}$, $c < d$, and $v : [c, d] \rightarrow \mathbb{R}$ be such that $V_{p}(v, [c, d]) < \infty$. Let $\alpha : [c, d] \rightarrow \mathbb{R}$ be an increasing function such that $\text{Discont}(v) \cap \text{Discont}(\alpha) = \emptyset$. Moreover assume that

$$v \text{ monotone on } [\alpha(t-), \alpha(t+)] \quad \forall t \in \text{Discont}(\alpha).$$

(4.1)

Then $V_{p}(v \circ \alpha, [c, d]) = V_{p}(v, [\alpha(c), \alpha(d)])$.

Next we recall some consequences of the strict convergence.

**Lemma 4.2.** Let $v \in BV(I)$ and let $(v_n)$ be a sequence in $BV(I)$ that strictly converges to $v$. Then

(i) $|Dv_n|([a, t]) \rightarrow |Dv|([a, t])$ and $Dv_n([a, t]) \rightarrow Dv([a, t])$ as $n \nearrow \infty$ for every $t \in \text{Cont}(v');$

(ii) $v_n(t) \rightarrow v(t)$ for $L^1$-a.e. $t \in I;$
(iii) if in addition $v$ and $v_n$ are continuous for every $n \in \mathbb{N}$, then $v_n$ converges uniformly on $I$.

For the proof of the previous lemma see [8, Lemma 4.2]. We used Lemma 4.2 in [8] to prove that if a sequence $u_n$ is strictly converging in $BV(I)$ to $u$, then the sequence of the reparametrized functions $\tilde{u}_n$ is strictly converging to $\tilde{u}$. Now we are going to refine this result showing that in fact the sequence of reparametrizations converges in the strong topology of $W^{1,p}(I)$ for every $p \in [1, +\infty[$. To this aim we need the following

**Lemma 4.3.** Assume that $u \in BV(I)$ and let $\tilde{u}$ be its reparametrization defined by Proposition 3.1. Then we have that

$$|\tilde{u}'(\sigma)| = \frac{\|Du\|}{b-a} \quad \text{for } L^1\text{-a.e. } \sigma \in I. \quad (4.2)$$

In particular $\|Du\| = \|D\tilde{u}\|$.

**Proof.** Observe that by (3.4) and Lemma 4.1 we have that

$$V_p(\tilde{u}, [a, \ell_u(t)]) = V_p(\tilde{u} \circ \ell_u, [a, t]) = V_p(\tilde{u} \circ \ell_u, [a, t]) \quad \forall t \in \mathcal{T}, \quad (4.3)$$

last equality holding because $\tilde{u} \circ \ell_u$ is left-continuous, hence a good representative. But $u = \tilde{u} \circ \ell_u$ $L^1$-almost everywhere, therefore, by (3.1),

$$V_p(\tilde{u}, [a, \ell_u(t)]) = V_p(u, [a, t]) = \frac{\|Du\|}{b-a} (\ell_u(t) - a) \quad \forall t \in \mathcal{T}. \quad (4.4)$$

In particular, for $t = b$, this yields the equality $\|Du\| = \|D\tilde{u}\|$. More generally if $\sigma \in \ell_u(\mathcal{T})$, i.e. $\sigma = \ell_u(t)$ for some $t \in \mathcal{T}$, then $V_p(\tilde{u}, [a, \sigma]) = \frac{\|Du\|}{b-a} (\sigma - a)$. But $\sigma \mapsto V_p(\tilde{u}, [a, \sigma])$ is continuous on $I$ and affine on $I \setminus \ell_u(I)$, hence we get that

$$V_p(\tilde{u}, [a, \sigma]) = \frac{\|Du\|}{b-a} (\sigma - a) \quad \forall \sigma \in I. \quad (4.5)$$

Therefore, as $\tilde{u}$ is Lipschitz continuous, we have

$$\frac{\|Du\|}{b-a} (\sigma - a) = V_p(\tilde{u}, [a, \sigma]) = \int_a^\sigma |\tilde{u}'(\tau)| \, d\tau \quad \forall \sigma \in \mathcal{T},$$

thus differentiating we infer that $|\tilde{u}'(\sigma)| = \frac{\|Du\|}{b-a}$ for $L^1$-a.e. $\sigma \in I$. \hfill \Box

The previous lemma allows us to improve [8, Proposition 4.1], stating that $\tilde{u}_n \to \tilde{u}$ strictly on $I$ whenever $u_n \to u$ strictly on $I$. Indeed we have

**Proposition 4.1.** Assume $u, u_n \in BV(I)$ for every $n \in \mathbb{N}$ and $u_n \to u$ strictly on $I$. Let $\ell$ and $\ell_n$ be the “normalized” arc length functions of $u$ and $u_n$ defined as in (3.1), and let $\tilde{u}$ and $\tilde{u}_n$ be the unique Lipschitz functions satisfying (3.3)-(3.4) with $u, \tilde{u}, \ell_u$ replaced respectively by $u, \tilde{u}, \ell$ and $u_n, \tilde{u}_n, \ell_n$, as given by Proposition 3.1. Then

$$\ell_n(t) \to \ell(t) \quad \forall t \in \text{Cont}(u^I), \quad (4.6)$$

$$\tilde{u}_n \to \tilde{u} \quad \text{in } W^{1,p}(I) \quad \forall p \in [1, +\infty[. \quad (4.7)$$
Proof. Convergence (4.6) is a direct consequence of Lemma 4.2-(i). The first step to prove (4.7) consists in proving that
\[ \tilde{u}_n \to \tilde{u} \quad \text{uniformly on } I. \] (4.8)
Since, by (4.2), \( \text{Lip}(\tilde{u}_n) = \|\tilde{u}_n\|_\infty = \|Du_n\|/(b-a) \) and \( \|Du_n\| \to \|Du\| \) by the assumption, we infer that, at least for a subsequence not relabeled, \( \tilde{u}_n \) is uniformly convergent to some \( \tilde{u} \in \text{Lip}(I) \). This, together with (4.6), yields \( u_n(t) = \tilde{u}_n(t_n(t)) \to \tilde{u}(t) \) for \( L^1 \)-a.e. \( t \in I \). But by Lemma 4.2-(ii), \( u_n \to u \) \( L^1 \)-almost everywhere in \( I \), hence \( u(t) = \tilde{u}(t) \) for \( L^1 \)-a.e. \( t \in I \), and this implies, by construction of \( \tilde{u} \) that
\[ \tilde{u} = \tilde{u} \quad \text{on } \ell(I). \]
Now let us recall that affine functions on an interval \([c,d]\) are the unique minimizers of the functional \( v \mapsto \|v'\|_{L^2([c,d])} \) with given Dirichlet boundary conditions \( v(c),v(d) \) (cf., e.g., [2, Chapter 8]). It follows that \( \tilde{u} \) is the unique minimizer of the norm \( \|v\|_{L^2([c,d])} \) in the set of all functions \( v \in \text{Lip}(I) \) such that \( u = v \circ \ell \), because of (3.3)-(3.4). But \( \|\tilde{u}_n\|_{L^2(I)} \) is bounded and \( \tilde{u}_n \to \tilde{u} \) uniformly, hence \( \tilde{u}_n' \to \tilde{u}' \) in \( L^2(I) \) and, by lower semicontinuity and (4.2),
\[ \|\tilde{u}'\|_{L^2(I)}^2 \leq \liminf_{n \to \infty} \|\tilde{u}_n'\|_{L^2(I)}^2 = \liminf_{n \to \infty} \|Du_n\|^2 \frac{b-a}{b-a} = \|Du\|^2 \frac{b-a}{b-a} = \|\tilde{u}'\|_{L^2(I)}^2. \]
Thus \( \tilde{u} \) minimizes the norm \( \|\cdot\|_{L^2(I)} \), therefore \( \tilde{u} = \tilde{u} \) and convergence (4.8) is proved.
Now we prove that \( \tilde{u}_n' \to \tilde{u}' \) in \( W^{1,p}(I) \) for every \( p \in [1,\infty[ \). For every \( n \in \mathbb{N} \) we have that
\[ \|\tilde{u}_n'\|_{L^p(I)}^p = \int_a^b |\tilde{u}_n'(\sigma)|^p d\sigma = \int_a^b \left( \frac{\|Du_n\|}{b-a} \right)^p d\sigma. \]
Hence, since \( \|Du_n\| \to \|Du\| \) as \( n \to \infty \), we get that
\[ \lim_{n \to \infty} \|\tilde{u}_n'\|_{L^p(I)}^p = \int_a^b \left( \frac{\|Du\|}{b-a} \right)^p d\sigma = \int_a^b |\tilde{u}'(\sigma)|^p d\sigma. \]
Therefore we have shown that
\[ \|\tilde{u}_n'\|_{L^p(I)} \to \|\tilde{u}'\|_{L^p(I)} \quad \text{as } n \to \infty, \] (4.9)
on the other hand we know that \( \tilde{u}_n \to \tilde{u} \) in \( L^p(I) \), therefore
\[ \tilde{u}_n' \to \tilde{u}' \quad \text{in } L^p(I) \] (4.10)as \( n \to \infty \). Hence, as \( L^p(I) \) is uniformly convex for \( p \in ]1,\infty[ \), we have that (4.9)-(4.10) imply that
\[ \tilde{u}_n' \to \tilde{u}' \quad \text{in } L^p(I) \] (4.11)for every \( p \in ]1,\infty[ \) as \( n \to \infty \) (cf. e.g. [2, Proposition III.30]). Since \( I \) is bounded we get that (4.11) holds also for \( p = 1 \) and we are done. \( \square \)
Remark 4.1. In general \( \tilde{u}_n \) does not converge in \( W^{1,\infty}(I) \), indeed let \( g_n : ]0,1[ \rightarrow \mathbb{R} \) be defined by \( g_n := -\chi_{]0,1/n]} + \chi_{]1/n,1[} \) and set \( u_n(t) := \int_0^t g_n(s) \, ds, \ t \in ]0,1[, \ n \in \mathbb{N} \). If \( g(t) := 1 \) and \( u(t) := t \) for \( t \in ]0,1[ \), then we have \( u_n \to u \) strictly on \( ]0,1[ \) and \( g_n \to g \) in \( L^p(]0,1[) \) for every \( p \in [1, +\infty[ \) but \( g_n \) does not converge in \( L^\infty(]0,1[) \). Observe that \( \tilde{u} = u \) and \( \tilde{u}_n = u_n \), and \( \tilde{u}' = g, \ \tilde{u}_n' = g_n \) \( L^1 \)-a. e. for every \( n \in \mathbb{N} \), therefore \( \tilde{u}_n' \) does not converge to \( \tilde{u}' \) in \( L^\infty(I) \).

Now we can give the proof of the main theorem.

Proof of Theorem 3.1. Assume that \( u_n, u \in F(I) \) and \( u_n \to u \) strictly on \( I \). For simplicity we set \( \ell := \ell_u \) and \( \ell_n := \ell_{u_n} \) for every \( n \in \mathbb{N} \), where \( \ell_u \) and \( \ell_{u_n} \) are the “normalized” arc length functions of \( u \) and \( u_n \) defined as in (3.1), and \( \tilde{u} \) and \( \tilde{u}_n \) are the reparametrizations satisfying (3.3)-(3.4) with \( u, \tilde{u}, \ell_u \) replaced respectively by \( u, \tilde{u}_n, \ell_n \), as given by Proposition 3.1. Rate independence implies that

\[
R(u_n) = R(\tilde{u}_n \circ \ell_n) = R(\tilde{u}_n) \circ \ell_n \quad \forall n \in \mathbb{N}.
\]

The continuity of \( u \) and Proposition 4.1 let us infer that

\[
\ell_n(t) \to \ell(t) \quad \forall t \in I, \quad (4.13)
\]

\[
\tilde{u}_n \to \tilde{u} \quad \text{in } W^{1,1}(I) \quad (4.14)
\]

as \( n \to \infty \). Hence by the property (3.5) we get that

\[
R(\tilde{u}_n) \to R(\tilde{u}) \quad \text{strictly on } I \quad (4.15)
\]

for \( n \to \infty \). From this convergence, the continuity of \( R(u_n) \) and \( R(u) \), and Lemma 4.2-(iii) we get that \( R(u_n) \to R(u) \) uniformly on \( I \), therefore \( R(\tilde{u}_n) \circ \ell_n \to R(\tilde{u}) \circ \ell \) pointwise in \( I \). Finally, since \( \|R(\tilde{u}_n) \circ \ell_n\|_\infty \leq \|R(\tilde{u}_n)\|_\infty < +\infty \), by (4.12) and the dominated convergence theorem we infer that \( R(u_n) \to R(\tilde{u}) \circ \ell \) in \( L^1(I) \). Now, by the continuity of \( u \) and by rate independence, we have \( R(\tilde{u}) \circ \ell = R(\tilde{u} \circ \ell) = R(u) \), therefore we have proved that

\[
R(u_n) \to R(u) \quad \text{in } L^1(I) \quad (4.16)
\]

as \( n \to \infty \). It is left to prove the convergence of the variations. By (4.12), the continuity of \( \ell_u \), and by Lemma 4.1 we have that

\[
\|D(R(u_n))\| = \|D(R(\tilde{u}_n))\| \quad (4.17)
\]

and

\[
\|D(R(u))\| = \|D(R(\tilde{u}))\|. \quad (4.18)
\]

for every \( n \in \mathbb{N} \). Therefore by convergence (4.14) and assumption (3.5) we have that \( \|D(R(\tilde{u}_n))\| \to \|D(R(\tilde{u}))\| \), hence \( \|D(R(u_n))\| \to \|D(R(u))\| \) and we are done. \( \square \)

Before going on, let us recall the following extension theorem proved in [8, Theorem 3.1].
Theorem 4.1. Assume that $R : \text{Lip}(I) \longrightarrow BV(I) \cap C(I)$ is a rate independent operator which is continuous with respect to the strict topology, i.e. has the following property:

$$u_n \to u \quad \text{strictly on } I \quad \implies \quad R(u_n) \to R(u) \quad \text{strictly on } I$$

(4.19) as $n \to \infty$. Then $R$ can be continuously extended to an operator $\overline{R} : BV(I) \cap C(I) \longrightarrow BV(I) \cap C(I)$. Moreover $\overline{R}$ can be further continuously extended to an operator $\overline{R} : BV(I) \longrightarrow BV(I)$ if and only if $R$ is locally isotone. If we do not distinguish functions which are equal $L^1$-almost everywhere, then the extension $\overline{R}$ is unique and is given by the formula

$$\overline{R}(u) = R(\tilde{u}) \circ \ell_u \quad \forall u \in BV(I).$$

(4.20)

Remark 4.2. The statement of [8, Theorem 3.1] concerns only with the extension of $R$ to $BV(I)$. However by a simple inspection of the proof it is straightforward to see that $R$ can be continuously extended to an operator $\overline{R} : BV(I) \cap C(I) \longrightarrow BV(I) \cap C(I)$ without requiring that it is locally isotone.

Recalling that $C_{f}(\overline{I})$ is dense in $W^{1,1}(I)$ and collecting Theorems 3.1 and 4.1 we deduce Theorem 3.2.

Now we can finish this section with the

Proof of Theorem 3.3. First of all let us prove that

$$R(u) = R(\tilde{u}) \circ \ell_u \quad L^1\text{-a.e. in } I,$$

(4.21) for every $u \in BV(I)$. To this aim take $u \in BV(I)$ and $u_n \in \text{Lip}(I)$ such that $u_n \to u$ strictly on $I$. Let us give to $\ell_n$, $\ell$, $\tilde{u}_n$, $\tilde{u}$ the same meaning given in the proof of Theorem 3.1. From (3.9) and Theorem 3.1 it follows that the restriction of $R$ to $\text{Lip}(I)$ is strictly continuous, therefore thanks to Theorem 4.1 and formula (4.20)

$$R(u_n) \to \overline{R}(u) = R(\tilde{u}) \circ \ell_u \quad \text{strictly on } I.$$

(4.22)

On the other hand $R(u_n) = R(\tilde{u}_n \circ \ell_n) = R(\tilde{u}_n) \circ \ell_n$, thanks to the rate independence. But from Proposition 4.1 $\ell_n \to \ell$ $L^1$-a.e. and $\tilde{u}_n \to \tilde{u}$ in $W^{1,1}(I)$, hence by (3.9) $R(\tilde{u}_n) \to R(\tilde{u})$ strictly and hence uniformly (thanks to Lemma 4.2-(iii)). Therefore $R(u_n) = R(\tilde{u}_n) \circ \ell_n \to R(\tilde{u}) \circ \ell$ $L^1$-a.e. in $I$ and by the dominated convergence theorem

$$R(u_n) = R(\tilde{u}_n) \circ \ell_n \to R(\tilde{u}) \circ \ell \quad \text{in } L^1(I)$$

(4.23) and (4.21) is proved.

Now we can prove (3.10). Let $u, u_n \in BV(I)$ such that $u_n \to u$ strictly on $I$. Let us adopt the usual convention for $\ell_n$, $\ell$, $\tilde{u}_n$, $\tilde{u}$. By the same argument as above and by (4.21) we find that $R(u_n) \to R(u)$ in $L^1(I)$. As regards to the variations, observe that since $R$ is locally isotone, Lemma 4.1 and formulas (3.3)-(3.4) yield

$$\| D(R(u_n)) \| = \| D(R(\tilde{u}_n) \circ \ell_{u_n}) \| = \| D(R(\tilde{u}_n)) \|$$

(4.24)
5. Applications

In this section we show that Theorem 3.1 applies to several hysteresis operators that occur in the applications. In all the examples $I = [0,T]$, where $T \in (0,\infty]$. We will not present the operators in their more general form in order to simplify the presentation.

5.1. The play operator. Let $r > 0$ and $f_r : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_r(u,w) := \max\{u - r, \min\{u + r, w\}\}, \quad (u,w) \in \mathbb{R}^2.$$  

The play operator $P_r : C_{pl}(I) \to \mathbb{R}^I$ is defined by setting $P_r(u) := w$ where $w \in \mathbb{R}^I$ is defined by

$$w(0) := f_r(u(0),0),$$

$$w(t) := f_r(u(t),w(t_j)) \quad \text{for } t \in [t_{j-1},t_j]$$

where $(t_j)_{j=0}^m$ is a subdivision of $I$ such that $u$ is affine on each $[t_{j-1},t_j]$, $t_0 = 0$, $t_m = T$. In [3, §2.3] it is proved that the range of $P_r$ is contained in $C_{pl}(I)$ and that $P_r$ is continuous hysteresis operator with respect to $W^{1,1}$-topology, therefore we can apply Theorem 3.1 (or Corollary 3.1) and infer that $P_r$ is also continuous with respect to the strict metric. Moreover $P_r$ is locally monotone, hence by Theorem 3.2 it admits an extension $\overline{P}_r : BV(I) \to BV(I)$ which is continuous with respect to the strict metric. If we identify functions belonging to the same $L^1$-class of equivalence, then this extension is unique.

5.2. The stop operator. Let $r > 0$ and $e_r : \mathbb{R} \to \mathbb{R}$ be defined by

$$e_r(u) := \min\{r, \max\{-r,u\}\}, \quad u \in \mathbb{R}.$$  

For every $u \in C_{pl}(I)$ let us define the function $w : I \to \mathbb{R}$ by

$$w(0) := e_r(u(0)),$$

$$w(t) := e_r(u(t) - u(t_j) + w(t_j)) \quad \text{for } t \in [t_{j-1},t_j]$$

where $(t_j)_{j=0}^m$ is a subdivision of $I$ such that $u$ is affine on each $[t_{j-1},t_j]$, $t_0 = 0$, $t_m = T$. Setting $S_r(u) = w$ we have defined a hysteresis operator $S_r : C_{pl}(I) \to \mathbb{R}^I$ which is usually called stop operator. In [3, Section 2.1] it is proved that $S_r(u) = u - P_r(u)$ for every $u \in C_{pl}(I)$, where $P_r$ denotes the play operator. Hence the range of $S_r$ is contained in $C_{pl}(I)$ and $S_r$ is continuous with respect to $W^{1,1}$-topology, therefore we can apply Theorem 3.1 and infer that $S_r$ is also continuous with respect to the strict metric. Moreover $S_r$ is locally monotone, hence it admits an essentially unique
extension $\tilde{S}_r : BV(I) \to BV(I)$ which is continuous with respect to the strict metric (the term ‘essentially’ here means that the extension is unique if consider $L^1$-class of functions rather than functions defined everywhere).

5.3. The Prandtl operator. Given $p \in L^1([0, +\infty[)$, the Prandtl operator $F : W^{1,1}(I) \to \mathbb{R}^I$ is defined by setting

$$F(u)(t) := \int_0^{+\infty} p(r)P_r(u)(t) \, dr, \quad u \in W^{1,1}(I),$$

where $P_r$ denotes the play operator. In [3, Section 2.4] it is shown that $F$ is a hysteresis operator, its range is contained in $W^{1,1}(I)$ and that it is continuous with respect to the $W^{1,1}$-topology. Therefore by Theorem 3.1 it is continuous also with respect to the strict metric and can be continuously extended to $BV(I) \cap C(I)$.

Moreover if $p$ is positive then the Prandtl operator is locally monotone, hence it can be extended to $BV(I)$ in a continuous and essentially unique manner.

5.4. The Preisach operator. We begin by describing the relay operators. Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ be such that $\rho_1 < \rho_2$. Let $\eta \in \{-1, 1\}$. The relay operator $R_{\rho, \eta} : W^{1,1}(I) \to \mathbb{R}^I$ is defined by setting

$$R_{\rho, \eta}(u)(0) := \begin{cases} -1 & \text{if } u(0) \leq \rho_1 \\ \eta & \text{if } \rho_1 < u(0) < \rho_2 \\ 1 & \text{if } u(0) \geq \rho_2 \end{cases}$$

and

$$R_{\rho, \eta}(u)(t) := \begin{cases} R_{\rho, \eta}(u)(0) & \text{if } u(s) \not\in \{\rho_1, \rho_2\} \quad \forall s \in [0, t] \\ -1 & \text{if } \max\{s \in [0, t] : u(s) \in \{\rho_1, \rho_2\}\} = \rho_1 \\ 1 & \text{if } \max\{s \in [0, t] : u(s) \in \{\rho_1, \rho_2\}\} = \rho_2 \end{cases}$$

Let $P = \{(\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}$ be the so-called Preisach plane and let $\mu$ be a finite Borel measure on $P$ satisfying $|\mu|\left(\{\rho \in P : \rho_i = r\}\right) = 0$ for every $r \in \mathbb{R}$ and for $i = 1, 2$. Let $\eta : P \to \{-1, 1\}$ be a Borel measurable function. The hysteresis Preisach operator $W : W^{1,1}(I) \to \mathbb{R}^I$ is defined by

$$W(u)(t) := \int_P R_{\rho, \eta}(\rho)(u)(t) \, d\mu(\rho)$$

In [3, Proposition 2.4.11] it is proved that $W$ maps $W^{1,1}(I)$ into itself and that it is a hysteresis operator. Moreover $W$ is continuous with respect to the $W^{1,1}$-topology. Hence by virtue of Theorem 3.1 $W$ is also continuous with respect to the strict topology and can be continuously extended to $BV(I) \cap C(I)$. If $\mu$ is a positive measure, then $W$ is locally monotone (cf. [11, Theorem IV.2.1]), hence we can apply Theorem 3.2 and extend continuously $W$ to the space $BV(I)$ in a unique way.
5.5. The Duhem operator. Let \( w_0 \in \mathbb{R} \) and let \( g_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) be continuous, \( i = 1, 2 \), such that there exists \( L \in C(\mathbb{R}) \) positive satisfying \( |g_i(r, s_1) - g_i(r, s_2)| \leq L(r)|s_1 - s_2| \) for every \( r, s_1, s_2 \in \mathbb{R} \) and \( i = 1, 2 \). For any \( u \in W^{1,1}(I) \), let \( w =: D(u) \) be the unique solution of the Cauchy problem

\[
\begin{cases}
  w'(t) = g_1(u(t), w(t))(u'(t))^+ - g_2(u(t), w(t))(u'(t))^- & \text{in } I \\
  w(0) = w_0
\end{cases}
\]

(here \( x^+ \) and \( x^- \) represent the positive and the negative part of a number \( x \in \mathbb{R} \)). In this way we have defined a hysteresis operator \( D : W^{1,1}(I) \rightarrow \mathbb{R}^I \) (cf. [11, Theorem V.1.1]). By [11, Theorem V.1.2] \( D \) maps \( W^{1,1}(I) \) into itself and it is continuous with respect to the \( W^{1,1} \)-topology. Hence in order to prove that \( D \) is continuous with respect to the strict metric we only need to apply Theorem 3.1. Finally if \( g_1 \) and \( g_2 \) are positive then \( D \) is locally monotone (see [11, p. 140]). Under these assumptions we can apply Theorem 3.2 and infer the existence of a unique continuous extension of \( D \) to the whole space \( BV(I) \). Instead the existence of a unique continuous extension to \( BV(I) \cap C(I) \) is guaranteed without any restriction on \( g_1 \) and \( g_2 \).

References


Vincenzo Recupero, Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy.
Phone:+39 011564 7542; Fax: +39 0115647599.
E-mail address: vincenzo.recupero@polito.it