THE PLAY OPERATOR ON THE RECTIFIABLE CURVES IN A HILBERT SPACE

VINCENZO RECUPERO

Abstract. The vector play operator is the solution operator of a class of evolution variational inequalities arising in continuum mechanics. For regular data the existence of solutions is easily obtained from general results on maximal monotone operators. If the datum is a continuous function of bounded variation, then the existence of a weak solution is usually proved by means of a time discretization procedure. In this paper we give a short proof of the existence of the play operator on rectifiable curves making use of basic facts of measure theory. We also drop the separability assumptions usually made by other authors.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $Z \subseteq \mathcal{H}$ be a closed convex subset. Given $T > 0$ and $u : [0, T] \rightarrow \mathcal{H}$, it is well known that the evolution variational inequality in the unknown $y : [0, T] \rightarrow \mathcal{H}$

$$\langle u(t) - y(t) - z, y'(t) \rangle \geq 0 \quad \forall z \in Z$$

(1.1)

represents the constitutive law for several models in plasticity and elastoplasticity (see [1] and [2] for a survey). If the initial condition

$$u(0) - y(0) = z_0 \in Z$$

(1.2)

is prescribed, then the solution of problem (1.1)–(1.2) can be easily obtained from the theory of evolution equations governed by maximal monotone operators developed in [3]. To be more precise, if $u$ belongs to some Sobolev space $W^{1,p}([0, T]; \mathcal{H})$, $p \in [1, \infty]$, then there exists a unique function $y \in W^{1,p}([0, T]; \mathcal{H})$ satisfying (1.1)–(1.2). The solution operator $P : W^{1,p}([0, T]; \mathcal{H}) \rightarrow W^{1,p}([0, T]; \mathcal{H}) : u \mapsto P(u) := y$ is usually called play operator or vector play operator, to emphasize the fact that it acts on $\mathcal{H}$-valued curves. Often the suggestive terms input and output are used for the functions $u$ and $y$ respectively. A natural question which arises is to find a model for these problems in the case $u$ is less regular. The first step is to consider $u$ continuous and with bounded variation, in other words to extend in a suitable way $P$ to some operator $\tilde{P}$ defined on $BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H})$. In this case we cannot differentiate $u$, therefore formulation (1.1) makes no sense, and another

2000 Mathematics Subject Classification. 74N30, 47H30, 26A45.

Key words and phrases. Problems involving hysteresis, particular nonlinear operators, functions of bounded variation, rate independence, evolution variational inequalities.
notion of solution has to be found. In the monograph [1] a natural notion of generalized solution is defined in the case \( u \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \): a map \( y \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \) is called a generalized solution of (1.1)–(1.2) if (1.2) holds together with the integral inequality
\[
\int_0^T \langle u(t) - y(t) - z(t), dy(t) \rangle \geq 0 \quad \forall z \in C([0, T]; \mathcal{H}).
\] (1.3)

Here the integral has to be understood in the sense of Riemann-Stieltjes. In [1, Theorem 3.1, Section I.3] it is proved that problem (1.3)–(1.2) admits a unique solution by means of a time discretization technique. Essentially (1.1) is replaced by an Euler implicit scheme which is solved using Hilbert space techniques. Once one finds a priori estimates for the piecewise linear interpolation of the discrete solutions, the limit procedure is carried out by using suitable theorems for taking the limit under the Riemann-Stieltjes integral sign.

The main aim of our paper is to give a short proof of the existence-uniqueness theorem of a generalized solution of (1.3)–(1.2) using basic properties from measure theory, and exploiting the fact that solvability of (1.1)–(1.2) is well known in the regular case. Then the elementary character of our proof allows to state that in a certain sense the solvability of the generalized problem is a direct consequence of the solvability of the classical problem. Let us also point out that we do not require \( \mathcal{H} \) to be separable, as assumed in [1].

The idea for our proof is suggested by a well-known feature of the play operator, namely rate independence, which can be expressed in this way: for any \( u \in W^{1,p}([0, T]; \mathcal{H}) \) and for every increasing surjective Lipschitz reparametrization \( \varphi : [0, T] \rightarrow [0, T] \) one has
\[
\mathcal{P}(u \circ \varphi) = \mathcal{P}(u) \circ \varphi
\] (1.4)
(rate independent operators, hysteresis, and other related problems are studied in [4, 5, 6] for the scalar case and in [1] for the vector case). Whatever definition of generalized solution of problem (1.1)–(1.2) one may define, it is natural to expect the resulting operator solution \( \tilde{\mathcal{P}} \) has still a rate independence property. Therefore if \( u \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \), we can consider the particular reparametrization
\[
\varphi(t) := \ell_u(t) := \frac{T}{V(u, [0, t])} V(u, [0, t]), \quad t \in [0, T],
\] (1.5)
where \( V(u, [0, t]) \) denotes the variation of \( u \) on the interval \([0, t]\). Since one has the factorization
\[
u = \tilde{u} \circ \ell_u
\] (1.6)
for a suitable Lipschitz map \( \tilde{u} \in W^{1,\infty}([0, T]; \mathcal{H}) \) (see Proposition 3.1), it is natural to expect that
\[
\tilde{\mathcal{P}}(u) = \tilde{\mathcal{P}}(\tilde{u} \circ \ell_u) = \mathcal{P}(\tilde{u}) \circ \ell_u.
\] (1.7)
As \( \tilde{u} \) is Lipschitz, the output \( P(\tilde{u}) \) is known, and this suggests that formula (1.7) should be the generalized solution of the problem with continuous \( BV \) data. Hence we prove directly that \( P(\tilde{u}) \circ \ell_u \) is the solution of (1.3)–(1.2), thereby showing that \emph{the action of} \( P \) \emph{on Lipschitz functions uniquely determines the action of} \( P \) \emph{on} \( BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \).

We found formula (1.7) in [7] where we look for sufficient conditions in order to extend a general \emph{scalar} rate independent operator \( R : \text{Lip}([0, T]; \mathbb{R}) \rightarrow \text{Lip}([0, T]; \mathbb{R}) \) which is continuous with respect to the strict metric, i.e. such that \( \| R(u_n) - R(u) \|_{L^1} + |V(R(u_n), [0, T]) - V(R(u), [0, T])| \rightarrow 0 \) whenever \( \| u_n - u \|_{L^1} + |V(u_n, [0, T]) - V(u, [0, T])| \rightarrow 0 \). Precisely in [7] and [8] we prove that given a rate independent operator which is continuous with respect to the strict metric, then it can be continuously extended to \( BV([0, T]; \mathbb{R}) \) if and only if it is \emph{locally isotone}, i.e. if

\[
\text{if } u \text{ monotone on } [c, d] \implies R(u) \text{ monotone on } [c, d] \quad (1.8)
\]

for \( 0 \leq c \leq d \leq T \). If one wants to extend \( R \) only to the space \( BV([0, T]; \mathbb{R}) \cap C([0, T]; \mathbb{R}) \), the condition (1.8) is not needed and the existence of the continuous extension \( \bar{R} : BV([0, T]; \mathbb{R}) \cap C([0, T]; \mathbb{R}) \rightarrow BV([0, T]; \mathbb{R}) \cap C([0, T]; \mathbb{R}) \) is always granted. Concerning the vector case, the notion (1.8) makes no sense and has to be replaced by the condition ([7, Remark 4.2])

\[
V(u, [c, d]) = \| u(d) - u(c) \|_{\mathcal{H}} \implies V(R(u), [c, d]) = \| R(u)(d) - R(u)(c) \|_{\mathcal{H}}. \quad (1.9)
\]

The vector play operator \( P \) is continuous with respect to the strict metric but it does not satisfy (1.9), therefore it cannot be continuously extended to all of \( BV \) (cf. [9]). Nevertheless, as in the scalar case, the existence of the continuous extension is granted to the space of continuous functions of bounded variation, and such extension is given by the formula \( \bar{R}(u) = R(\tilde{u}) \circ \ell_u \) where \( \ell_u \) is defined in (1.5) and \( \tilde{u} \in \text{Lip}([0, T]; \mathcal{H}) \) satisfies (1.6) (in fact in [9] we prove these facts for general vector rate independent operators in a metric space setting). Anyway, we will show that the proof that (1.7) solves (1.3)–(1.2) does not require any analysis of continuity properties of \( P \) but relies only on basic facts about measure theory (of course continuity properties are important for physical applications and to give further motivations of the notion of generalized solution).

One may wonder if formula (1.7) may be chosen as a generalized solution of (1.1)–(1.2) for \( u \in BV([0, T]; \mathcal{H}) \), since (1.7) makes sense also in this case. In [10] a definition of solution for \( u \) with bounded variation is given and it is very similar to the integral condition (1.3): the integral has to be understood in the Young sense, and the integrand map has to be replaced with its right continuous representative. But it is possible to prove that in general for discontinuous \( u \in BV([0, T]; \mathcal{H}) \), the unique solution found in [10] is not given by formula (1.7). The relations between \( P(\tilde{u}) \circ \ell_u \) and the solution of [10] is investigated in [9].
Let us conclude with a brief plan of the paper. In the following section we recall some basic facts about functions of bounded variation and Riemann-Stieltjes integral. In Section 3 we state the precise results of the paper and in Section 4 we review the existence results for the play operator with regular data. In Section 5 we give the detailed proofs.

2. Preliminaries

We assume that $\mathcal{H}$ is a real Hilbert space with scalar product $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$: $(x, y) \mapsto \langle x, y \rangle$ and norm $\|x\|_\mathcal{H} := \langle x, x \rangle^{1/2}$. We fix $a, b \in \mathbb{R}$ with $a < b$, and $I := [a, b]$, the compact interval with endpoints $a, b$. We will deal with functions in $\mathcal{H}^I$, the set of $\mathcal{H}$-valued functions defined on $I$. Most of times we will consider the subspace $C(I; \mathcal{H})$ of continuous maps. In the sequel $\mathbb{N}$ is the set of strictly positive integers $\{1, 2, \ldots\}$, $\mathcal{B}(I)$ denotes the family of all Borel subsets of $I$, and $\mathcal{L}^1$ is the Lebesgue measure on $\mathbb{R}$. We do not identify two functions agreeing $\mathcal{L}^1$-a.e.. If $f \in \mathcal{H}^I$ is given, Lip$(f)$ denotes its Lipschitz constant. In the case $\mathcal{H} = \mathbb{R}$ we say that $f$ is increasing if $(f(t_1) - f(t_2))(t_1 - t_2) \geq 0$ for every $t_1, t_2 \in I$.

2.1. Functions of bounded variation. Let $J$ be a closed interval with $J \subseteq I$. We recall that a subdivision of $J$ is a finite set of points $\mathfrak{s} = \{t_j\}_{j=0}^m$, $m \in \mathbb{N}$, with the property that $\inf J = t_0 < t_1 < \cdots < t_m = \sup J$. The size of $\mathfrak{s}$ is defined as $\max\{|t_j - t_{j-1}| : j = 1, \ldots, m\}$. If $\delta > 0$, the family of subdivisions of $J$ whose size is less or equal than $\delta$ is denoted by $\mathcal{S}_\delta(J)$. We set $\mathcal{S}(J) := \bigcup_{\delta > 0} \mathcal{S}_\delta(J)$, the set of all subdivisions of $J$. If $f \in \mathcal{H}^I$ the variation of $f$ on $J$ is defined as

$$V(f, J) := \sup \left\{ \sum_{j=1}^m \| f(t_j) - f(t_{j-1}) \|_\mathcal{H} : m \in \mathbb{N}, \{t_j\}_{j=0}^m \in \mathcal{S}(J) \right\}.$$ 

A function $f \in \mathcal{H}^I$ such that $V(f, I) < \infty$ is called of bounded variation and we set $BV(I; \mathcal{H}) := \{ f \in \mathcal{H}^I : V(f, I) < \infty \}$. If $V(f, I) < \infty$, it is well known that $f$ is bounded and that $V(f \circ \phi, I) = V(f, I)$ for every $\phi : I \rightarrow I$ increasing and surjective (see e.g. [13, Chapter X]). We do not distinguish between pointwise variation and essential variation because we will only consider the variation of continuous maps, for which these two notions coincide (cf. [11, Section 4.5.10]).

2.2. The Riemann-Stieltjes integral. Let us now recall the definition of Riemann-Stieltjes integral. There are many different ways to define this integral (see e.g. [12] for the scalar case), but in the context we are dealing with (that is when the integrand is continuous) they are all equivalent. We follow [13, Chapter X]. Let $f, g \in \mathcal{H}^I$ be bounded maps and let $\mathfrak{s} = \{t_0, \ldots, t_m\} \in \mathcal{S}(I)$ be a subdivision of $I$. If the family of numbers $c = (c_j)_{j=1}^m$ is consistent with $\mathfrak{s}$, i.e. $t_{j-1} \leq c_j \leq t_j$ for every...
$j = 1, \ldots, m$, we define the Riemann-Stieltjes sum

$$S(f, g, s, c) := \sum_{j=1}^{m} \langle f(c_j), g(t_j) - g(t_{j-1}) \rangle.$$  \hfill (2.1)

We say that $f$ is Riemann-Stieltjes integrable with respect to $g$ if there exists an element $L \in \mathbb{R}$ such that given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $s \in S_{\delta}(a,b)$ and $c$ is consistent with $s$ then

$$|L - S(f, g, s, c)| < \varepsilon.$$ 

This element $L$ is clearly unique and is called Riemann-Stieltjes integral of $f$ with respect to $g$ and is denoted by one of the symbols

$$\int_{a}^{b} \langle f, dg \rangle, \quad \int_{a}^{b} \langle f(t), dg(t) \rangle.$$ 

The Riemann-Stieltjes integral is linear in the integrator function $g$. The proof of the following proposition can be found in [13, Propositions 1.4 and 1.5, Chapter X].

**Proposition 2.1.** Let $f, g \in \mathcal{H}^{I}$ be bounded maps.

(i) If $f \in C(I; \mathcal{H})$ and $g \in BV(I; \mathcal{H})$, then $f$ is Riemann-Stieltjes integrable with respect to $g$.

(ii) We have that $f$ is Riemann-Stieltjes integrable with respect to $g$ if and only if $g$ is Riemann-Stieltjes integrable with respect to $f$ and in that case the following formula for integration by parts holds:

$$\int_{a}^{b} \langle f, dg \rangle = - \int_{a}^{b} \langle g, df \rangle + \langle f(b), g(b) \rangle - \langle f(a), g(a) \rangle.$$  \hfill (2.2)

The definition of Riemann-Stieltjes integral given in [1] is slightly different: it is assumed that $f \in C(I; \mathcal{H})$, $g \in BV(I; \mathcal{H})$, and $c_j = t_j$ for every $j = 1, \ldots, m$ in (2.1). It is a simple exercise to verify that in this case the two definitions are equivalent. It is possible to set the Riemann-Stieltjes integral in the framework of the theory of integration with respect to a vector measure (cf. [14]), but we don’t need such a generality. For the sake of completeness we recall instead some notions about the Bochner integral with respect to positive measure (see, e.g., [13, Chapter VI] or [15, Appendix] for details).

**2.3. The Bochner integral.** Suppose we are given a bounded measure $\mu : \mathcal{B}(I) \rightarrow [0, \infty]$ and a step function $f : I \rightarrow \mathcal{H}$, that is $f = \sum_{j=1}^{m} \chi_{B_j} f_j$ for some $m \in \mathbb{N}$, $B_j \in \mathcal{B}(I)$, $f_j \in \mathcal{H}$, $j = 1, \ldots, m$ ($\chi_{B}$ is the characteristic function of $B$).

The integral of $f$ with respect to $\mu$ is defined as $\int_{I} f \, d\mu := \sum_{j=1}^{m} \mu(B_j) f_j$. The space $L^{1}(\mu; \mathcal{H}) = L^{1}(\mu; \mathcal{H})$ is defined as the set of functions $f : I \rightarrow \mathcal{H}$ which admit a $L^{1}$-approximating sequence of step functions, i.e. a sequence $f_n$ of step functions
that converges $\mu$-a.e. to $f$ and that is $L^1$-Cauchy, that is $\int_I \|f_n(t) - f_m(t)\|_H \, dt \to 0$ as $n, m \to \infty$. For any $f \in L^1(\mu; H)$ we define its integral with respect to $\mu$ as

$$\int_I f \, d\mu := \int_I f(t) \, d\mu(t) := \lim_{n \to \infty} \int_I f_n \, d\mu,$$

once it is proved that this limit exists and is independent of the choice of the approximating sequence. As usual $\int_B f \, d\mu := \int_I \chi_B f \, d\mu$ for $B \in \mathcal{B}(I)$. For any $x \in H$ and $f \in L^1(\mu; H)$ we have that $t \mapsto \|f(t)\|_H$ and $t \mapsto \langle x, f(t) \rangle$ are integrable with respect to $\mu$, we define the seminorm $\|f\|_{L^1(\mu; H)} := \int_I \|f(t)\|_H \, d\mu$, and the following fundamental formula holds ([13, Theorem 4.1, Chapter VI]):

$$\langle x, \int_I f(t) \, d\mu(t) \rangle = \int_I \langle x, f(t) \rangle \, d\mu(t). \quad (2.3)$$

We will need the following easy property of Bochner integral. If $f \in L^1(\mu; H)$, $\nu : \mathcal{B}(I) \to [0, \infty]$ is a measure, and $\psi : I \to I$ is such that $\nu(B) = \mu(\psi^{-1}(B))$ for every $B \in \mathcal{B}(I)$, then

$$\int_A f(s) \, d\nu(s) = \int_{\psi^{-1}(A)} f(\psi(t)) \, d\mu(t) \quad \forall A \in \mathcal{B}(I). \quad (2.4)$$

This formula is obvious if $f$ is a step function. In the general case we approximate $f$ by a $L^1$-approximating sequence and we observe that $f_n \circ \psi \to f \circ \psi$ in $L^1(\mu; H)$, hence we conclude by taking the limit in (2.4) with $f$ replaced by $f_n$.

Positive measures on the real line can be obtained from increasing functions. Indeed if $h : I \to \mathbb{R}$ is a right-continuous increasing function, then there exists a unique measure $\mu_h : \mathcal{B}(I) \to [0, \infty]$ with the property that $\mu_h([c, d]) = h(d) - h(c)$ whenever $a \leq c \leq d \leq b$. The measure $\mu_h$ is called Lebesgue-Stieltjes measure associated to $h$ (see e.g. [16, Theorem 1.16, Section 1.5] or [17, Theorem 8.12, Chapter 8]). If $h(t) = t$ for every $t$ then $\mu_h = L^1$ and the notations $\int_I f(t) \, dt := \int_I f \, dL^1$ and $L^1(I, H) := L^1(L^1, I, H)$ are used. It is well known ([16, Section 9.1]) that $\mu_h$ turns out to be the distributional derivative of $h$, therefore the notation $\mu_h = Dd$ is also customary. Hence for the integral of $f \in L^1(Dd, H)$ one can use the symbols

$$\int_I f \, dDd := \int_I f \, d\mu_h.$$

We recall that if $t \in I$, then $Dd\{t\} \neq 0$ if and only if $t$ is a jump point of $h$. In particular if $h$ is continuous, any singleton has $Dd$-measure zero and notations like $\int_c^d f \, dDd$ are not ambiguous. If $\nu = L^1$, $\psi = h : I \to I$ is continuous and surjective, and $\mu = Dd$, then formula (2.4) applies with $A = [h(c), h(d)]$ and reads

$$\int_{h(c)}^{h(d)} f(s) \, ds = \int_{\inf h^{-1}(h(c))}^{\sup h^{-1}(h(d))} f(h(t)) \, dDd(t) = \int_c^d f(h(t)) \, dDd(t) \quad (2.5)$$

for every $f \in L^1(I, H)$. Another straightforward consequence of formula (2.4) is the following
Proposition 2.2. Let \( h : I \rightarrow I \) be increasing and continuous. Assume \( N \subseteq I \) has Lebesgue measure zero and let \( E := \{ t \in I : h(t) \in N \} \). Then \( \text{Dh}(E) = 0 \).

We recall that the Sobolev space \( W^{1,p}(I; \mathcal{H}) \), \( p \in [1, \infty] \), consists of the functions \( f : I \rightarrow \mathcal{H} \) such that there exists \( g \in L^p(I; \mathcal{H}) \) satisfying the equality

\[
 f(t) = f(a) + \int_a^t g(s) \, ds \quad \forall t \in I. \tag{2.6}
\]

This is equivalent to say that \( f \in C(I; \mathcal{H}) \) and its distributional derivative \( f' \in L^p(I; \mathcal{H}) \), so that \( f' = g \) \( L^1 \)-a.e. in \( I \) (for this and the following properties see [3, Appendix]). It is well known that if \( f \in W^{1,p}(I; \mathcal{H}) \) and (2.6) holds then it is differentiable \( L^1 \)-almost everywhere and its differential is \( L^1 \)-a.e. equal to \( g \). If \( 1 \leq p \leq q \leq \infty \) we have \( W^{1,q}(I; \mathcal{H}) \subseteq W^{1,p}(I; \mathcal{H}) \subseteq BV(I; \mathcal{H}) \cap C(I; \mathcal{H}) \). Moreover we have that \( W^{1,\infty}(I; \mathcal{H}) = \text{Lip}(I; \mathcal{H}) \), the set of maps with finite Lipschitz constant.

3. Main results

Within this section we state the main result of the paper. We need some preliminary notions. Following [11, Section 2.5.16], we define a sort of arc length of a function of bounded variation. Recall that \( I = [a, b] \). If \( u \in BV(I; \mathcal{H}) \cap C(I; \mathcal{H}) \), we define the normalized arc length function \( \ell_u : I \rightarrow I \) by setting

\[
 \ell_u(t) := \begin{cases} 
 a + \frac{b - a}{V(u, I)} V(u, [a, t]) & \text{if } V(u, I) \neq 0, \\
 a & \text{if } V(u, I) = 0,
\end{cases} \quad t \in I. \tag{3.1}
\]

The function \( \ell_u \) is increasing, (continuous) and surjective. If \( t_1 < t_2 \) we have

\[
 \| u(t_1) - u(t_2) \|_{\mathcal{H}} \leq V(u, [t_1, t_2]) = V(u, [a, t_2]) - V(u, [a, t_1])
\]

therefore

\[
 \| u(t_1) - u(t_2) \|_{\mathcal{H}} \leq \frac{V(u, I)}{b - a} |\ell_u(t_1) - \ell_u(t_2)| \quad \forall t_1, t_2 \in I. \tag{3.2}
\]

This inequality yields that \( u(\ell_u^{-1}(\sigma)) \) is a singleton for every \( \sigma \in \ell_u(I) \), therefore there is a unique function \( \tilde{u} : I \rightarrow \mathcal{H} \) such that \( \tilde{u} \circ \ell = u \). From (3.2) it also follows that \( \tilde{u} \) is Lipschitz continuous. Hence we have the first part of the following proposition, whose proof will be completed in Section 5.

Proposition 3.1. Assume \( u \in BV(I; \mathcal{H}) \cap C(I; \mathcal{H}) \) and let \( \ell_u : I \rightarrow I \) be its “normalized” arc length defined by (3.1). Then there exists a unique function \( \tilde{u} \in \text{Lip}(I; \mathcal{H}) \) such that

\[
 u = \tilde{u} \circ \ell_u.
\]

If \( p \in [1, \infty] \) and \( u \in W^{1,p}(I; \mathcal{H}) \) then \( \ell_u \in W^{1,p}(I; \mathbb{R}) \). Moreover \( V(\tilde{u}, I) = V(u, I) \). Finally if \( \phi : I \rightarrow I \) is increasing and surjective, and \( v := u \circ \phi \), then \( \ell_v = \ell_u \circ \phi \) and \( \tilde{v} = \tilde{u} \), or in other terms \( u \circ \phi = \tilde{u} \).
Throughout the paper we assume that
\[ Z \] is a closed convex subset of \( H \), \( (3.3) \)
\[ 0 \in Z. \] \( (3.4) \)

Let us recall the classical existence theorem of the play operator on Lipschitz maps.

**Proposition 3.2.** Assume \((3.3)-(3.4)\) hold. If \( z_0 \in Z \) and \( u \in \text{Lip}([0,T];H) \) then there exists a unique \( y = P(u) \in \text{Lip}([0,T];H) \) such that
\[
\begin{align*}
 u(t) - y(t) &\in Z \quad \forall t \in [0,T], \\
 \langle u(t) - y(t) - z, y'(t) \rangle &\geq 0 \quad \forall z \in Z, \text{ for } L^1\text{-a.e. } t \in [0,T], \\
 u(0) - y(0) &= z_0.
\end{align*}
\]
\( (3.5) \quad (3.6) \quad (3.7) \)

The previous result allows us to define the operator \( P : \text{Lip}([0,T];H) \rightarrow \text{Lip}([0,T];H) \) associating with \( u \) the unique solution \( y \) satisfying \((3.6)-(3.7)\). Now we can state the main theorem.

**Theorem 3.1.** Assume \((3.3)-(3.4)\) hold. Consider \( z_0 \in Z, u \in \text{BV}([0,T];H) \cap C([0,T];H) \), and let \( \ell_u \) and \( \bar{u} \) be the functions given by Proposition 3.1 with \( a = 0, b = T \). If we define
\[
\tilde{P}(u) := P(\bar{u}) \circ \ell_u,
\]
\( (3.8) \)
then \( y := \tilde{P}(u) \in \text{BV}([0,T];H) \cap C([0,T];H) \) and \( y \) is the unique map such that
\[
\begin{align*}
 u(t) - y(t) &\in Z \quad \forall t \in [0,T], \\
 \int_0^t \langle u(s) - y(s) - z(s), dy(s) \rangle &\geq 0 \quad \forall z \in C([0,T];Z), \forall t \in [0,T], \\
 u(0) - y(0) &= z_0.
\end{align*}
\]
\( (3.9) \quad (3.10) \quad (3.11) \)
If \( u \in \text{Lip}([0,T];H) \) then \( \tilde{P}(u) \) is the unique solution of \((3.5)-(3.7)\), in other words \( \tilde{P} \) extends \( P \).

**Remark 3.1.** As we outlined in the Introduction the main novelty is the proof of Theorem 3.1. Indeed the existence and uniqueness of Problem \((3.9)-(3.11)\) were proved in [1, Section I.3] by means of time discretization procedure, whereas we present a short proof based only on some basic facts of measure theory. We also point out that formula \((3.8)\) seems to be new, though not surprising. Finally let us observe that in [1] \( H \) is supposed to be separable, but we do not need such assumption.

Formula \((3.8)\) allows us to deduce that if one knows how \( P \) acts on Lipschitz maps, then one directly infers the action of the play operator on the larger space \( \text{BV}([0,T];H) \cap C([0,T];H) \). Thanks to \((3.8)\) we will also easily show (Proposition 5.1) that the operator \( \tilde{P} \) defined by Theorem 3.1 is rate independent. We have
not found an explicit proof of this fact in the literature and it seems it is not straightforward to prove it starting from the integral formulation (3.10). Finally in Proposition 5.2 we see that \( \tilde{\mathcal{P}}(u) \) satisfies (3.5)–(3.7) also for \( u \in W^{1,p}([0,T]; \mathcal{H}) \), \( p \in [1,\infty] \).

4. THE PLAY OPERATOR WITH REGULAR INPUTS

In this section we recall some well known classical facts about the play operator on spaces of regular functions.

**Problem (P).** Assume \( p \in [1,\infty] \) and (3.3)–(3.4) hold. Given \( z_0 \in \mathcal{Z} \) and \( u \in W^{1,p}([0,T]; \mathcal{H}) \) find \( y \in W^{1,p}([0,T]; \mathcal{H}) \) such that

\[
\begin{align*}
    u(t) - y(t) &\in \mathcal{Z} \quad \forall t \in [0,T], \\
    \langle u(t) - y(t) - z, y'(t) \rangle &\geq 0 \quad \forall z \in \mathcal{Z}, \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \\
    u(0) - y(0) &= z_0.
\end{align*}
\]  

Strictly related to the previous problem is the following

**Problem (S).** Assume \( p \in [1,\infty] \) and (3.3)–(3.4) hold. Given \( z_0 \in \mathcal{Z} \) and \( u \in W^{1,p}([0,T]; \mathcal{H}) \) find \( x \in W^{1,p}([0,T]; \mathcal{H}) \) such that

\[
\begin{align*}
    x(t) &\in \mathcal{Z} \quad \forall t \in [0,T], \\
    \langle x(t) - z, u'(t) - x'(t) \rangle &\geq 0 \quad \forall z \in \mathcal{Z}, \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \\
    x(0) &= z_0.
\end{align*}
\]

Observe that the two problems are related by the formula \( u = x + y \), indeed if \( y \) is a solution of problem (P), then \( x := u - y \) is a solution of problem (S). Viceversa given a solution \( x \) of problem (S), then a solution of the problem (P) is given by \( y := u - x \).

Problem (S) can be immediately solved if set in the framework of the theory of evolution equations governed by a maximal monotone operator. Indeed let \( I_{\mathcal{Z}} : \mathcal{H} \rightarrow [0,\infty] \) denote the indicator function of \( \mathcal{Z} \), defined by

\[
I_{\mathcal{Z}}(x) := \begin{cases} 
0 & \text{if } x \in \mathcal{Z} \\
\infty & \text{if } x \notin \mathcal{Z}.
\end{cases}
\]

Then \( I_{\mathcal{Z}} \) is convex and lower semicontinuous and its subdifferential, given by \( \partial I_{\mathcal{Z}}(x) := \{ y \in \mathcal{H} : \langle x - z, y \rangle \geq 0 \ \forall z \in \mathcal{Z} \} \), \( x \in \mathcal{H} \), defines a multivalued map \( \partial I_{\mathcal{Z}} : \mathcal{H} \rightarrow \mathcal{P}(\mathcal{H}) \) with closed domain \( D(\partial I_{\mathcal{Z}}) := \{ x \in \mathcal{H} : \partial I_{\mathcal{Z}} \neq \emptyset \} = \mathcal{Z} \).

Hence, as \( u' \in L^p([0,T]; \mathcal{H}) \) and \( z_0 \in \mathcal{Z} \), thanks to [3, Proposition 3.4 and Remark...
there exists a unique $x \in W^{1,p}([0,T];H)$ such that
\begin{align*}
x(t) &\in Z \quad \forall t \in [0,T], \\
x'(t) + \partial I_Z(x(t)) &\ni u'(t) \quad \text{for } L^1\text{-a.e. } t \in [0,T], \\
x(0) & = z_0.
\end{align*}

Hence from the definition of subdifferential of the indicator function it follows that $x$ satisfies (4.4)–(4.6). Therefore we can state the following theorem, which in particular yields Proposition 3.2.

**Theorem 4.1.** Both problems (P) and (S) admit a unique solution for every $p \in [1,\infty]$.

The previous theorem allows us to define two solution operators
\begin{align*}
P : & \ W^{1,p}([0,T];H) \rightarrow W^{1,p}([0,T];H), \\
S : & \ W^{1,p}([0,T];H) \rightarrow W^{1,p}([0,T];H)
\end{align*}

associating with every $u \in W^{1,p}([0,T];H)$ the solutions $y$ and $x$ of Problems (P) and (S) respectively. The operator $S$ is usually called *stop operator* and has an important role in many physical applications. The play and stop operators are related by the formula
\[ P(u) + S(u) = u \quad \forall u \in W^{1,p}([0,T];H). \]

5. The play operator with continuous inputs of bounded variation

Let us start with a property of the Riemann-Stieltjes integral.

**Lemma 5.1.** Assume $f \in C(I;H)$, $g \in W^{1,1}(I;H)$ and $h : I \rightarrow I$ is continuous and increasing. Then
\[ \int_a^b \langle f, d(g \circ h) \rangle = \int_a^b \langle f(t), g'(h(t)) \rangle \ dDh(t). \]

**Proof.** Fix $\delta > 0$. If $s = \{t_j\}_{j=0}^m \in \mathcal{G}_\delta(I)$ and if $c = \{c_j\}_{j=1}^m$ is consistent with $s$, then using (2.6) and (2.5) we have that
\[
S(f, g \circ h, s, c) = \sum_{j=1}^m \left( f(c_j), g(h(t_j)) - g(h(t_{j-1})) \right)
= \sum_{j=1}^m \left( f(c_j), \int_{h(t_{j-1})}^{h(t_j)} g'(s) \ ds \right) = \sum_{j=1}^m \left( f(c_j), \int_{t_{j-1}}^{t_j} g'(h(t)) \ dDh(t) \right).
\]
Therefore, if $\omega_f$ is the modulus of continuity of $f$, thanks to (2.3) we get
\[
\left| \int_a^b \langle f(t), g'(h(t)) \rangle \, dDh(t) - S(f, g, h, s, c) \right| \\
= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \langle f(t), g'(h(t)) \rangle \, dDh(t) - \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \langle f(c_j), g'(h(t)) \rangle \, dDh(t) \\
= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \langle f(t) - f(c_j), g'(h(t)) \rangle \, dDh(t) \\
\leq \omega_f(\delta) \int_a^b \|g'(h(t))\|_H \, dDh(t).
\]
We can conclude thanks to the uniform continuity of $f$. $\square$

The previous Lemma and Proposition 2.5 allow us to give the following

**Proof of Theorem 3.1.** Uniqueness is straightforward: we take $z = u - (y_1 + y_2)/2$ where $y_1, y_2$ are two solutions. Hence for every $t \in [0, T]$, using (3.10), (2.2) and (3.11) we find that $0 \geq \int_0^t \langle y_1 - y_2, d(y_1 - y_2) \rangle = \|y_1(t) - y_2(t)\|_H^2/2$. Concerning existence, if we set $y := P(\bar{u}) \circ \ell_u$ then $V(y, [0, T]) = V(P(\bar{u}), [0, T]) < \infty$, hence $y \in BV([0, T] ; H) \cap C([0, T] ; H)$ and the integral in (3.10) makes sense. Condition (3.9) is obvious because $u(t) - y(t) = \bar{u}(\ell_u(t)) - P(\bar{u})(\ell_u(t))$ for every $t \in [0, T]$. The same calculation gives $u(0) - y(0) = \bar{u}(0) - P(\bar{u})(0) = z_0$, hence the initial condition holds. Now take $t \in [0, T]$. Thanks to Lemma 5.1 we have
\[
\int_0^t \langle u(s) - y(s) - z(s), dy(s) \rangle \\
= \int_0^t \langle \bar{u}(\ell_u(s)) - P(\bar{u})(\ell_u(s)) - z(s), d(P(\bar{u}) \circ \ell_u)(s) \rangle \\
= \int_0^t \langle \bar{u}(\ell_u(s)) - P(\bar{u})(\ell_u(s)) - z(s), (P(\bar{u}))'(\ell_u(s)) \rangle \, dD\ell_u(s).
\]
Now let
\[
A = \{ \tau \in [0, \ell_u(t)] : \langle \bar{u}(\tau) - P(\bar{u})(\tau) - z, (P(\bar{u}))'(\tau) \rangle \geq 0 \, \forall z \in Z \}.
\]
Thanks to Proposition 3.2 (or formula (4.8)) we know that $L^1([0, \ell_u(t)] \setminus A) = 0$. If $E = \{ s \in [0, t] : \ell_u(s) \in [0, \ell_u(t)] \setminus A \}$, then in view of Proposition 2.2 we get that
\[
D\ell_u(E) = 0 \text{ therefore }
\]
\[
D\ell_u([0, t]) = D\ell_u([0, t] \setminus E) = D\ell_u(\{ s \in [0, t] : \ell_u(s) \in A \}) \\
\leq D\ell_u(\{ s \in [0, t] : \langle \bar{u}(\ell_u(s)) - P(\bar{u})(\ell_u(s)) - z(s), (P(\bar{u}))'(\ell_u(s)) \rangle \geq 0 \}).
\]
This implies that $\langle \bar{u}(\ell_u(s)) - P(\bar{u})(\ell_u(s)) - z(s), (P(\bar{u}))'(\ell_u(s)) \rangle \geq 0$ for $D\ell_u$-a.e. $s \in [0, t]$. Therefore $\int_0^t \langle u(t) - y(t) - z(t), dy(t) \rangle \geq 0$ and (3.10) is proved. The proof
of the last part of Theorem 3.1 is proved in the following Proposition 5.2 where a more general property is proved. \qed

Now we show that the operator \( \tilde{P} \) is actually rate independent. To this purpose we need to conclude the

**Proof of Proposition 3.1.** If \( u \in W^{1,p}(I; \mathcal{H}) \) then by [3, Corollary A.1, Appendix] we have \( V(u, [a, t]) = \int_a^t \| u'(s) \|_{\mathcal{H}} \, ds \) for every \( t \), hence we get that \( \ell_u \in W^{1,p}(I; \mathbb{R}) \). The equality \( V(u, I) = \tilde{V}(u, I) \) follows from the surjectivity of \( \ell_u \). Now let us observe that the assumptions on \( \phi \) yield \( V(v, [a, t]) = V(u \circ \phi, [a, t]) = V(u, [a, \phi(t)]) \) for every \( t \in I \), therefore \( V(v, I) = V(u, I) \) and

\[
\ell_v(t) = \frac{b-a}{V(v, I)} V(v, [a, t]) = \frac{b-a}{V(u, I)} V(u, [a, \phi(t)]) = (\ell_u \circ \phi)(t) \quad \forall t \in I
\]

Thus we have \( \tilde{v} \circ \ell_v = v = u \circ \phi = \tilde{u} \circ \ell_u \circ \phi = \tilde{u} \circ \ell_v \) and the thesis follows from the uniqueness of \( \tilde{v} \). \( \Box \)

**Proposition 5.1.** For every \( u \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \) and for every increasing surjective function \( \varphi : [0, T] \to [0, T] \) we have that \( u \circ \varphi \in BV([0, T]; \mathcal{H}) \) and

\[
\tilde{P}(u \circ \varphi) = \tilde{P}(u) \circ \varphi.
\]

**Proof.** Set \( v := u \circ \varphi \). Clearly \( u \circ \varphi \in BV([0, T]; \mathcal{H}) \) and from the last statement of Proposition 3.1 we infer that

\[
\tilde{P}(u \circ \varphi) = P(\tilde{u} \circ \varphi) \circ \ell_v = P(\tilde{u}) \circ \ell_u \circ \varphi = \tilde{P}(u) \circ \varphi.
\]

\( \Box \)

We finish with a proof that the extension \( \tilde{P} \) is actually the operator solution of the Problem (P) for \( p \in [1, \infty] \).

**Proposition 5.2.** If \( p \in [1, \infty] \) and \( u \in W^{1,p}([0, T]; \mathcal{H}) \) then \( \tilde{P}(u) = \tilde{P}(\tilde{u}) \circ \ell_u \) is the unique solution of Problem (P).

**Proof.** Conditions (4.1) and (4.3) are checked as in the proof of Theorem 3.1. If \( y := \tilde{P}(u) \), by (2.5)–(2.6) one has that \( y \in W^{1,p}([0, T]; \mathcal{H}) \) and \( y'(t) = \ell_u'(t)(P(\tilde{u}))(\ell_u(t)) \) for \( \mathcal{L}^1 \)-a.e. \( t \). Hence we infer that for every \( z \in \mathcal{Z} \) and for \( \mathcal{L}^1 \)-a.e. \( t \)

\[
\left\langle u(t) - y(t) - z, y'(t) \right\rangle = \left\langle u(t) - P(\tilde{u})(\ell_u(t)) - z, \ell_u'(t)(P(\tilde{u}))(\ell_u(t)) \right\rangle = \ell_u'(t)\left\langle \tilde{u}(\ell_u(t)) - P(\tilde{u})(\ell_u(t)) - z, (P(\tilde{u}))(\ell_u(t)) \right\rangle.
\]

(5.1)

Since \( \ell_u \) is absolutely continuous, it has the “property N”, that is \( \mathcal{L}^1(\ell_u(N)) = 0 \) whenever \( \mathcal{L}^1(N) = 0 \). Hence if we take

\[
A = \{ s \in [0, T] : \left\langle \tilde{u}(s) - P(\tilde{u})(s) - z, (P(\tilde{u}))(s) \right\rangle \geq 0 \ \forall z \in \mathcal{Z} \}
\]
and if $N := [0, T] \setminus A$ we have $L^1(N) = 0$ and therefore $L^1(\ell_u(N)) = 0$, which together with (5.1) implies that the complement of $\{ t \in [0, T] : \langle u(t) - y(t) - z, y'(t) \rangle \geq 0 \ \forall z \in Z \}$ is Lebesgue negligible. Thus (4.2) is proved. □

References


Vincenzo Recupero, Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy. 
Phone: +39 011564 7542; Fax: +39 0115647599. 
E-mail address: vincenzo.recupero@polito.it