GLOBAL SOLUTION TO A PENROSE-FIFE MODEL WITH SPECIAL HEAT FLUX LAW AND MEMORY EFFECTS

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Abstract. This paper deals with a phase-field model of Penrose-Fife type in which the heat flux law takes account of memory effects in the material where the phase transition occurs. Under a quite general setting, which includes a wide and meaningful class of heat fluxes, we prove an existence result for a related initial-boundary value problem. Strengthening the assumptions on the memory term, we also get the uniqueness of the solution and an improved regularity theorem, generalizing a recent result by Colli, Laurençot, and Sprekels.

1. Introduction

This paper deals with the following initial-boundary value problem in the cylinder $Q := \Omega \times (0, T)$, where $T > 0$ and $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \leq 3$, with sufficiently smooth boundary $\Gamma$. We look for two functions $\theta$ and $\chi$ defined over $Q$, satisfying

\begin{align*}
\partial_t (\theta + \lambda(\chi)) - \Delta (\alpha(\theta) + k * \alpha(\theta)) &= g \quad \text{in } Q, \\
\mu \partial_t \chi - \nu \Delta \chi + \beta(\chi) + \sigma'(\chi) &\geq -\frac{\lambda'(\chi)}{\theta} \quad \text{in } Q, \\
-\partial_n (\alpha(\theta) + k * \alpha(\theta)) &= \gamma (\alpha(\theta) + k * \alpha(\theta) - h), \quad \partial_n \chi = 0 \quad \text{on } \Sigma := \Gamma \times (0, T), \\
\theta(\cdot, 0) &= \theta_0, \quad \chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega.
\end{align*}

In (1.1–4) $\partial_t$ and $\Delta$ stand respectively for the time derivative and the Laplacian with respect to the space variables, whereas $\partial_n$ denotes the outer normal derivative. In (1.1) there is a memory term given by the convolution product with respect to time, that is

\begin{equation}
(a * b)(t) := \int_0^t a(s)b(t - s)ds, \quad t \in [0, T], \tag{1.5}
\end{equation}
where \(a\) and \(b\) may also depend on the space variables. Concerning the data, \(\mu, \nu, \) and \(\gamma\) are positive constants, \(\lambda\) and \(\sigma\) are smooth functions, as well as the nonlinear function \(\alpha : (0, \infty) \to \mathbb{R}\). Finally, \(k : [0, T] \to \mathbb{R}\) is an integration kernel and \(\beta\) is a possibly multivalued map which gives place to the inclusion (1.2).

Equations (1.1–2) provide a quite general version of a thermodynamically consistent model of phase-field type for phase transition phenomena with non-conserved order parameter. Such a model, which has been first proposed by Penrose and Fife \[17, 18\], describes, for example, the dynamics of a solid-liquid material or, more generally, of a material exhibiting two different phases, and that is contained in the domain \(\Omega\) during the time interval \([0, T]\). The unknown functions \(\theta\) and \(\chi\) represent respectively the absolute temperature and the order parameter (for example the fraction of one of the phases); the data \(g\) and \(h\) stand for the heat supply and the outer temperature. The functions \(\lambda\) and \(\sigma\) come from the smooth part of the free energy, while the multivalued map \(\beta\) is derived from its non-smooth but convex part (usually \(\beta\) is the inverse of the Heaviside graph).

The term \(-\Delta(\alpha(\theta) + k \ast \alpha(\theta))\) in the equation (1.1) is the divergence of the heat flux, which is given by

\[
q = -\nabla \left( \alpha(\theta) + k \ast \alpha(\theta) \right).
\]

The constitutive law (1.6) is not usual and recently the Penrose-Fife model has been widely studied with different choices for \(q\), among which the most common has been

\[
q = -\nabla \left( \frac{c_1}{\theta} \right),
\]

where \(c_1\) is a positive constant (cf. \[9, 12, 14, 19, 20\]).

The law (1.7) turns out to be satisfactory for low and intermediate temperatures and offers some advantages from the mathematical point of view, but unfortunately it does not look acceptable for high temperatures because it does not provide any coerciveness as \(\theta\) becomes larger and larger. These considerations suggest to replace (1.7) by

\[
q = -\nabla \left( -\frac{c_1}{\theta} + c_2 \theta \right),
\]

for some \(c_2 > 0\). Concerning this case, an existence result is given in \[7\], where, more generally, the heat flux is given by

\[
q = -\nabla \alpha(\theta),
\]

where \(\alpha\) is a nonlinear function chosen in such a way that the system (1.1-2) is still consistent with the second principle of thermodynamics.

In the paper \[7\], the great generality of \(\alpha\) does not allow to achieve uniqueness of the solution; hence, in \[8\], a more particular case is considered and a uniqueness result is proved, always permitting \(\alpha\) to belong to a wide class of nonlinearities that includes (1.8) and other important cases.
In our paper we make the same assumptions on $\alpha$ as in [8], but generalize the problem by using the constitutive law (1.6), i.e. by inserting a memory term in the heat flux.

The first work in which the Penrose-Fife model is coupled with memory effects is [10], where the abovementioned inconveniences given by (1.7) are overcome by using the law $q = -\nabla (-c_1/\theta + k * \theta)$. Here, instead, we choose another approach generalizing directly (1.9) and allowing the occurrence of memory effects in the phase transition. For a justification of (1.5) and for other related works where memory effects are concerned, we refer to [5, 6, 10, 11] and the references therein.

In our paper we are going to prove the existence of a weak solution to (1.1–4) under a very general setting. In particular, no convexity property is assumed for $\lambda$ and the kernel $k$ need not to be differentiable, but is supposed to be only integrable. Note that both these hypotheses of convexity and regularity are assumed in [10]. In the present framework, the proof is not so straightforward, mainly because it is impossible to derive any spatial regularity for $\alpha(\theta)$ in order to handle the convolution product. Instead, the differentiability of the kernel is required in order to prove the uniqueness of the solution and a better regularity, thereby generalizing the result of [8].

**Acknowledgments.** The author wishes to thank Professor Pierluigi Colli for proposing him the problem and for useful discussions and valuable suggestions.

## 2. Main results

In this section we are going to introduce the assumptions on the data, give the variational formulation of the problem (1.1–4), and then state our main results.

Recalling the physical settings outlined in the Introduction, we first specify the following related assumptions:

(H1) $\beta : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a maximal monotone operator such that $0 \in \beta(0)$ and $D(\beta)$ denotes its domain. Hence, there exists a proper, convex, and lower semicontinuous function $\hat{\beta} : \mathbb{R} \to [0, \infty]$ such that $\beta = \partial \hat{\beta}$ (the subdifferential of $\hat{\beta}$) and $\hat{\beta}(0) = 0$.

(H2) $\lambda, \sigma \in C^2(\mathbb{R})$, $\lambda', \lambda'', \sigma'' \in L^\infty(\mathbb{R})$.

(H3) $\alpha \in C^1(0, \infty)$ such that

\begin{equation}
\alpha \text{ is concave};
\end{equation}

there exist a function $\ell \in C^1(0, \infty)$ and two constants $d_0, d_\infty \in (0, \infty)$ such that

\begin{equation}
\ell' \in L^\infty(0, \infty),
\end{equation}

\begin{equation}
-\frac{1}{r} = d_0 \alpha(r) + \ell(r) \quad \forall \ r \in (0, \infty),
\end{equation}
\[
\lim_{r \to \infty} d_{\infty} \alpha'(r) = 1. \tag{2.4}
\]

(H4) \( k \in L^1(0, T) \).

(H5) \( g \in L^2(0, T; L^2(\Omega)), \ h \in L^2(0, T; L^2(\Gamma)) \).

(H6) \( \theta_0 \in L^2(\Omega), \ \theta_0 > 0 \) a.e. in \( \Omega \), \( u_0 := \alpha(\theta_0) \in L^2(\Omega) \).

(H7) \( \chi_0 \in H^1(\Omega), \ \hat{\beta}(\chi_0) \in L^1(\Omega) \).

Let us remark that (2.1) and (2.4) yield
\[
\alpha' \geq d_{\infty}^{-1} > 0. \tag{2.5}
\]
Moreover, (2.3) implies that
\[
\lim_{r \to 0^+} r^2 \alpha'(r) = d_0^{-1}. \tag{2.6}
\]
Therefore, we can deduce from (2.6) and (2.4) that \( \lim_{r \to 0^+} \alpha(r) = -\infty \) and \( \lim_{r \to \infty} \alpha(r) = \infty \).
So, we get that \( \alpha \) is invertible and
\[
\rho := \alpha^{-1} : \mathbb{R} \to (0, \infty) \text{ is increasing and Lipschitz continuous,} \tag{2.7}
\]
because (2.5) gives \( \rho' \leq d_{\infty} \).
It will be useful to consider the function defined by
\[
\hat{\alpha}(r) := \int_{r_0}^{r} \alpha(s)ds, \quad r \in (0, \infty),
\]
where \( r_0 > 0 \) is such that \( \alpha(r_0) = 0 \). Let us note that
\[
0 \leq \hat{\alpha}(r) \leq |\alpha(r)||r - r_0| \quad \forall \ r \in (0, \infty). \tag{2.8}
\]

Concerning the notation, we set \( H := L^2(\Omega) \) and \( V := H^1(\Omega) \) (every space we deal with is real). The usual inner product in \( H \) will be denoted by \( (\cdot, \cdot) \). We will use the same symbol for a function \( v \in V \) and for its trace on \( \Gamma \). For the sake of convenience, \( V \) will be endowed with the inner product \( ((\cdot, \cdot)) \), defined by
\[
((v_1, v_2)) := \int_{\Omega} \nabla v_1 \cdot \nabla v_2 + \gamma \int_{\Gamma} v_1 v_2, \quad v_1, v_2 \in V,
\]
where \( \gamma \) is the positive constant appearing in the first boundary condition of (1.3).

Let us indicate by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( V' \) and \( V \). Hence, the Riesz isomorphism \( J : V \to V' \) and the inner product \( ((\cdot, \cdot))_* \) in \( V' \) are given by
\[
\langle Jv_1, v_2 \rangle = ((v_1, v_2)), \quad v_1, v_2 \in V,
\]
\[(w_1, w_2)_* = \langle w_1, J^{-1}w_2 \rangle, \ w_1, w_2 \in V'.\]

Let us observe that the norm \(\| \cdot \|_V\) in \(V\) related to the inner product defined above is equivalent to the usual norm in \(V\); i.e. there exist two constant \(C_\Omega, C'_\Omega > 0\) such that, if \(| \cdot |_V\) is the usual norm in \(H^1(\Omega)\), we have

\[C'_\Omega |v|_V \leq \|v\|_V \leq C_\Omega |v|_V \quad \forall \ v \in V.\]

Similar considerations hold also for \(V'\). In general, if \(X\) and \(Y\) are two normed spaces, the notation \(X \hookrightarrow Y\) means that \(X\) is contained in \(Y\) with continuous injection, whereas \(X \hookrightarrow\hookrightarrow Y\) indicates that this injection is also compact. If we identify \(H\) with \(H'\), we get in this way that \(V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V'\) with dense inclusions, whence it follows that

\[\langle v_1, v_2 \rangle = (v_1, v_2) \quad \forall \ v_1 \in H, \ \forall \ v_2 \in V.\]

We will also exploit the operator \(A \in L(V, V')\) defined by

\[\langle Av_1, v_2 \rangle := \int_\Omega \nabla v_1 \cdot \nabla v_2, \ v_1, v_2 \in V.\]

Finally let us set

\[L := \max\{\|\lambda'\|_{L^\infty(\mathbb{R})}, \|\lambda''\|_{L^\infty(\mathbb{R})}, \|\sigma''\|_{L^\infty(\mathbb{R})}, \|\rho'\|_{L^\infty(\mathbb{R})}, \|\ell'\|_{L^\infty(0, \infty)}\}\]

and let \(f \in L^2(0, T; V')\) be defined by

\[\langle f(t), v \rangle := \int_\Omega g(t)v + \gamma \int_\Gamma h(t)v, \ v \in V, \text{ for a.a. } t \in (0, T).\]

**Remark.** Thanks to (2.3), if we set \(u := \alpha(\theta)\), it is possible to write the right hand side of (1.2) in the form \(\lambda(\chi)(d_0u + \ell(\rho(u)))\). Indeed, the role played by (2.3) is fundamental in view of the resolution of problem (1.1–4), and in the following variational formulation it is convenient to write the equations in terms of the unknown \(u\) rather than in terms of the unknown \(\theta\).

**Problem (P).** Find a couple of functions \((\theta, \chi)\) satisfying the following conditions:

\[\theta \in H^1(0, T; V') \cap L^2(0, T; V), \ \theta > 0 \ a.e. \ in \ Q, \quad (2.9)\]

\[u := \alpha(\theta) \in L^2(0, T; V), \ k \ast u \in L^2(0, T; V), \quad (2.10)\]

\[\chi \in H^1(0, T; H) \cap L^2(0, T; H^2(\Omega)), \ \chi \in D(\beta) \ a.e. \ in \ Q, \quad (2.11)\]

\[\exists \xi \in L^2(0, T; H), \ \xi \in \beta(\chi) \ a.e. \ in \ Q, \quad (2.12)\]

\[(\rho(u) + \lambda(\chi))' + Ju + J(k \ast u) = f \ in \ V', \ a.e. \ in \ (0, T), \quad (2.13)\]
\[ \mu \chi' + \nu A \chi + \xi + \sigma' (\chi) = \lambda' (\chi) (d_0 u + \ell (\rho (u))) \quad \text{in} \ V', \ a.e. \ in \ (0, T), \quad (2.14) \]

\[ \theta (0) = \theta_0, \ \chi (0) = \chi_0. \quad (2.15) \]

The main result in this paper is the following

**Theorem 2.1.** Suppose that (H1)–(H7) are satisfied. Then Problem (P) admits at least one solution.

Concerning the uniqueness of solution, we have the following result:

**Theorem 2.2.** Suppose that (H1)–(H7) are satisfied. Assume in addition that

(H8) \( k \in W^{1,1} (0, T) \),

(H9) \( f \in W^{1,1} (0, T; V') \),

(H10) \( u_0 \in V \),

(H11) \( \chi_0 \in H^2(\Omega) \), \( \partial_n \chi_0 = 0 \) a.e. in \( \Gamma \), \( \chi_0 \in D(\beta) \) a.e. in \( \Omega \), \( \beta^o (\chi_0) \in H \),

where for each \( r \in D(\beta) \), \( \beta^o (r) \) denotes the element of \( \beta (r) \) having minimum modulus. Then, Problem (P) has a unique solution \((\theta, u)\). Moreover, this solution satisfies the following conditions:

\[ \theta \in W^{1,\infty} (0, T; V'), \quad (2.16) \]

\[ u := \alpha (\theta) \in L^\infty (0, T; V), \quad (2.17) \]

\[ \chi \in W^{1,\infty} (0, T; H) \cap H^1 (0, T; V) \cap L^\infty (0, T; H^2 (\Omega)). \quad (2.18) \]

**Remark.** Note that (H9) is satisfied if \( g \in W^{1,1} (0, T; H) \) and \( h \in W^{1,1} (0, T; L^2 (\Gamma)) \) and that (H10) yields that \( \theta_0 = \rho (u_0) \in V \) because \( \rho \) is Lipschitz continuous.

The existence of a solution to problem (P) is shown via an approximation method. We regularize the data \( k \in L^1 (0, T) \), \( f \in L^2 (0, T; V') \), \( u_0 \in L^2 (\Omega) \), \( \chi_0 \in H^1 (\Omega) \) respectively with families of functions \((k_\varepsilon)_\varepsilon\), \((f_\varepsilon)_\varepsilon\), \((u_0_\varepsilon)_\varepsilon\), \((\chi_0_\varepsilon)_\varepsilon\) such that

\[ k_\varepsilon \in W^{1,1}_0 (0, T) \forall \varepsilon > 0, \quad (2.19) \]

\[ f_\varepsilon \in W^{1,1} (0, T; V') \forall \varepsilon > 0, \quad (2.20) \]

\[ u_0_\varepsilon \in V \forall \varepsilon > 0, \quad (2.21) \]

\[ \chi_0_\varepsilon \in H^2 (\Omega) \], \( \partial_n \chi_0_\varepsilon = 0 \) a.e. in \( \Gamma \), \( \chi_0_\varepsilon \in D(\beta) \) a.e. in \( \Omega \), \( \beta^o (\chi_0_\varepsilon) \in H \forall \varepsilon > 0, \)

\[ (\| \beta (\chi_0_\varepsilon) \|_{L^1 (\Omega)})_\varepsilon \text{ is bounded}, \quad (2.22) \]
Problem (P). Let us note that the proof of the existence of a family $(\chi_{0\varepsilon})_{\varepsilon}$ satisfying (2.22) and (2.24) is not trivial and is postponed to the Appendix. To be more precise, we can state our regularized problem as

**Problem (P).** Find a couple of functions $(\theta, \chi)$ satisfying the following relations.

$$\theta \in W^{1,\infty}(0,T;V') \cap C([0,T];H), \quad \theta > 0 \text{ a.e. in } Q,$$

$$u := \alpha(\theta) \in L^\infty(0,T;V), \quad k_\varepsilon \ast u \in L^2(0,T;V), \quad \chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \quad \chi \in D(\beta) \text{ a.e. in } Q,$$

$$\exists \, \xi \in L^\infty(0,T;H), \quad \xi \in \beta(\chi) \text{ a.e. in } Q,$$

$$\rho(u) + \lambda(\chi) + Ju + J(k_\varepsilon \ast u) = f_\varepsilon \text{ in } V', \quad \text{a.e. in } (0,T),$$

$$\mu \chi' + \nu A\chi + \xi + \sigma'(\chi) = \lambda'(\chi) (d_0 u + \ell(\rho(u))) \text{ in } V', \quad \text{a.e. in } (0,T),$$

$$\theta(0) = \theta_{0\varepsilon}, \quad \chi(0) = \chi_{0\varepsilon}.$$  

Provided that we are able to solve (P), a solution to Problem (P) is then obtained via a passage to the limit as $\varepsilon \to 0^+$. In fact, the following theorem holds:

**Theorem 2.3.** Suppose (H1)–(H3) hold and assume that (2.19–22) are satisfied. Then, Problem (P) has a unique solution.

In the proof of Theorem 2.3 it will be useful to consider an auxiliary problem, that is the following one.

**Problem (P^a).** Let $F \in W^{1,1}(0,T;V')$. Find a couple of functions $(\theta, \chi)$ satisfying the conditions

$$\theta \in W^{1,\infty}(0,T;V') \cap C([0,T];H), \quad \theta > 0 \text{ a.e. in } Q,$$

$$u := \alpha(\theta) \in L^\infty(0,T;V),$$

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \quad \chi \in D(\beta) \text{ a.e. in } Q,$$

$$\exists \, \xi \in L^\infty(0,T;H), \quad \xi \in \beta(\chi) \text{ a.e. in } Q,$$

$$(\rho(u) + \lambda(\chi))' + Ju = F \text{ in } V', \quad \text{a.e. in } (0,T),$$

$$\mu \chi' + \nu A\chi + \xi + \sigma'(\chi) = \lambda'(\chi) (d_0 u + \ell(\rho(u))) \text{ in } V', \quad \text{a.e. in } (0,T),$$

$$k_\varepsilon \to k \text{ in } L^1(0,T), \quad f_\varepsilon \to f \text{ in } L^2(0,T;V'), \quad u_{0\varepsilon} \to u_0 \text{ in } H \text{ as } \varepsilon \to 0^+$$  

$$\chi_{0\varepsilon} \to \chi_0 \text{ in } V \text{ as } \varepsilon \to 0^+.$$
\[ \theta(0) = \theta_0, \quad \chi(0) = \chi_0. \] (2.38)

To conclude this section, let us recall some formulas concerning the convolution product which hold whenever they make sense. Precisely, we need the identities

\[ a \ast b = a(0)(1 \ast b) + a' \ast (1 \ast b), \] (2.39)

\[ (a \ast b)' = a(0)b + a' \ast b \] (2.40)

and the Young theorem

\[ \|a \ast b\|_{L^r(0,T;X)} \leq \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)}, \]

with \(1 \leq p, q, r \leq \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,\) (2.41)

where \(X\) is a normed space. We will also need the elementary Young inequality

\[ ab \leq \frac{\delta}{2} a^2 + \frac{1}{2\delta} b^2 \quad \forall \ a, b \in \mathbb{R}, \ \forall \ \delta > 0. \] (2.42)

The symbols \(C_i, \hat{C}_i, \) where \(i \in \mathbb{N},\) will denote constants which do not depend on \(\varepsilon,\) on \(t \in (0, T)\) (but may depend on \(T\)), and on any norm of \(F.\)

3. Analysis of Problem \((P^a)\)

In this section we analyze the Problem \((P^a).\) We start by recalling the main result of [8].

**Proposition 3.1.** Assume that \((H1)-(H3), \ (H6)-(H7), \ (H10) \) and \((H11)\) hold. Then Problem \((P^a)\) has a unique solution.

**Proof.** This is in fact the main result of [8]. In that paper the only difference with our Problem \((P^a)\) is that the datum \(F\) is replaced by \(g + h\) with \(g \in W^{1,1}(0, T; H)\) and \(h \in W^{1,1}(0, T; L^2(\Gamma)).\) But it is easy to see that few changes are needed in order to adapt the argument to our case. \(\blacksquare\)

Now, let be \(\omega \in [0, 1] \) a number we will fix later. We introduce the notation

\[ \varphi^\omega(t) := e^{-\omega t} v(t) \quad \forall \ t \in [0, T], \]

which holds for functions \(v : [0, T] \to X,\) where \(X\) is a normed space.
Lemma 3.2. Let \((\theta, \chi)\) be the solution to Problem \((P^a)\) and let \(u\) satisfy (2.33). Then there exists a constant \(C_1 > 0\) such that

\[
\|\overline{\theta}\|_{L^2(0,t;V)}^2 + \|\overline{\chi}\|_{L^2(0,t;H)}^2 + \int_0^t |\nabla \overline{\chi}(t)|^2 + \|\overline{\theta'}(t)\|^2_H \leq C_1 \left(1 + \|\overline{F}\|_{L^2(0,t;V')}^2\right) \quad \forall t \in [0,T]. \tag{3.1}
\]

Proof. Let us start by applying (2.36), evaluated at the time \(s \in (0,t)\), to the function \(e^{-2\omega s} u(s) \in V\). Then, we integrate from 0 to \(t \in [0,T]\) and multiply the result by \(d_0\) (in symbols, \(d_0 \int_0^t (\rho(u)\)'(s), e^{-2\omega s} u(s))ds\)). This procedure yields

\[
d_0 \int_0^t (\rho(u)\)'(s), e^{-2\omega s} u(s))ds + d_0 \int_0^t (\lambda(\chi))'(s)e^{-2\omega s} u(s)ds + d_0 \|\overline{\theta}\|_{L^2(0,t;V)}^2 \]

\[= d_0 \int_0^t \langle F(s), e^{-2\omega s} u(s) \rangle ds. \]

Exploiting a regularization method before integrating by parts, and noting that \(\tilde{\alpha}(\theta_0) \in L^1(\Omega)\), thanks to (2.8), we are led to the identity

\[
\int_0^t (\rho(u)\)'(s), e^{-2\omega s} u(s))ds = \int_\Omega (\tilde{\alpha}(\theta(t)) e^{-2\omega t} - \int_\Omega (\tilde{\alpha}(\theta_0) + \int_0^t 2\omega \tilde{\alpha}(\theta(s)) e^{-2\omega s} ds.
\]

Recalling then the Hölder and Young inequalities and the fact that \(\tilde{\alpha} \geq 0\), we get

\[
\frac{d_0}{2} \|\overline{\theta}\|_{L^2(0,t;V)}^2 + d_0 \int_0^t (\lambda(\chi))'(s)e^{-2\omega s} u(s)ds \leq d_0 \|\tilde{\alpha}(\theta_0)\|_{L^1(\Omega)} + \frac{d_0}{2} \|\overline{F}\|_{L^2(0,t;V')}. \tag{3.2}
\]

Now let \(\xi\) be the function appearing in (2.35) e (2.37). Let us apply (2.37), evaluated at the time \(s \in (0,t)\), to the function \(e^{-2\omega s} \chi'(s) \in V\) and integrate from 0 to \(t\) (i.e. we perform \(\int_0^t ((\chi)'(s), e^{-2\omega s} \chi'(s))ds\)). We get

\[
\mu \|\overline{\chi}\|_{L^2(0,t;H)}^2 + \nu \int_0^t \int_\Omega e^{-2\omega s} \nabla \chi(s) \cdot \nabla \chi'(s)ds + \int_0^t \int_\Omega (\xi(s)) e^{-2\omega s} \chi'(s)ds \\
= -\int_0^t \int_\Omega \sigma'(\chi(s)) e^{-2\omega s} \chi'(s)ds + d_0 \int_0^t \int_\Omega (\chi'(s))^2 e^{-2\omega s} u(s)ds \\
+ \int_0^t \int_\Omega (\lambda(\chi))'(s)e^{-2\omega s} \ell(\rho(u(s)))ds. \tag{3.3}
\]

Recalling (2.35) and (2.38) and adapting the proof of [4, Lemma 3.3, p. 73], one can check that

\[
\int_0^t \int_\Omega (\xi(s)) e^{-2\omega s} \chi'(s)ds = \int_\Omega e^{-2\omega t} \overline{\beta}(\chi(t)) - \int_0^t \overline{\beta}(\chi(t)) + 2\omega \int_0^t \int_\Omega e^{-2\omega s} \beta'(\chi(s))ds. \tag{3.4}
\]
Thanks to (H2) we deduce that
\[- \int_0^t \int_\Omega \sigma'(\chi(s))e^{-\omega s}\chi'(s)ds \leq \int_0^t \int_\Omega |\nabla\chi|^2(s)e^{-\omega s}(L|\chi(s)| + |\sigma'(0)|)ds \]
\[\leq \frac{\mu}{4} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{2}{\mu} \int_0^t \int_\Omega e^{-\omega s}(L^2|\chi(s)|^2 + |\sigma'(0)|^2)ds \]
\[\leq \frac{\mu}{4} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{2}{\mu} \left( L^2 \int_0^t \int_\Omega e^{-\omega s}|\chi(s)|^2_H ds + T|\Omega| |\sigma'(0)|^2 \right) \]
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. Now, since we have $\|\chi(s)\|_H \leq \|\chi|_H + \int_0^s \|\chi'(\tau)\|_H d\tau$, taking the squares and exploiting the Hölder inequality, it is not hard to get
\[- \int_0^t \int_\Omega \sigma'(\chi(s))e^{-\omega s}\chi'(s)ds \]
\[\leq \frac{\mu}{4} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{4}{\mu} \left( L^2 T\|\chi|_H^2 + L^2 \int_0^t \int_\Omega e^{-\omega \tau}|\chi'(\tau)|^2_H d\tau ds + T|\Omega| |\sigma'(0)|^2 \right) \]
\[\leq \frac{\mu}{4} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{4T}{\mu} \left( L^2 \|\chi|_H^2 + L^2 \int_0^t \|\nabla\chi\|_{L^2(\Omega,s;H)}^2 ds + |\Omega| |\sigma'(0)|^2 \right). \tag{3.5} \]

From (H2) we also deduce that
\[\int_0^t \int_\Omega (\lambda(s))^\prime e^{-\omega s}\ell(\rho(u(s)))ds \leq \int_0^t \int_\Omega L|\nabla\chi|e^{-\omega s}|\ell(\rho(u(s)))|ds \]
\[\leq \frac{\mu}{4} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{2L^4}{\mu} \int_0^t \|\rho(u)\|_{H}^2 ds + \frac{2L^2 \|\ell(0)\|^2 T|\Omega|}{\mu}. \tag{3.6} \]

Collecting (3.3–6), thanks to (H7) and to the fact that $\hat{\beta} \geq 0$, we find that there exists a constant $\hat{C}_1 > 0$ such that
\[\frac{\mu}{2} \|\nabla\chi\|_{L^2(\Omega,t;H)}^2 + \frac{\nu}{2} \int_0^t \text{div}(\nabla\chi)^2(t) + \nu \omega \int_0^t \int_\Omega |\nabla\chi|^2(s)^2 ds \]
\[\leq \hat{C}_1 + d_0 \int_0^t \int_\Omega (\lambda(s))^\prime e^{-\omega s}u(s)ds \]
\[+ \frac{2L^2}{\mu} \left( 2T \int_0^t \|\nabla\chi\|_{L^2(\Omega,s;H)}^2 ds + L^2 \int_0^t \|\rho(u)\|_{H}^2 ds \right). \tag{3.7} \]

Since $\rho$ is Lipschitz continuous, we have that $e^{-\omega s}(\rho(u(s)) - r_0) \in V$, whence we can apply (2.36) evaluated at the time $s \in (0,T)$ to $e^{-\omega s}(\rho(u(s)) - r_0)$ and then integrate from $0$ to $t \in [0,T]$ (in symbols $\int_0^t \langle (2.36)(s), e^{-\omega s}(\rho(u(s)) - r_0) \rangle ds$; note that $\rho(0) = r_0$ and $\rho(u) = \theta \in H^1(0,T;V^r) \cap L^2(0,T;V)$). We are led to the equality
\[\int_0^t \langle (\rho(u))^\prime(s), e^{-\omega s}(\rho(u(s)) - r_0) \rangle ds + \int_0^t \int_\Omega (\lambda(s))^\prime e^{-\omega s}(\rho(u(s)) - r_0)ds \]
\[+ \int_0^t \int_\Omega e^{-\omega s} \rho'(u(s))|\nabla u(s)|^2 ds + \gamma \int_0^t \int_\Gamma e^{-\omega s}(\rho(u(s)) - r_0)u(s)ds \]
\[= \int_0^t \langle F(s), e^{-\omega s}(\rho(u(s)) - r_0) \rangle ds. \]
Integrating by parts in the first term, exploiting the Hölder inequality and observing that the function $\rho - r_0$ is increasing and null in $0$, with the help of (H6), we find

$$
\frac{1}{2} \| \rho(u)^\omega(t) \|^2_H + \omega \| \rho(u)^\omega \|^2_{L^2(0,t;H)} \leq \frac{1}{2} \| \theta_0 \|^2_H + r_0 \| \theta_0 \|_{L^1(\Omega)}
$$

$$
- \int_0^t \int_\Omega (\lambda(\chi))'(s) e^{-2\omega s}(\rho(u(s)) - r_0) \, ds + r_0 \int_\Omega e^{-2\omega t}\rho(u(t)) + 2\omega r_0 \int_0^t e^{-2\omega s}\rho(u(s)) \, ds + \| \bar{F}^\omega \|_{L^2(0,t;V')} \| \rho(u)^\omega \|_{L^2(0,t;V)} + \| \bar{F}^\omega \|_{L^2(0,t;V')} \| \tau_0^\omega \|_{L^2(0,t;V)}. \tag{3.8}
$$

The assumption (H2) allows us to recover the first of the following three inequalities:

$$
- \int_0^t \int_\Omega (\lambda(\chi))'(s) e^{-2\omega s}(\rho(u(s)) - r_0) \, ds \leq \int_0^t \int_\Omega L \chi^\omega(s) \| \rho(u)^\omega(s) - \tau_0^\omega \| ds
$$

$$
\leq \frac{\mu}{4} \| \chi^\omega \|^2_{L^2(0,t;H)} + \frac{2L^2}{\mu} \int_0^t \| \rho(u)^\omega(s) \|^2_H + \frac{2L^2T|\Omega|r_0^2}{\mu}; \tag{3.9}
$$

$$
\int_\Omega e^{-2\omega t}\rho(u(t)) = \int_\Omega e^{-2\omega t}\rho(u)^\omega(t) \leq \| \rho(u)^\omega(t) \|_{L^1(\Omega)} \leq \frac{1}{4r_0} \| \rho(u)^\omega(t) \|^2_H + r_0|\Omega|; \tag{3.10}
$$

$$
2\omega \int_\Omega \int_0^t e^{-2\omega s}\rho(u(s)) \, ds \leq 2\omega \int_0^t \int_\Omega |\rho(u)^\omega(s)| \, ds \leq \frac{1}{2r_0} \int_0^t \| \rho(u)^\omega(s) \|^2_H ds + 2\omega^2 r_0|\Omega|T. \tag{3.11}
$$

Let us now remark that the Lipschitz continuity of $\rho$ yields

$$
\| \rho(u(s)) \|^2_V = \int_\Omega |\nabla \rho(u(s))|^2 + \gamma \int_\Gamma |\rho(u(s))|^2
$$

$$
\leq L^2 \int_\Omega |\nabla u(s)|^2 + 2L^2 \gamma \int_\Gamma |u(s)|^2 + 2\gamma \int_\Gamma r_0^2
$$

$$
\leq 2L^2 \| u(s) \|^2_V + 2\gamma \| r_0 \|^2_{L^2(\Gamma)} \text{ for a.a. } s \in (0, T), \tag{3.12}
$$

and so

$$
\| \bar{F}^\omega \|_{L^2(0,t;V')} \| \rho(u)^\omega \|_{L^2(0,t;V)} \leq \frac{d_0}{8L^2} \| \rho(u)^\omega \|^2_{L^2(0,t;V)} + \frac{2L^2}{d_0} \| \bar{F}^\omega \|^2_{L^2(0,t;V')} \tag{3.13}
$$

Inequality (3.8), together with the estimates (3.9–13), allows us to deduce the existence of a constant $\tilde{C}_2 > 0$ such that

$$
\frac{1}{4} \| \rho(u)^\omega(t) \|^2_H + \omega \| \rho(u)^\omega \|^2_{L^2(0,t;H)} \leq \frac{\mu}{4} \| \chi^\omega \|^2_{L^2(0,t;H)}
$$

$$
+ \frac{d_0}{4} \| \tau^\omega \|^2_{L^2(0,t;V)} + \tilde{C}_2 \left( 1 + \int_0^t \| \rho(u)^\omega(s) \|^2_H ds + \| \bar{F}^\omega \|^2_{L^2(0,t;V')} \right). \tag{3.14}
$$
Let us sum (3.2), (3.7), and (3.14), and notice that there is a cancellation. Therefore, recalling that \( \theta = \rho(u) \), one finds a constant \( \tilde{C}_3 > 0 \) such that

\[
\frac{d_0}{4} \| \theta \|_{L^2(0,t;V)}^2 + \frac{\mu}{4} \| \theta' \|_{L^2(0,t;H)}^2 + \frac{\nu}{2} \int_{\Omega} | \nabla \theta|^2(t)^2 \\
+ \nu \omega \int_0^t \int_{\Omega} | \nabla \theta(s)|^2 ds + \frac{1}{4} \| \theta' \|_H^2 + \omega \| \theta' \|_{L^2(0,t;H)}^2 \\
\leq \tilde{C}_3 \left( 1 + \| F \|_{L^2(0,t;V')}^2 \right) \forall t \in [0,T].
\]

(3.15)

**Proof.** First of all, let us note that (3.1) entails, taking \( \omega = 0 \),

\[
\| u \|_{L^2(0,t;V)}^2 + \| \chi \|_{H^1(0,t;H)}^2 + \int_{\Omega} \| \nabla \chi(t) \|^2 + \| \theta(t) \|_H^2 \\
\leq C_1 \left( 1 + \| F \|_{L^2(0,t;V')}^2 \right) \forall t \in [0,T].
\]

(3.16)

Moreover, from (2.36) we derive that

\[
\| \theta' \|_{L^2(0,t;V')} \leq L \| \chi \|_{L^2(0,t;H)}^2 + \| u \|_{L^2(0,t;V)} + \| F \|_{L^2(0,t;V')}.
\]

(3.17)

On the other hand, arguing exactly as in [10, Lemma 3.3] (cf. in particular formulas (3.24) and (3.25) of that paper), thanks to (H1) and (H7) one finds a constant \( \tilde{C}_4 > 0 \) such that

\[
\| \chi \|_{L^2(0,t;H^2(\Omega))}^2 + \| \xi \|_{L^2(0,t;H)}^2 \leq \tilde{C}_4 \left( 1 + || - \sigma'(\chi) + \lambda'(\chi)(d_0 u + \ell(\rho(u)))\|_{L^2(0,t;H)}^2 \right)
\]

for all \( t \in [0,T] \). Since we have

\[
\| \sigma'(\chi) \|_{L^2(0,t;H)}^2 \leq 2L^2 \| \chi \|_{L^2(0,t;H)}^2 + 2T|\Omega|\| \sigma'(0) \|^2
\]

and

\[
\| \lambda'(\chi)(d_0 u + \ell(\rho(u)))\|_{L^2(0,t;H)}^2 \\
\leq 2L^2 d_0^2 \| u \|_{L^2(0,t;H)}^2 + 2L^2 \int_0^t \int_{\Omega} \| \ell(\rho(u(s))) \|^2 d s
\]

\[
\leq 2L^2 d_0^2 \| u \|_{L^2(0,t;H)}^2 + 8L^2 \| u \|_{L^2(0,t;H)}^2 + 4L^2 |\Omega|T(2\rho_0^2 + |\ell(0)|^2),
\]

Lemma 3.3 Let \((\theta, \chi)\) solve Problem \((\text{P}^a)\). Let \( u \) be defined by (2.33) and \( \xi \) satisfying (2.35) and (2.37). Then there exists a constant \( C_2 > 0 \) such that

\[
\| u \|_{L^2(0,t;V)}^2 + \| \chi \|_{H^1(0,t;H)}^2 \leq C_2 \left( 1 + \| F \|_{L^2(0,t;V')}^2 \right) \forall t \in [0,T].
\]
we infer that there exists a constant $\hat{C}_5 > 0$ such that
\[ \|\chi\|_{L^2(0, t; H^2(\Omega))}^2 + \|\xi\|_{L^2(0, t; H)}^2 \leq \hat{C}_5 \left( 1 + \|\chi\|_{L^2(0, t; H)}^2 + \|u\|_{L^2(0, t; H)}^2 \right). \] (3.18)

Finally, recalling (3.12), we can easily deduce (3.15) from (3.16–18).

**Lemma 3.4** Let $(\theta, \chi)$ be the solution to Problem (P$a$) and let $u$ be defined by (2.33). Then there exists a constant $C_3 > 0$ such that
\[
\frac{d_0}{2} \int_0^t \int_{\Omega} \rho'(u(s))|u'(s)|^2 ds + \frac{d_0}{2} \|u\|_{L^\infty(0, t; V)}^2 + \frac{\mu}{2} \|\chi\|_{L^\infty(0, t; H)}^2 + \frac{\nu}{2} \int_0^t \int_{\Omega} |\nabla \chi'(s)|^2 ds
\leq C_3 e^{C_3 \left( 1 + \|F\|_{L^\infty(0, t; V')}^2 \right)} \left( 1 + \|F\|_{W^{1, 1}(0, t; V')}^2 \right) \quad \forall \ t \in (0, T].
\] (3.19)

**Proof.** In order to obtain the estimate (3.19) we proceed by performing formally the following two calculations:
(i) we multiply (2.36) at the time $s \in (0, t)$, $0 \leq t \leq T$, by $d_0$ and then we apply it to the function $u'(s)$;
(ii) we take the time derivative of (2.37) at the time $s$ and we apply it to the function $\chi'(s)$.

Finally we sum the results of (i) and (ii) and then integrate from 0 to $t$ (in short: $d_0 \int_0^t \langle (2.36)(s), u'(s) \rangle ds + \int_0^t \langle (2.37)'(s), \chi'(s) \rangle ds$). As we pointed out before, these calculations are just formal; indeed, $u$, $\chi'$, and $\xi$ need not to be differentiable. Nonetheless, everything can be made rigorous by suitably regularizing our problem: a possible approximation is given in [8, Section 3], where a similar estimate is performed. Anyway, for the sake of clarity, we proceed with our calculations. Hence, in view of (H10) and (H11), we infer the existence of a constant $\hat{C}_6 > 0$ such that (cf. [8, Section 3, formula (32)])
\[
d_0 \int_0^t \int_{\Omega} \rho'(u(s))|u'(s)|^2 ds + \frac{d_0}{2} \|u(t)\|_{V}^2 + \frac{\mu}{2} \|\chi'(t)\|_{H}^2 + \nu \int_0^t \int_{\Omega} |\nabla \chi'(s)|^2 ds
\leq \hat{C}_6 \left( 1 + \|\chi\|_{L^2(0, t; H)}^2 + \|F(t)\|_{V'}^2, + \int_0^t \|F'(s)\|_{V'} \|u(s)\|_{V} ds
\]
\[
+ \int_0^t \int_{\Omega} |u(s)||\chi'(s)|^2 ds + \int_0^t \int_{\Omega} \rho'(u(s))|u'(s)||\chi'(s)| ds \right).
\] (3.20)

As $N \leq 3$, we have that $V \hookrightarrow \hookrightarrow L^4(\Omega)$; hence, we can exploit formula (16.16) in [16, p. 102] and find two constants $\hat{C}_7, \hat{C}_8 > 0$ such that
\[
\hat{C}_6 \int_0^t \int_{\Omega} |u(s)||\chi'(s)|^2 ds
\]
Now (3.19) is an easy consequence of the last inequality.

\[
\frac{1}{2} \int_0^t \left( \nu \int_\Omega |\nabla \chi'(s)|^2 + \tilde{C}_8 \|\chi'(s)\|_H^2 \right) ds
\]

Thanks to (2.5) we readily get

\[
\int_0^t \int_\Omega \rho'(u(s))|u'(s)||\chi'(s)| ds \leq \frac{d_0}{2} \int_0^t \int_\Omega \rho'(u(s))|u'(s)|^2 ds + \frac{d_\infty}{2d_0} \|\chi'\|_{L^2(0,t,H)}^2.
\]

Therefore, if we define, for \( t \in [0,T] \), \( \psi(t) \geq 0 \) such that

\[
(\psi(t))^2 = \frac{d_0}{2} \int_0^t \int_\Omega \rho'(u(s))|u'(s)|^2 ds + \frac{d_0}{2} \|u(t)\|_{V'}^2 + \frac{\mu}{2} \|\chi'(t)\|_H^2 + \frac{\nu}{2} \int_0^t \int_\Omega |\nabla \chi'(s)|^2 ds,
\]

thanks to (3.20–22) we deduce that there exists \( \tilde{C}_9 > 0 \) such that

\[
(\psi(t))^2 \leq \tilde{C}_9 \left( 1 + \|F(t)\|_{V'}^2 + \|\chi'\|_{L^2(0,t,H)}^2 \right. \\
+ \left. \int_0^t \|F'(s)\|_{V'} \psi(s) ds + \int_0^t \|\chi'(s)\|_H^2 (\psi(s))^2 ds \right)
\]

Then, recalling the estimate (3.15) (note in particular that the function \( s \mapsto \|\chi'(s)\|_H^2 \) belongs to \( L^1(0,t) \)) and applying an extended version of the Gronwall lemma (see e.g. [2, Theorem 2.1, formulas (2.7) and (2.12)]), we find a constant \( \tilde{C}_{10} > 0 \) such that

\[
(\psi(t))^2 \leq \tilde{C}_{10} e^{\tilde{C}_{10} \left( 1 + \|F\|_{L^\infty(0,t;V')}^2 + \|F\|_{L^2(0,t;V')}^2 + \|F'\|_{L^1(0,t;V')}^2 \right)}
\]

Now (3.19) is an easy consequence of the last inequality.

**Lemma 3.5.** There exists a constant \( d_1 > 0 \) such that

\[
\frac{d_1 |r_1 - r_2|^2}{1 + r_1^2 + r_2^2} \leq |\rho(r_1) - \rho(r_2)| \quad \forall \ r_1, r_2 \in \mathbb{R}.
\]

**Proof.** Exploiting (2.3), (2.2), and (2.6), it is easily seen that \( \lim_{r \to 0^+} \frac{\alpha'(r)}{1 + |\alpha(r)|^2} = d_0 \), and moreover \( \lim_{r \to \infty} \frac{\alpha'(r)}{1 + |\alpha(r)|^2} = 0 \). It follows that, as \( \alpha \in C^1(0,\infty) \), there exists a constant \( d_1 > 0 \) such that \( \alpha'/(1 + |\alpha|^2) \leq d_1^{-1} \). Then, by the mean value theorem, we find an \( r_* \) between \( r_1 \) and \( r_2 \) such that

\[
|\rho(r_1) - \rho(r_2)| = |\rho'(r_*)(r_1 - r_2)| = \frac{|r_1 - r_2|}{\alpha'(\alpha^{-1}(r_*))} \geq \frac{d_1 |r_1 - r_2|}{1 + r_*^2} \geq \frac{d_1 |r_1 - r_2|}{1 + r_1^2 + r_2^2}.
\]

Lemma 3.6. Let \((\theta_j, \chi_j)\) be the solution to Problem \((P^a)\) corresponding to the datum 
\(F_j, j = 1, 2\). Moreover, suppose that \(u_j = \alpha(\theta_j)\) is such that \(\|u_j\|_{L^\infty(0,t;V)} \leq M\). If we set 
\[ \tilde{u} := u_1 - u_2, \quad \tilde{\chi} := \chi_1 - \chi_2, \quad \tilde{F} := F_1 - F_2, \]
then there exists a constant \(C(M) > 0\), depending on \(M\), but not on \(\varepsilon, t \in (0,T)\), and on 
\(\tilde{F}\), such that 
\[
\frac{1}{2} \int_0^t \int_\Omega \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{1}{4} \|(1 \ast \tilde{u})(t)\|_V^2 + \frac{\mu}{2} \|\tilde{\chi}(t)\|_H^2 + \frac{\nu}{4} \int_0^t \int_\Omega |\nabla \tilde{\chi}(s)|^2 ds 
\leq C(M) \|\tilde{F}\|_{L^1(0,t;V')}^2 \quad \forall t \in [0,T]. \tag{3.23}
\]

Proof. At first, let us subtract the respective equations \((2.36)\) for \((\theta_j, \chi_j), j = 1, 2\), from each other and integrate from \(0\) to \(s \in (0,t)\), \(0 \leq t \leq T\). Then we apply the result to \(\tilde{u}(s)\) and finally integrate over \((0,t)\) (shortly \(\int_0^t (\int_0^s (2.36)_1(\tau) - (2.36)_2(\tau)) d\tau, \tilde{u}(s))ds\). We get 
\[
\int_0^t \int_\Omega (\rho(u_1(s)) - \rho(u_2(s))) (u_1(s) - u_2(s)) ds + \frac{1}{2} \left\| \int_0^t \tilde{u}(s)ds \right\|_V^2 
\leq L \int_0^t \int_\Omega |\tilde{\chi}(s)||\tilde{u}(s)|ds + \int_0^t \langle \int_0^s \tilde{F}(\tau) d\tau, \tilde{u}(s) \rangle ds. \tag{3.24}
\]
Performing an integration by parts and noticing that \(\int_0^t \tilde{u}(s)ds = (1 \ast \tilde{u})(t)\), we get 
\[
\int_0^t \langle \int_0^s \tilde{F}(\tau)d\tau, \tilde{u}(s) \rangle ds = \langle \int_0^t \tilde{F}(ds), (1 \ast \tilde{u})(t) \rangle - \int_0^t \langle \tilde{F}(s), (1 \ast \tilde{u})(s) \rangle ds 
\leq \left\| \int_0^t \tilde{F}(s)ds \right\|_{V'} \|(1 \ast \tilde{u})(t)\|_V + \int_0^t \left\| \tilde{F}(s) \right\|_{V'} \|(1 \ast \tilde{u})(s)\|_V ds 
\leq \left\| \tilde{F} \right\|_{L^1(0,t;V')}^2 + \frac{1}{4} \|(1 \ast \tilde{u})(t)\|_V^2 + \int_0^t \left\| \tilde{F}(s) \right\|_{V'} \|(1 \ast \tilde{u})(s)\|_V ds,
\]
therefore from \((3.24)\), using Lemma 3.5, we obtain 
\[
\int_0^t \int_\Omega \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{1}{4} \|(1 \ast \tilde{u})(t)\|_V^2 
\leq L \int_0^t \int_\Omega |\tilde{u}(s)||\tilde{\chi}(s)|ds + \left\| \tilde{F} \right\|_{L^1(0,t;V')}^2 + \int_0^t \left\| \tilde{F}(s) \right\|_{V'} \|(1 \ast \tilde{u})(s)\|_V ds. \tag{3.25}
\]
Let us now take the difference of the equations \((2.37)\) corresponding to the solutions 
\((\theta_j, \chi_j), j = 1, 2\), at the time \(s\), and let us apply this difference to the function \(\tilde{\chi}(s)\). Then 
we let us integrate over \((0,t)\) (i.e. we compute \(\int_0^t ((2.37)_1(s) - (2.37)_2(s), \tilde{\chi}(s)) ds\). We infer 
\[
\frac{\mu}{2} \|\tilde{\chi}(t)\|_H^2 + \nu \int_0^t \int_\Omega |\nabla \tilde{\chi}(s)|^2 ds \leq L \int_0^t \|\tilde{\chi}(s)\|_H^2 ds 
+ \int_0^t \int_\Omega \left[ \lambda'(\chi_1(s))(u_1(s) + \ell(\rho(u_1(s)))) - \lambda'(\chi_2(s))(u_2(s) + \ell(\rho(u_2(s)))) \right] \tilde{\chi}(s) ds. \tag{3.26}
\]
Thanks to (H2), we have

\[ \lambda'(\chi_1(s))(u_1(s) + \ell(\rho(u_1(s)))) - \lambda'(\chi_2(s))(u_2(s) + \ell(\rho(u_2(s)))) \]
\[ = \lambda'(\chi_1(s))(u_1(s) - u_2(s)) + (\lambda'(\chi_1(s)) - \lambda'(\chi_2(s)))u_2(s) \]
\[ + \lambda'(\chi_1(s))(\ell(\rho(u_1(s))) - \ell(\rho(u_2(s)))) + \left(\lambda'(\chi_1(s)) - \lambda'(\chi_2(s))\right)\ell(\rho(u_2(s))) \]
\[ \leq L|\tilde{u}(s)| + L|\tilde{x}(s)||u_2(s)| + L^3|\tilde{u}(s)| + L^3|\tilde{x}(s)||u_2(s)| + L|\ell(\rho(0))||\tilde{x}(s)|; \]

therefore, from (3.26), we deduce that

\[ \frac{\mu}{2}\|\tilde{x}(t)\|_H^2 + \nu \int_0^t \int_{\Omega} |\nabla \tilde{x}(s)|^2 ds \leq L(1 + |\ell(\rho(0))|) \int_0^t \|\tilde{x}(s)\|_H^2 ds \]
\[ + (L + L^3) \int_0^t \int_{\Omega} |\tilde{u}(s)||\tilde{x}(s)| ds + (L + L^3) \int_0^t \int_{\Omega} |\tilde{x}(s)|^2 |\tilde{u}(s)| ds. \quad (3.27) \]

Adding (3.25) and (3.27), we find a constant \( \hat{C}_{11} > 0 \) such that

\[ \int_0^t \int_{\Omega} \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{1}{4} \|(1 * \tilde{u})(t)\|_V^2 + \frac{\mu}{2}\|\tilde{x}(t)\|_H^2 + \nu \int_0^t \int_{\Omega} |\nabla \tilde{x}(s)|^2 ds \leq \||\tilde{F}|_{L^2(0,t;V')} + \int_0^t \||\tilde{F}(s)||\tilde{V}'||1 * \tilde{u}(s)||_V ds \]
\[ + \hat{C}_{11} \left( \int_0^t \|\tilde{x}(s)\|_H^2 ds + \int_0^t \int_{\Omega} |\tilde{u}(s)||\tilde{x}(s)| ds + \int_0^t \int_{\Omega} |\tilde{x}(s)|^2 |\tilde{u}(s)| ds \right). \quad (3.28) \]

Let us point out that, since \( V \hookrightarrow L^4(\Omega) \), there exists \( \hat{C}_{12} > 0 \) such that

\[ \hat{C}_{11} \int_0^t \int_{\Omega} |u_2(s)||\tilde{x}(s)|^2 ds \leq \hat{C}_{11} \int_0^t \|u_2(s)||\tilde{x}(s)||L^4(\Omega)||\tilde{x}(s)||H ds \]
\[ \leq \hat{C}_{12} \||u_2||L^\infty(0,t;V) \int_0^t \|\tilde{x}(s)||V||\tilde{x}(s)||H ds \]
\[ \leq \frac{\nu}{4\hat{C}_{12}^2} \int_0^t \|\tilde{x}(s)||V^2 ds + \frac{\nu C^2_{12} C^2 M^2}{\nu} \int_0^t \|\tilde{x}(s)||H^2 ds \]
\[ \leq \frac{\nu}{4} \int_0^t \int_{\Omega} |\nabla \tilde{x}(s)|^2 ds + \frac{\nu}{4} \int_0^t \|\tilde{x}(s)||H^2 ds + \frac{\nu C^2_{12} C^2 M^2}{\nu} \int_0^t \|\tilde{x}(s)||H^2 ds. \quad (3.29) \]

Thanks to (2.42) we deduce

\[ \hat{C}_{11} \int_0^t \int_{\Omega} |\tilde{u}(s)||\tilde{x}(s)| ds \]
\[ = \hat{C}_{11} \int_0^t \int_{\Omega} \frac{|\tilde{u}(s)|}{(1 + |u_1(s)|^2 + |u_2(s)|^2)^{1/2}} (1 + |u_1(s)|^2 + |u_2(s)|^2)^{1/2} |\tilde{x}(s)| ds \]
\[ \leq \frac{1}{2} \int_0^t \int_{\Omega} \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{\hat{C}_{11}^2}{2d_1} \int_0^t \int_{\Omega} \left( \sum_{j=1}^2 |u_j(s)|^2 \right) |\tilde{x}(s)|^2 ds. \quad (3.30) \]
By [16, p. 102, formula (16.16)], we find a constant \( \hat{C}_{13} > 0 \) and a constant \( \hat{C}(M) > 0 \), depending on \( M \), but not on \( \varepsilon, t, \) and \( \tilde{F} \), in such a way that

\[
\frac{\hat{C}_{11}^2}{2d_1} \sum_{j=1}^{2} \int_0^t \int_\Omega |u_j(s)|^2 |\tilde{x}(s)|^2 ds \leq \hat{C}_{13} \sum_{j=1}^{2} \|u_j\|_{L^\infty(0,t;V)}^2 \int_0^t \|\tilde{x}(s)\|_{L^4(\Omega)}^2 ds \leq \hat{C}_{13}(1 + 2M^2) \int_0^t \|\tilde{x}(s)\|_{L^4(\Omega)}^2 ds \leq \hat{C}_{13}(1 + 2M^2) \int_0^t \left( \frac{\nu}{4\hat{C}_{13}(1 + 2M^2)} \int_\Omega |\nabla\tilde{x}(s)|^2 + \hat{C}(M)\|\tilde{x}(s)\|_H^2 \right) ds. \tag{3.31}
\]

Collecting together (3.28–31) we infer the existence of a constant \( \hat{C}'(M) > 0 \), depending on \( M \), but not on \( \varepsilon, t, \) and \( \tilde{F} \), such that

\[
\int_0^t \int_\Omega \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{1}{4}\|1 \ast \tilde{u}(t)\|_V^2 + \frac{\mu}{2}\|\tilde{x}(t)\|_H^2 + \frac{\nu}{4} \int_0^t \int_\Omega |\nabla\tilde{x}(s)|^2 ds \leq \|\tilde{F}\|_{L^1(0,t;V')} + \hat{C}'(M) \int_0^t \|\tilde{x}(s)\|_H^2 ds + \int_0^t \|\tilde{F}(s)\|_{V'} \|1 \ast \tilde{u}(s)\|_V ds.
\]

Thus a generalized version of the Gronwall lemma (cf. [4, pp. 156, 157] or [2, Theorem 2.1]) enables us to conclude. ■

4. Existence for the Problem \((P_\varepsilon)\)

Now we are going to exploit the results of the previous section in order to solve the Problem \((P_\varepsilon)\), \( \varepsilon > 0 \), and prove Theorem 2.3.

**Lemma 4.1.** Under the assumptions of Theorem 2.3, the Problem \((P_\varepsilon)\) has at most one solution.

**Proof.** If \((\theta_1, \chi_1)\) and \((\theta_2, \chi_2)\) are two solutions to Problem \((P_\varepsilon)\), then they also solve problem \((P^a)\) with \( \theta_0 = \theta_0\varepsilon \) and \( \chi_0 = \chi_0\varepsilon \), and where \( F \) is replaced respectively by \( f_\varepsilon - J(k_\varepsilon \ast u_1) \) and \( f_\varepsilon - J(k_\varepsilon \ast u_2) \), with \( u_j = \alpha(\theta_j), \ j = 1, 2 \). Set \( \tilde{u} := u_1 - u_2 \) and \( \tilde{x} := \chi_1 - \chi_2 \). Since, thanks to (2.39), (2.41), and (2.19)

\[
\|J(k_\varepsilon \ast \tilde{u})\|_{L^1(0,t;V')} \leq \|k_\varepsilon\|_{L^1(0,t)} \|1 \ast \tilde{u}\|_{L^1(0,t;V')} \quad \forall \ t \in [0,T], \tag{4.1}
\]

then, applying Lemma 3.6, we find a constant \( C(M_{1,2}) > 0 \), which may depend on \( M_{1,2} = \max_{j=1,2} \|u_j\|_{L^\infty(0,T;V')} \), but not on \( \varepsilon \) and on \( t \in (0,T) \), such that

\[
\frac{1}{2} \int_0^t \int_\Omega \frac{d_1|\tilde{u}(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} ds + \frac{1}{4}\|1 \ast \tilde{u}(t)\|_V^2 + \frac{\mu}{2}\|\tilde{x}(t)\|_H^2 + \frac{\nu}{4} \int_0^t \int_\Omega |\nabla\tilde{x}(s)|^2 ds \leq C(M_{1,2})\|k_\varepsilon\|_{L^1(0,T)} \int_0^t \|1 \ast \tilde{u}(s)\|_V^2 ds \quad \forall \ t \in [0,T]. \tag{4.2}
\]
Hence, it is sufficient to apply the Gronwall lemma to deduce that \( \tilde{x}(t) = 0 \) for all \( t \in [0, T] \), and that \( \tilde{u} = 0 \) a.e. in \( (0, T) \). The thesis easily follows. ■

Now we are going to prove Theorem 2.3. To this goal let us consider the operator \( \Sigma_\varepsilon : L^\infty(0, T; V) \to L^\infty(0, T; V) \) defined in this way. Given \( U \in L^\infty(0, T; V) \), we set \( \Sigma_\varepsilon(U) := u = \alpha(\theta) \), where \( (\theta, \chi) \) is the unique solution to Problem \((P^\alpha)\) with the choices of \( F = f_\varepsilon - J(k_\varepsilon * U) \), \( \theta_0 = \theta_{0\varepsilon} \), and \( \chi_0 = \chi_{0\varepsilon} \). The operator \( \Sigma_\varepsilon \) is well defined thanks to Proposition 3.1.

**Lemma 4.2.** Suppose that \( U \in L^\infty(0, T; V) \) and \( u = \Sigma_\varepsilon(U) \). Then there exists a constant \( R_1(\varepsilon) > 0 \) which may depend on \( \varepsilon \), but not on \( t \in (0, T) \), such that

\[
\|u\|_{L^\infty(0, t; V)} \leq R_1(\varepsilon) e^{R_1(\varepsilon)(1 + \|k_\varepsilon\|_{W^{1,1}(0, t)}^2 \|U\|_{L^\infty(0, t; V)}^2)} \quad \forall \ t \in [0, T], \tag{4.3}
\]

Moreover let \( U_j \in L^\infty(0, T; V) \) and \( u_j = \Sigma_\varepsilon(U_j) \), \( j = 1, 2 \), and assume \( \|u_j\|_{L^\infty(0, T; V)} \leq M \). Then there exists a constant \( R_2(\varepsilon) > 0 \), with the same dependences as \( R_1(\varepsilon) \), such that

\[
\|(1 * u_1) - (1 * u_2)\|_{C([0, t]; V)}^2 \leq R_2(\varepsilon) C(M) t^2 \|(1 * U_1) - (1 * U_2)\|_{C([0, t]; V)}^2 \quad \forall \ t \in [0, T], \tag{4.4}
\]

where \( C(M) \) is the constant appearing in (3.23).

**Proof.** Take \( t \in [0, T] \). From (2.39-41) one deduces that

\[
\|J(k_\varepsilon * U)\|_{W^{1,1}(0, t; V')} = \|k_\varepsilon * U\|_{L^1(0, t; V')} + \|(k_\varepsilon * U)'\|_{L^1(0, t; V'}) \leq \|k_\varepsilon\|_{L^1(0, t)} \|U\|_{L^1(0, t; V')} + \|k_\varepsilon'\|_{L^1(0, t)} \|U\|_{L^1(0, t; V')} \tag{4.5}
\]

Then (3.19) and (4.5) yield (4.3) (note that to recover (4.3) it is essential the fact that \( k_\varepsilon(0) = 0 \)). Inequality (4.4) is obtained instead applying Lemma 3.6 and exploiting (4.1) with \( \tilde{u} \) replaced by \( \tilde{U} \), or exploiting (4.5) and the Sobolev embedding theorems. ■

**Lemma 4.3.** There exists \( t_0 \in (0, T) \) such that Problem \((P_\varepsilon)\) has a unique solution on \([0, t_0]\).

**Proof.** Choose

\[
M = 2R_1(\varepsilon) e^{2R_1(\varepsilon)}.
\]

Then let \( t_1, t_2 \in [0, T] \) be such that

\[
\|k_\varepsilon\|_{W^{1,1}(0, t_1)} \leq \frac{1}{M} \quad \tag{4.6}
\]

and

\[
R_2(\varepsilon) C(M) t_2^2 \leq \frac{1}{2}. \quad \tag{4.7}
\]
So let us take \( t_0 = \min\{t_1, t_2\} \) and define \( Y_0 = B_{L^\infty(0,t_0;V)}[0,M] \), the closed ball of radius \( M \) in \( L^\infty(0,t_0;V) \). Inequalities (4.3) and (4.6) imply that \( \Sigma_\varepsilon \) maps \( Y_0 \) into itself. Let us endow \( Y_0 \) with the metric
\[
d(v_1, v_2) := \|1 * (v_1 - v_2)\|_{L^\infty(0,t_0;V)}, \quad v_1, v_2 \in Y_0.
\]
Then \( Y_0 \) turns out to be a complete metric space, and by (4.4) and (4.7) it results that \( \Sigma_\varepsilon \) is a contraction mapping in \( Y_0 \), so that it admits a unique fixed point. On account of Lemma 4.1 we can conclude.

**Proof of Theorem 2.3.** Lemma 4.3 tells us that \((P_\varepsilon)\) has a unique local solution. Then to prove the theorem it is enough to find an estimate which does not depend on \( t_0 \). Let us remark that using (2.41) with \( r = \infty \) and \( p = q = 2 \), we get, if \( F = f_\varepsilon - J(k_\varepsilon * u) \),
\[
\|F\|_2^2 \leq 2 \left( \|f_\varepsilon\|_2^2 + \int_0^t \|(k_\varepsilon * u)(s)\|_V^2 ds \right) \leq 2 \left( \|f_\varepsilon\|_2^2 \|k_\varepsilon\|_{L^2(0,T)} \|u\|_{L^2(0,s;V)}^2 ds \right). \tag{4.8}
\]
Hence, to obtain the estimate we are looking for, it suffices to exploit (3.15) together with (4.8), and to apply the Gronwall lemma.

5. Existence for the Problem \((P)\)

Within this section we recover some estimates which are uniform with respect to \( \varepsilon \) and that will allow us to find a solution to Problem \((P)\) via a passage to the limit. It is not restrictive to assume \( \varepsilon < \varepsilon_0 \), where \( \varepsilon_0 \) is such that
\[
\|k_\varepsilon - k\|_{L^1(0,T)} \leq \frac{1}{4} \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{5.1}
\]

**Lemma 5.1.** Let \( (\theta_\varepsilon, \chi_\varepsilon) \) be the solution to Problem \((P_\varepsilon)\) and let be \( u_\varepsilon := \alpha(\theta_\varepsilon) \). Then there exists a constant \( C_4 > 0 \) such that
\[
\frac{1}{2} \|u_\varepsilon\|_{L^2(0,t;V)}^2 + \|\chi_\varepsilon'\|_{L^2(0,t;H)}^2 + \int_\Omega |\nabla \chi_\varepsilon(t)|^2 + \|\theta_\varepsilon(t)\|_H^2 \leq C_4 \quad \forall \ t \in [0,T]. \tag{5.2}
\]

**Proof.** \((\theta_\varepsilon, \chi_\varepsilon)\) is also a solution to Problem \((P^a)\) with \( F = f_\varepsilon - J(k_\varepsilon * u_\varepsilon), \theta_0 = \theta_{0\varepsilon} \) and \( \chi_0 = \chi_{0\varepsilon} \). Now let us choose \( \omega \in [0,1] \) such that
\[
\|k^\omega\|_{L^1(0,T)} \leq \frac{1}{4}.
\]
We have, thanks to (2.41) and (5.1),
\[
\|F^\omega\|_{L^2(0,t;V')}^2 \leq 2 \left( \|\tilde{f}_\varepsilon^\omega\|_{L^2(0,t;V')}^2 + \|k_\varepsilon^\omega \ast \tilde{u}_\varepsilon^\omega\|_{L^2(0,t;V)}^2 \right) \\
\leq 2 \left( \|\tilde{f}_\varepsilon^\omega\|_{L^2(0,t;V')}^2 + \|k_\varepsilon^\omega\|_{L^1(0,T)} \|u_\varepsilon^\omega\|_{L^2(0,t;V)}^2 \right) \\
\leq 2 \left( \|\tilde{f}_\varepsilon^\omega\|_{L^2(0,t;V')}^2 + (\|k_\varepsilon - k\|_{L^1(0,T)} + \|k_\varepsilon^\prime\|_{L^1(0,T)})^2 \|u_\varepsilon^\omega\|_{L^2(0,t;V)}^2 \right) \\
\leq 2 \|\tilde{f}_\varepsilon^\omega\|_{L^2(0,t;V')}^2 + \frac{1}{2} \|u_\varepsilon^\omega\|_{L^2(0,t;V)}^2. \tag{5.3}
\]

Applying Lemma 3.2 and taking (5.3) into account, since \(\|F^\omega\|_{L^2(0,t;V')}\) is bounded (cf. (2.23)), we determine a constant \(\hat{C}_{14} > 0\) such that
\[
\frac{1}{2} \|u_\varepsilon^\omega\|_{L^2(0,t;V)}^2 + \|u_\varepsilon^\omega\|_{L^2(0,t;H)}^2 + \int_\Omega |\nabla u_\varepsilon^\omega(t)|^2 + \|u_\varepsilon(t)\|_H^2 \leq \hat{C}_{14} \quad \forall \ t \in [0,T].
\]
Hence we infer (5.2) recalling that \(e^{-2\omega T} \leq e^{-2\omega t}\) for all \(t \in [0,T]\). 

**Lemma 5.2.** Let \((\theta_\varepsilon, \chi_\varepsilon)\) be the solution to Problem \((P_\varepsilon)\), \(u_\varepsilon := \alpha(\theta_\varepsilon)\) and let \(\xi_\varepsilon\) be a function satisfying the same properties as \(\xi\) in (2.35) and (2.37), where \(\theta, \chi,\) and \(u\) are replaced respectively by \(\theta_\varepsilon, \chi_\varepsilon\), and \(u_\varepsilon\). Then there exists a constant \(C_5 > 0\) such that
\[
\|u_\varepsilon\|_{L^2(0,t;V)}^2 + \|\chi_\varepsilon\|_{H^1(0,t;H) \cap C([0,t];V) \cap L^2(0,t;H^2(\Omega))}^2 + \|\theta_\varepsilon\|_{H^1(0,t;V') \cap C([0,t];H) \cap L^2(0,t;H)}^2 + \|\xi_\varepsilon\|_{L^2(0,t;H)}^2 \\
\leq C_5 \quad \forall \ t \in [0,T]. \tag{5.4}
\]

**Proof.** It suffices to apply Lemma 3.3 with \(F = f_\varepsilon - J(k_\varepsilon \ast u_\varepsilon)\) and exploit (5.2) together with the estimate
\[
\|k_\varepsilon \ast u_\varepsilon\|_{L^2(0,t;V')} \leq \|k_\varepsilon\|_{L^1(0,T)} \|u_\varepsilon\|_{L^2(0,t;V)} \leq (\|k\|_{L^1(0,T)} + \hat{C}_{15}) \|u_\varepsilon\|_{L^2(0,t;V)},
\]
where \(\hat{C}_{15}\) is a constant depending only on \(\varepsilon_0\) (cf. also (2.23)). 

Now we have all the ingredients to perform the passage to the limit.

**Proof of Theorem 2.1.** Lemma 5.2 allows us to say that there exist four functions \(u, \chi, \xi, \theta\) such that, possibly taking subsequences,
\[
u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(0,T;V), \tag{5.5}
\]
\[
\chi_\varepsilon \rightharpoonup \chi \quad \text{in} \quad H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \tag{5.6}
\]
\[
\xi_\varepsilon \rightarrow \xi \quad \text{in} \quad L^2(0,T;H), \tag{5.7}
\]
\[
\theta_\varepsilon \rightharpoonup \theta \quad \text{in} \quad H^1(0,T;V') \cap L^2(0,T;V). \tag{5.8}
\]
Since for all \( t \in [0, T] \) \( \{\chi_\varepsilon(t) : \varepsilon > 0\} \) is a precompact set in \( H \) and \( \{\chi_\varepsilon : \varepsilon > 0\} \) is an equicontinuous set in \( C([0, T]; H) \), thanks to the generalized Ascoli theorem (cf. [13, Theorem 3.1, p. 57]) we deduce that \( \{\chi_\varepsilon : \varepsilon > 0\} \) is precompact in \( C([0, T]; H) \). Moreover by the Aubin lemma (cf. [15, p. 58]) we see that \( H^1(0, T; H) \cap L^2(0, T; H^2(\Omega)) \hookrightarrow L^2(0, T; V) \); hence \( \{\chi_\varepsilon : \varepsilon > 0\} \) is precompact in \( L^2(0, T; V) \). Therefore, at least for subsequences, we get
\[
\chi_\varepsilon \to \chi \quad \text{in} \quad C([0, T]; H) \cap L^2(0, T; V). \tag{5.9}
\]

Similarly it is easily seen that
\[
\theta_\varepsilon \to \theta \quad \text{in} \quad C([0, T]; V') \cap L^2(0, T; H). \tag{5.10}
\]

Thanks to (5.10) and (H2), we have
\[
\ell(\theta_\varepsilon) \to \ell(\theta) \quad \text{in} \quad L^2(0, T; H), \tag{5.11}
\]

and in view of (5.9) and (H2) one can see that
\[
\sigma'(\chi_\varepsilon) \to \sigma'(\chi) \quad \text{in} \quad C([0, T]; H),
\]
\[
\lambda'(\chi_\varepsilon) \to \lambda'(\chi) \quad \text{in} \quad C([0, T]; H). \tag{5.12}
\]

From (5.12), (5.5–6), and (5.11) we deduce:
\[
\lambda'(\chi_\varepsilon)u_\varepsilon \rightharpoonup \lambda'(\chi)u, \quad \lambda'(\chi_\varepsilon)\chi_\varepsilon' \rightharpoonup \lambda'(\chi)\chi', \quad \text{in} \quad L^1(Q), \tag{5.13}
\]

\[
\lambda'(\chi_\varepsilon)\ell(\theta_\varepsilon) \rightharpoonup \lambda'(\chi)\ell(\theta), \quad \text{in} \quad L^1(Q), \tag{5.14}
\]

hence, since the sequences in (5.13) and (5.14) are bounded in \( L^2(Q) \),
\[
\lambda'(\chi_\varepsilon)u_\varepsilon \rightharpoonup \lambda'(\chi)u, \quad \lambda'(\chi_\varepsilon)\chi_\varepsilon' \rightharpoonup \lambda'(\chi)\chi', \quad \text{in} \quad L^2(0, T; H), \tag{5.15}
\]

\[
\lambda'(\chi_\varepsilon)\ell(\theta_\varepsilon) \rightharpoonup \lambda'(\chi)\ell(\theta), \quad \text{in} \quad L^2(0, T; H). \tag{5.16}
\]

Finally from (5.5) and (2.23) we deduce that
\[
k_\varepsilon \ast u_\varepsilon \rightharpoonup k \ast u \quad \text{in} \quad L^2(0, T; V). \tag{5.17}
\]

As \( \rho \) is maximal monotone in \( \mathbb{R} \) and \( \theta_\varepsilon = \rho(u_\varepsilon) \), thanks to (5.5) and (5.10), we have that
\[
(\rho(u_\varepsilon), u_\varepsilon)_{L^2(Q)} = \int_0^T \langle \rho(u_\varepsilon(t)), u_\varepsilon(t) \rangle \, dt \to \int_0^T \langle \theta(t), u(t) \rangle \, dt = (\rho(u), u)_{L^2(Q)}. \tag{5.18}
\]

Thus (cf., e.g., [3, Lemma 1.3, p. 42]), we get \( \theta = \rho(u) \) a.e. in \( Q \). Similarly one sees that \( \xi \in \beta(\chi) \) a.e. in \( Q \). At this point it is sufficient to pass to the limit to see that \( (\theta, \chi) \) is a solution to Problem \((P)\). \( \blacksquare \)
Now let us come to the proof of Theorem 2.2. Due to the higher regularity of data, in the approximation procedure it is possible to choose

\[ f_\varepsilon = f, \quad u_{0\varepsilon} = u_0, \quad \chi_{0\varepsilon} = \chi_0 \quad \forall \varepsilon > 0, \]
\[ k_\varepsilon \in W^{1,1}_0(0, T) \quad \forall \varepsilon > 0, \]
\[ k_\varepsilon \to k \quad \text{in} \ L^1(0, T), \quad \text{as} \ \varepsilon \to 0^+, \]
\[ (\|k_\varepsilon\|_{W^{1,1}_0(0, T)}) \varepsilon \text{ bounded.} \quad (5.19) \]

This is possible taking, for any \( \varepsilon > 0 \), the function \( k_\varepsilon \) defined by

\[ k_\varepsilon(t) := k(t)\chi_{[\varepsilon, T-\varepsilon]}(t) + (k(\varepsilon)/\varepsilon)t\chi_{[0,\varepsilon]}(t) + (k(T-\varepsilon)/\varepsilon)(T-t)\chi_{(T-\varepsilon,T]}(t), \]
where \( \chi_S \) denotes the characteristic function of a set \( S \).

**Lemma 5.3.** Suppose that the assumptions of Theorem 2.2 hold. If \((\theta_\varepsilon, \chi_\varepsilon)\) is a solution to Problem \((P_\varepsilon)\) and if \( u_\varepsilon = \alpha(\theta_\varepsilon) \), then there exists a constant \( C_6 > 0 \) such that:

\[ \|u_\varepsilon\|_{L^\infty(0,T;V)} + \|\chi_\varepsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C_6. \quad (5.20) \]

**Proof.** If \( F = f - J(k_\varepsilon \ast u_\varepsilon) \), arguing as in (4.5) and exploiting (5.19) we get

\[ \|F\|_{W^{1,1}(0,T;V')} \leq 2\|f\|_{W^{1,1}(0,T;V')} + 2\|k_\varepsilon\|_{W^{1,1}(0,T)} \|u_\varepsilon\|_{L^1(0,T;V)} \]
\[ \leq 2\|f\|_{W^{1,1}(0,T;V')} + 2T\hat{C}_{16}\|u_\varepsilon\|_{L^2(0,T;V)}, \quad (5.21) \]

Where \( \hat{C}_{16} \) is a constant which, thanks to (5.19), is independent of \( \varepsilon \). Now the assertion is implied by (3.19), (5.21), and (5.2). \( \blacksquare \)

Now we can conclude with the

**Proof of Theorem 2.2.** The last lemma allows us to add to the limit procedure in the proof of Theorem 2.1 the further convergences

\[ u_\varepsilon \rightharpoonup^* u \quad \text{in} \ L^\infty(0,T;V), \quad (5.22) \]
\[ \chi_\varepsilon \rightharpoonup^* \chi \quad \text{in} \ W^{1,\infty}(0,T;H) \cap H^1(0,T;V). \quad (5.23) \]

Therefore, taking the limit as \( \varepsilon \) goes to zero, we get that (2.17) holds and that \( \chi \in W^{1,\infty}(0,T;H) \cap H_0^1(0,T;V) \). To obtain that \( \chi \in L^\infty(0,T;H^2(\Omega)) \), we have to regularize the equations (2.12) and (2.14) by replacing \( \beta \) with its Yosida approximation \( \beta_\lambda \), which is a Lipschitz continuous function. So, by comparison in the approximated equation, we see that there exists \( G \in L^\infty(0,T;H) \) such that

\[ -\Delta \chi + \beta_\lambda(\chi) = G \quad \text{a.e. in} \ (0,T), \quad (5.24) \]
where $\chi$ is meant to be the solution of the approximated problem. Multiplying (5.24) by $-\Delta \chi$ and integrating over $\Omega$ we get the estimate

$$\frac{1}{2}||\Delta \chi||^2_H + \int_{\Omega} \beta'_{\chi}(\chi)|\nabla \chi|^2 \leq \frac{1}{2}||G||^2_{L^\infty(0,T;H)} \quad \text{a.e. in } (0,T).$$

Hence we obtain (2.18) (after taking the limit as $\lambda \to 0^+$, of course). Finally, (2.16) follows from a comparison in (2.13). To prove uniqueness, let us observe that if $(\theta_1, \chi_1)$ and $(\theta_2, \chi_2)$ are two solutions to Problem (P), then they are solutions to Problem $(P^\varepsilon)$ with $F$ replaced respectively by $f - J(k_\varepsilon * u_1)$ and by $f - J(k_\varepsilon * u_2)$, where $u_j = \alpha(\theta_j)$, $j = 1, 2$. Setting $\tilde{u} := u_1 - u_2$, $\tilde{\chi} := \chi_1 - \chi_2$, and $\tilde{F} = F_1 - F_2$, thanks to (2.39) we have

$$||\tilde{F}||^2_{L^1(0,t;V')} \leq 2(|k(0)|^2 + ||k'||^2_{L^1(0,T)}) ||1 * \tilde{u}||^2_{L^1(0,t;V)}$$

$$\leq 2(|k(0)|^2 + ||k'||^2_{L^1(0,T)}) T \int_0^t ||(1 * \tilde{u})(s)||^2_V ds. \quad (5.25)$$

Now, recalling (3.23) and exploiting the Gronwall lemma, we deduce that $\theta_1 = \theta_2$ and $\chi_1 = \chi_2$, and the proof of the theorem is complete.

**A. Appendix**

We give here the proof of the existence of a family of functions $(\chi_{0\varepsilon})_\varepsilon$ satisfying (2.22) and (2.24). We will consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, satisfying the assumptions of the Introduction. We denote with $\Gamma$ the boundary of $\Omega$ and set $H := L^2(\Omega)$ and $V := H^1(\Omega)$, endowed with the usual norms or equivalent ones.

**Proposition A.1.** Let $\varepsilon > 0$. Assume that hypotheses (H1) and (H7) of section 2 hold. There exists a unique $\chi_{0\varepsilon} \in H^2(\Omega)$ such that

$$\chi_{0\varepsilon} - \varepsilon \Delta \chi_{0\varepsilon} + \varepsilon \xi_{0\varepsilon} = \chi_0 \quad \text{a.e. in } \Omega, \quad (A.1)$$

$$\chi_{0\varepsilon} \in D(\beta), \quad \xi_{0\varepsilon} \in \beta(\chi_{0\varepsilon}) \quad \text{a.e. in } \Omega, \quad \beta^o(\chi_{0\varepsilon}) \in H, \quad (A.2)$$

$$\partial_n \chi_{0\varepsilon} = 0 \quad \text{a.e. in } \Gamma, \quad (A.3)$$

where, for each $r \in D(\beta)$, $\beta^o(r)$ denotes the element of $\beta(r)$ having minimum modulus. Moreover we have

$$\frac{1}{2}||\chi_{0\varepsilon}||^2_H + \varepsilon \int_{\Omega} |\nabla \chi_{0\varepsilon}|^2 \leq \frac{1}{2}||\chi_0||^2_H, \quad (A.4)$$

$$\frac{1}{2} \int_{\Omega} |\nabla \chi_{0\varepsilon}|^2 + \varepsilon ||\Delta \chi_{0\varepsilon}||^2_H \leq \frac{1}{2} \int_{\Omega} |\nabla \chi_0|^2, \quad (A.5)$$

$$\int_{\Omega} \tilde{\beta}(\chi_{0\varepsilon}) + \varepsilon ||\xi_{0\varepsilon}||^2_H \leq \tilde{\beta}(\chi_0)||L^1(\Omega). \quad (A.6)$$
Proof. Let $\beta_\lambda$ be the Yosida approximation of $\beta$ and $\hat{\beta}_\lambda$ the antiderivative of $\beta_\lambda$ such that $\hat{\beta}_\lambda(0) = 0$. Let $\chi_{0\varepsilon}^\lambda \in H^2(\Omega)$ be the unique solution of the problem

$$
\chi_{0\varepsilon}^\lambda - \varepsilon \Delta \chi_{0\varepsilon}^\lambda + \varepsilon \beta_\lambda(\chi_{0\varepsilon}^\lambda) = \chi_0^\varepsilon \quad \text{a.e. in } \Omega,
$$

(A.7)

$$
\partial_n \chi_{0\varepsilon}^\lambda = 0 \quad \text{a.e. in } \Gamma.
$$

(A.8)

Multiplying (A.7) by $\chi_{0\varepsilon}^\lambda$ and integrating over $\Omega$ we easily get

$$
\frac{1}{2} \| \chi_{0\varepsilon}^\lambda \|_H^2 + \varepsilon \int_\Omega |\nabla \chi_{0\varepsilon}^\lambda|^2 \leq \frac{1}{2} \| \chi_0^\varepsilon \|_H^2,
$$

(A.9)

multiplying (A.7) by $-\Delta \chi_{0\varepsilon}^\lambda$ and integrating in space we have

$$
\frac{1}{2} \int_\Omega |\nabla \chi_{0\varepsilon}^\lambda|^2 + \varepsilon \| \Delta \chi_{0\varepsilon}^\lambda \|_H^2 \leq \frac{1}{2} \| \nabla \chi_0^\varepsilon \|_2.
$$

(A.10)

As $\beta_\lambda(\chi_{0\varepsilon}^\lambda) \in \partial \hat{\beta}_\lambda(\chi_{0\varepsilon}^\lambda)$ we find

$$
\int_\Omega \hat{\beta}_\lambda(\chi_{0\varepsilon}^\lambda) \leq \int_\Omega \beta_\lambda(\chi_{0\varepsilon}^\lambda)(\chi_{0\varepsilon}^\lambda - \chi_0^\varepsilon) + \int_\Omega \hat{\beta}_\lambda(\chi_0^\varepsilon)
$$

$$
= -\varepsilon \int_\Omega \beta_\lambda(\chi_{0\varepsilon}^\lambda)|\nabla \chi_{0\varepsilon}^\lambda|^2 - \varepsilon \| \beta_\lambda(\chi_{0\varepsilon}^\lambda) \|_H^2 + \int_\Omega \hat{\beta}_\lambda(\chi_0^\varepsilon),
$$

whence we get

$$
\int_\Omega \hat{\beta}_\lambda(\chi_{0\varepsilon}^\lambda) + \varepsilon \| \beta_\lambda(\chi_{0\varepsilon}^\lambda) \|_H^2 \leq \| \hat{\beta}_\lambda(\chi_0^\varepsilon) \|_{L^1(\Omega)}.
$$

(A.11)

Thanks to (A.9–11), we get that there exist $\chi_{0\varepsilon} \in H^2(\Omega)$ and $\xi_{0\varepsilon} \in H$ such that, at least for subsequences,

$$
\chi_{0\varepsilon}^\lambda \rightharpoonup \chi_{0\varepsilon} \quad \text{in } H^2(\Omega),
$$

(A.12)

$$
\beta_\lambda(\chi_{0\varepsilon}^\lambda) \rightharpoonup \xi_{0\varepsilon} \quad \text{in } H.
$$

(A.13)

The last two convergences imply

$$
(\beta_\lambda(\chi_{0\varepsilon}^\lambda), \chi_{0\varepsilon}^\lambda)_{L^2(\Omega)} \rightarrow (\xi_{0\varepsilon}, \chi_{0\varepsilon})_{L^2(\Omega)},
$$

(A.14)

therefore (see [3, Lemma 1.3, p. 42]) $\xi_{0\varepsilon} \in \beta(\chi_{0\varepsilon})$ a.e. in $\Omega$. Hence, to obtain (A.1–6) it suffices to take the limit as $\lambda$ goes to 0 and to observe that $\beta^\circ(\chi_{0\varepsilon}) \in H$ because $|\beta^\circ(\chi_{0\varepsilon})| \leq |\xi_{0\varepsilon}|$ a.e. in $\Omega$ and $\beta_\lambda(\chi_{0\varepsilon}) \rightarrow \beta^\circ(\chi_{0\varepsilon})$ a.e. and in $H$ as $\lambda$ goes to 0. The uniqueness is a standard matter. 

Now, estimates (A.4–6) yield

$$
\varepsilon \Delta \chi_{0\varepsilon} \rightarrow 0 \quad \text{in } H,
$$

(A.15)

$$
\varepsilon \xi_{0\varepsilon} \rightarrow 0 \quad \text{in } H;
$$

(A.16)
so, by comparison in (A.1),

\[ \chi_{0\varepsilon} \to \chi_0 \quad \text{in } H, \quad (A.17) \]

and therefore, thanks to (A.4–5),

\[ \chi_{0\varepsilon} \to \chi_0 \quad \text{in } V \quad (A.18) \]

(actually, the last convergence is strong in \( V \), since from (A.4–5) it is easily seen that \( \|\chi_{0\varepsilon}\|_V \to \|\chi_0\|_V \) as \( \varepsilon \) goes to 0). Finally, (A.6) gives us that

\[ \left( \int_{\Omega} \tilde{\beta}(\chi_{0\varepsilon}) \right)_\varepsilon \text{ is bounded.} \quad (A.19) \]

Collecting (A.2–3) and (A.18–19), we finally deduce (2.22) and (2.24), as desired.

References


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