Convergence to the Stefan Problem of the Hyperbolic Phase Relaxation Problem and Error Estimates

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1. Introduction

When we consider the evolution of a material contained in a bounded domain \( \Omega \) of \( \mathbb{R}^n \) \((n \in \mathbb{N})\) during the time interval \([0, T]\), the usual equation for the energy balance reads

\[
\frac{\partial e}{\partial t} + \text{div} \, \mathbf{q} = g \quad \text{in} \ Q := \Omega \times (0, T),
\]

where \( e \) denotes the internal energy of the system, \( \mathbf{q} \) is the heat flux, and \( g \) represents the heat supply.

Let us suppose that the material is homogeneous and assume that it exhibits two phases. Then let us denote by \( \theta \) the relative temperature and by \( \chi \) the phase variable (e.g. the concentration of the more energetic phase). Thus a widely used model for describing the phase transition dynamics is obtained from (1) assuming the constitutive laws

\[
e = \theta + \chi,
\]

\[
\mathbf{q} = -\nabla \theta
\]

(for simplicity we have normalized to 1 all the physical constants). The relation (3) is usually called Fourier heat conduction law. This choice leads to the following equation for the energy balance

\[
\frac{\partial (\theta + \chi)}{\partial t} - \Delta \theta = g \quad \text{in} \ Q.
\]

In order to describe the evolution of the system, we have then to establish a further relation between the temperature and the phase. If \( \theta = 0 \) is the critical temperature of phase transition, we can take the following equilibrium condition of Stefan type (cf., e.g., [6, 9, 10] and their references)

\[
\chi \in H(\theta) \quad \text{in} \ Q
\]
where $H$ is the Heaviside graph (i.e., $H(r) = 0$ if $r < 0$, $H(0) = [0,1]$, $H(r) = 1$ if $r > 0$).

The problem of finding $\bar{\theta}$ and $\chi$ satisfying (4–5) is the so called Stefan problem. It has been widely investigated and several existence and uniqueness results have been proved, provided that (4–5) is coupled with suitable initial and boundary conditions (cf. e.g., [1, 4, 6, 9, 10]).

The inclusion (5) represents an equilibrium condition. If we want to take into account dissipation phenomena such as dynamical supercooling or superheating effects, we have to replace condition (5) by a non-equilibrium one. In [9], Visintin proposed to use the following relaxation dynamics for the phase variable $\chi$, that is,

\begin{equation}
\epsilon \frac{\partial \chi}{\partial t} + H^{-1}(\chi) \ni \bar{\theta} \quad \text{in } Q,
\end{equation}

$\epsilon$ being a small kinetic positive constant. The system (4), (6) is usually called phase relaxation problem. In paper [9], after coupling both problems (4–5) and (4), (6) with suitable initial and boundary conditions, it is proved that the last problem is well-posed and its solution converges, in a suitable sense, to the solution of the Stefan problem (4–5) as the parameter $\epsilon$ goes to 0.

Let us observe now that the Fourier law (3) leads to the parabolic equation (4), and it is well known that it allows the thermal disturbances to propagate at infinite speed. The first approach in order to overcome this feature, is due to Cattaneo, which in his work “Sulla conduzione del calore” [2], modified the Fourier law, originating the so-called Maxwell-Cattaneo law

\begin{equation}
\alpha \frac{\partial q}{\partial t} + q = -\nabla \theta \quad \text{in } Q,
\end{equation}

in which $\alpha$ represents a small positive relaxation parameter. Observe that a trivial integration of (7) with respect to time gives

\begin{equation}
q(t) = -\frac{1}{\alpha} \int_0^t \exp\left(\frac{s-t}{\alpha}\right) \nabla \theta(s) ds,
\end{equation}

so that (7) can be considered as a particular model of a material with memory (for updated reviews of the theory of Cattaneo see [3] and [8, Chapter 2]). With the constitutive assumptions (2) and (7), the energy balance (1) yields a hyperbolic equation predicting finite speed of propagation for the temperature field. If we couple this equation with the Stefan equilibrium condition we get system (1–2), (5), (7), which is also known as the hyperbolic Stefan problem. The existence of solutions of such a system is still an open problem. If we take account of both the relaxations (6–7), the following hyperbolic phase relaxation problem follows:

\begin{equation}
\frac{\partial (\bar{\theta} + \chi)}{\partial t} + \text{div } q = g \quad \text{in } Q,
\end{equation}

\begin{equation}
\alpha \frac{\partial q}{\partial t} + q = -\nabla \theta \quad \text{in } Q,
\end{equation}

\begin{equation}
\epsilon \frac{\partial \chi}{\partial t} + H^{-1}(\chi) \ni \bar{\theta} \quad \text{in } Q.
\end{equation}
This system was proposed for the first time in [9], where an existence result has been outlined when (9–11) is coupled with some initial and boundary condition. Moreover it is stated there that its solutions converge to the solution of the analogous problem for (4), (6), as the parameter $\alpha$ goes to zero, whereas $\varepsilon$ is fixed.

In [5] and in the present paper, (9–11) is supplied with rather general and meaningful initial-boundary conditions. More precisely, letting $\Gamma_0$ and $\Gamma_1$ denote two measurable subsets in which the boundary of $\Omega$ is partitioned, we take

\begin{align}
\theta &= \theta_D \text{ on } \Gamma_0 \times (0,T), \\
q \cdot n &= \varphi_N \text{ on } \Gamma_1 \times (0,T), \\
\theta(\cdot,0) &= \bar{\theta}_0, \quad \chi(\cdot,0) = \chi_0, \quad q(\cdot,0) = \bar{q}_0 \quad \text{in } \Omega,
\end{align}

where $\theta_D$, $\varphi_N$, $\bar{\theta}_0$, $\chi_0$, $\bar{q}_0$ are given functions and $n$ is the outward unit vector, normal to the boundary of $\Omega$. We assume that $\theta_D$ is a sufficiently smooth function defined on the whole $Q$ and that there exists a vector function $q_N : Q \to \mathbb{R}^n$ such that $q_N \cdot n = \varphi_N$ on $\Gamma_1 \times (0,T)$ in a suitable sense. Hence, setting $\theta_0 := \bar{\theta}_0 - \theta_D(0)$ and $q_0 := \bar{q}(0) - q_N(0)$, we rewrite the problem (9–14) in terms of the unknowns $\theta = \bar{\theta} - \theta_D$, $\chi$, and $q = \bar{q} - q_N$, obtaining the following equations and conditions:

\begin{align}
\partial_t (\theta + \chi) + \text{div} \ q &= g - \frac{\partial \theta_D}{\partial t} - \text{div} \ q_N \quad \text{in } Q, \\
\alpha \frac{\partial q}{\partial t} + q &= -\nabla \theta - \nabla \theta_D - \alpha \frac{\partial q_N}{\partial t} - q_N \quad \text{in } Q, \\
\varepsilon \frac{\partial \chi}{\partial t} + H^{-1}(\chi) \ni \theta + \theta_D \quad \text{in } Q, \\
\theta &= 0 \quad \text{on } \Gamma_0 \times (0,T), \quad q \cdot n = 0 \quad \text{on } \Gamma_1 \times (0,T), \\
\theta(\cdot,0) &= \theta_0, \quad \chi(\cdot,0) = \chi_0, \quad q(\cdot,0) = q_0 \quad \text{in } \Omega.
\end{align}

This new formulation turns out to be quite convenient to deal with, because of the homogeneous boundary conditions for $\theta$ and $q$ in (18). In the sequel the right hand side of (15) will be denoted by $f$ and, noting that the right hand side of (16) contains a factor $\alpha$, we will set

\begin{align}
\mathbf{h}_\alpha &= -\nabla \theta_D - \alpha (\partial q_N / \partial t) - q_N, \\
\mathbf{h} &= -\nabla \theta_D - q_N.
\end{align}

However the theorems stated in this paper will be valid for more general data $f$, $\mathbf{h}_\alpha$, and $\mathbf{h}$.

In most of physical applications the relaxation parameters introduced in (6) and (7) are very small with respect to the used length scale, so that the Stefan problem is often considered as approximation for the relaxed systems. Therefore it seems quite important to know the asymptotic behaviour of phase relaxation problems based on (4), (6) as $\varepsilon$ goes to zero, as well as the asymptotic investigation of thermodynamic models including (7) as $\alpha$ approaches 0.

In view of these facts it appears quite natural to wonder whether the solutions of the hyperbolic phase relaxation problem (15–19) converge, in a suitable topology, to the solution of the Stefan problem when both the two relaxation coefficients $\alpha$
and $\varepsilon$ tend to zero. In paper [5] P. Colli and the author answer affirmatively to this question. After proving in a rigorous way that (15–19) admits at least one solution $(\theta, \chi, q)$, they argue on a triplet $(\theta_{ae}, \chi_{ae}, q_{ae})$ which is any of its solutions, for $\alpha > 0$ and $\varepsilon > 0$. Then, they show that, as $\alpha, \varepsilon \searrow 0$, the family $(\theta_{ae}, \chi_{ae})$ converges to the solution $(\theta, \chi)$ of the Stefan problem (cf. (4–5) and (12–14))

\begin{align}
\frac{\partial(\theta + \chi)}{\partial t} - \Delta \theta &= g - \frac{\partial \theta_D}{\partial t} + \Delta \theta_D \quad \text{in } Q, \\
\chi &\in H(\theta + \theta_D) \quad \text{in } Q, \\
\theta &= 0 \quad \text{on } \Gamma_0 \times (0, T), \quad \partial_n \theta = 0 \quad \text{on } \Gamma_1 \times (0, T), \\
(\theta + \chi)(\cdot, 0) &= \theta_0 + \chi_0 \quad \text{in } \Omega,
\end{align}

where $\partial_n$ denotes the outward normal derivative to the boundary of $\Omega$ and the initial condition (25) is the one agreeing with (22). It is worth noting that in the asymptotic analysis performed in [5], no relation is required between $\alpha$ and $\varepsilon$ as they tend to zero.

In the present note we want to extend the asymptotic analysis studied in [5] deducing error estimates for the sequences $\theta_{ae} - \theta$ and $q_{ae} - q$ with respect to the relaxation parameters $\alpha$ and $\varepsilon$. To this aim in Section 2 we give the weak formulations of the Stefan problem and of the hyperbolic Stefan problem. Then we will recall the results obtained in paper [5] and we will exploit them in Section 3 to infer the desired error estimate.

2. – Variational formulations, known theorems and new results

In this section we give the variational formulations of the problems presented in the Introduction and we recall the related existence/uniqueness and convergence results proved in [5]. Finally we will state the theorem which is the object of our note. Concerning the data of the problems we assume that

(H1) $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with Lipschitz boundary $\Gamma := \partial \Omega$. The outward normal unit vector will be denoted by $n$.

(H2) $\Gamma_0$ and $\Gamma_1$ are open subsets of $\Gamma$ such that $\Gamma_0 \cup \Gamma_1 = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\Gamma_0 \cap \Gamma_1$ is of Lipschitz class.

(H3) $Q := \Omega \times (0, T)$, where $T$ is a positive number.

(H4) $\alpha$ and $\varepsilon$ are positive numbers.

(H5) $f \in L^1(0, T; L^2(\Omega)) \cap L^2(0, T; (H^1_{\Gamma_0}(\Omega))^\prime)$, where $H^1_{\Gamma_0}(\Omega) = \{ v \in H^1(\Omega) : v|_{\Gamma_0} = 0 \}$.

(H6) $\theta_D \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \equiv H^1(Q)$.

(H7) $h \in L^2(0, T; (L^2(\Omega))^n)$ and $(h_\alpha)_{\alpha > 0}$ is a family of functions in $L^2(0, T; (L^2(\Omega))^n)$ such that $h_\alpha \to h$ in $L^2(0, T; (L^2(\Omega))^n)$ as $\alpha \searrow 0$. 

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(H8) $\theta_0 \in L^2(\Omega)$, $\chi_0 \in L^\infty(\Omega)$ and $0 \leq \chi_0 \leq 1$ a.e. in $\Omega$, $q_0 \in (L^2(\Omega))^n$.

**Remark 2.1.** Concerning the data of the system (15–19), we point out that the assumption (H6) on $\theta_D$ and the regularities $g \in L^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))^n$ and $q_N \in H^1(0,T;(L^2(\Omega))^n) \cap L^2(0,T;L^2_{\text{div}}(\Omega))$ actually ensure that (H5)–(H7) hold (the definition of the space $L^2_{\text{div}}(\Omega)$ is recalled just below). Let us also observe that the $(n-1)$-dimensional Hausdorff measure of $\Gamma_0$ is not required to be strictly positive.

Concerning the notation, we set $H := L^2(\Omega)$ and $V := H^1_0(\Omega)$, endow $H$ and $V$ with the usual inner products, and identify $V$ with its dual space. Then we have $V \subset H \subset V'$ with dense and compact embeddings. We define the operator $A \in \mathcal{L}(V,V')$ by

$$v \mapsto \langle Av, v \rangle_V := \int_\Omega \nabla v \cdot \nabla v, \quad v \in V,$$

where the dot stands for the usual inner product in $\mathbb{R}^n$. Next, we consider the spaces $H := (L^2(\Omega))^n$ and $L^2_{\text{div}}(\Omega) := \{v \in H : \text{div } v \in H\}$, the latter endowed with the inner product

$$\langle v_1, v_2 \rangle_{L^2_{\text{div}}(\Omega)} := \langle v_1, v_2 \rangle_H + \langle \text{div } v_1, \text{div } v_2 \rangle_H, \quad v_1, v_2 \in L^2_{\text{div}}(\Omega).$$

It is well-known that if $v \in L^2_{\text{div}}(\Omega)$, then $v \cdot n \in H^{-1/2}(\Gamma)$ and the restriction $v \cdot n|_{\Gamma_i}$ makes sense in $(H^{1/2}(\Gamma_i))^n$ (see, e.g., [7]). In this functional framework we introduce the closed subspace of $L^2_{\text{div}}(\Omega)$

$$V := \{v \in L^2_{\text{div}}(\Omega) : v \cdot n|_{\Gamma_i} = 0\}.$$

If we identify $H$ with its dual space, we get $V \subset H \subset V'$ with dense and continuous embeddings. Moreover, we will consider the operators $B \in \mathcal{L}(H,V')$ and $L \in \mathcal{L}(H,H')$ defined by

$$v \mapsto \langle Bu, v \rangle_V := -\int_\Omega u \cdot \nabla v, \quad u \in H, \ v \in V,$$

$$v \mapsto \langle Lu, v \rangle_V := \int_\Omega \text{div } v, \quad u \in H, \ v \in V.$$

We now recall some well-known statements that will be useful in the sequel.

**Lemma 2.1** Let $v_0 \in H$. If there exists a function $u_0 \in H$ such that $Bu_0 = v_0$, i.e.,

$$\langle Bu_0, v \rangle_V = -\int_\Omega v_0 \cdot \nabla v = \int_\Omega u_0 v \quad \forall v \in V,$$

then $v_0 \in V$, div $v_0 = u_0$, and $\|v_0\|_V \leq \|v_0\|_H + \|u_0\|_H$.

**Lemma 2.2** Let $u_0 \in H$. If there is a function $v_0 \in H$ such that $Lu_0 = v_0$, i.e.,

$$\langle Lu_0, v \rangle_V = \int_\Omega \text{div } v_0 \cdot v \quad \forall v \in V,$$

then $u_0 \in V$, $v_0 = -\nabla u_0$, and $\|u_0\|_V \leq \|u_0\|_H + \|v_0\|_H$. 

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Then the weak formulation of the problem (15–19) reads as follows.

**Problem** (\(P_{\alpha\varepsilon}\)). Find a triplet \((\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, q_{\alpha\varepsilon})\) satisfying the following conditions

(33) \[\theta_{\alpha\varepsilon} \in H^1(0, T; V') \cap L^\infty(0, T; H),\]
(34) \[\chi_{\alpha\varepsilon} \in H^1(0, T; H), \quad 0 \leq \chi_{\alpha\varepsilon} \leq 1 \quad \text{a.e. in } Q,\]
(35) \[q_{\alpha\varepsilon} \in H^1(0, T; V') \cap L^\infty(0, T; H),\]
(36) \[(\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon})' + Bq_{\alpha\varepsilon} = f \quad \text{in } V', \quad \text{a.e. in } (0, T),\]
(37) \[\alpha q_{\alpha\varepsilon}' + q_{\alpha\varepsilon} = L\theta_{\alpha\varepsilon} + h_{\alpha} \quad \text{in } V', \quad \text{a.e. in } (0, T),\]
(38) \[\varepsilon \chi_{\alpha\varepsilon}' + H^{-1}(\chi_{\alpha\varepsilon}) \ni \theta_{\alpha\varepsilon} + \theta_D \quad \text{a.e. in } Q,\]
(39) \[\theta_{\alpha\varepsilon}(0) = \theta_0 \quad \text{in } V', \quad \chi_{\alpha\varepsilon}(0) = \chi_0 \quad \text{in } H, \quad q_{\alpha\varepsilon}(0) = q_0 \quad \text{in } V'.\]

Here and in what follows the symbol “\(\prime\)” will denote the derivative with respect to time of vector-valued functions. The boundary conditions in (18) are not included in (33) and (35), since the spaces \(V\) and \(V'\), respectively, do not appear there. On the other hand, the analogous homogeneous boundary conditions for the integrated variables \(\int_0^T \theta_{\alpha\varepsilon}, \int_0^T q_{\alpha\varepsilon}\) are collected into equations (36–37), and this can be easily checked with the help of integrations in time and using Lemmas 2.1–2.2.

The following existence theorem for Problem \((P_{\alpha\varepsilon})\) was proved in [5, Theorem 2.1].

**Theorem 2.1** Assume that \((H1)–(H8)\) hold. Then Problem \((P_{\alpha\varepsilon})\) admits at least one solution. Moreover there exists a constant \(C > 0\), independent of \(\alpha\) and \(\varepsilon\), such that for all solutions \((\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, q_{\alpha\varepsilon})\) of \((P_{\alpha\varepsilon})\) there holds

\[
\|\theta_{\alpha\varepsilon}\|_{L^\infty(0,T;H)} + \|\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon}\|_{H^1(0,T;V')} + \alpha^{1/2}\|q_{\alpha\varepsilon}\|_{L^\infty(0,T;H)} \\
+ \|q_{\alpha\varepsilon}\|_{L^2(0,T;H)} + \alpha\|q_{\alpha\varepsilon}'\|_{L^2(0,T;V')} + \varepsilon^{1/2}\|\chi_{\alpha\varepsilon}\|_{H^1(0,T;H)} + \|\chi_{\alpha\varepsilon}\|_{L^\infty(Q)} \leq C.
\]

We want to stress that a relevant feature of the analysis performed in [5] is that estimate (40) is fulfilled by any solution of Problem \((P_{\alpha\varepsilon})\).

Now let us state the weak formulation of the Stefan problem.

**Problem** \((P)\). Find a pair \((\theta, \chi)\) satisfying the following conditions

(41) \[\theta \in L^\infty(0, T; H) \cap L^2(0, T; V),\]
(42) \[\chi \in L^\infty(Q),\]
(43) \[\theta + \chi \in H^1(0, T; V'),\]
(44) \[(\theta + \chi)' + A\theta = f - Bh \quad \text{in } V', \quad \text{a.e. in } (0, T),\]
(45) \[\chi \in H(\theta + \theta_D) \quad \text{a.e. in } Q,\]
(46) \[(\theta + \chi)(0) = \theta_0 + \chi_0 \quad \text{in } V'.\]

The next result can be easily deduced by slightly adapting the arguments reported, e.g., in [4] (see also [10, Chapter II]).
Theorem 2.2 Assume that (H1)–(H8) hold. Then there exists a unique solution to Problem $(P)$.

Note that Problem $(P)$ can be equivalently formulated saying that a triplet $(\theta, \chi, q)$ is to be found in such a way that (41–43) and (45–46) are satisfied and

\begin{align}
q &\in L^2(0, T; \mathbf{H}) \\
(\theta + \chi)' + Bq &= f \quad \text{in } V', \text{ a.e. in } (0, T), \\
q &= -\nabla \theta + h \quad \text{a.e. in } Q.
\end{align}

Consequently Theorem 2.2 can be rephrased according to this equivalent formulation.

As we recalled in the Introduction, the object of paper [5] is the asymptotic behaviour of the solutions of Problem $(P_\alpha\varepsilon)$, as $\alpha$ and $\varepsilon$ tend to zero. The precise result is the following

Theorem 2.3 Assume that hypotheses (H1)–(H8) hold. Let $(\theta, \chi)$ be the unique solution to Problem $(P)$ and let $q$ be defined by (49). Moreover for any pair $\alpha, \varepsilon > 0$ let $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, q_{\alpha\varepsilon})$ denote an arbitrary solution to Problem $(P_{\alpha\varepsilon})$. Then, as $\alpha, \varepsilon \searrow 0$, we have that

\begin{align}
\theta_{\alpha\varepsilon} &\rightharpoonup \theta \quad \text{in } L^\infty(0, T; H), \\
\chi_{\alpha\varepsilon} &\rightharpoonup \chi \quad \text{in } L^\infty(Q), \\
q_{\alpha\varepsilon} &\rightharpoonup q \quad \text{in } L^2(0, T; \mathbf{H}).
\end{align}

We warn the reader that the part of the previous theorem concerning the convergence of $\theta_{\alpha\varepsilon}$ and $\chi_{\alpha\varepsilon}$ is stated in [5, Theorem 2.2]. Instead the convergence of $q_{\alpha\varepsilon}$ is proved in Section 5 of paper [5] where is exploited to prove (50–51). However, if we assume (50–51), then (52) can be easily inferred from (37), (40), (H7), and (41).

Let us introduce a general notation which will hold throughout the sequel. For a map $\psi \in L^1(0, T; X)$, where $X$ is a Banach space, we define $\hat{\psi} : [0, T] \to X$ by

\begin{equation}
\hat{\psi}(t) := \int_0^t \psi, \quad t \in [0, T].
\end{equation}

Now we can state the theorem concerning the error estimate. In order to prove this theorem we need to prescribe a certain rate of convergence for the sequence $h_\alpha - h$. Precisely we assume that there exists a constant $C_0 > 0$, independent of $\alpha$ and $\varepsilon$, such that

\begin{equation}
\|h_\alpha - h\|_{L^2(0, T; \mathbf{H})} \leq C_0 \alpha^{1/2}
\end{equation}

for all $\alpha > 0$. This assumption seems quite reasonable, in view of the fact that in applications the expression of $h_\alpha$ and $h$ are given respectively by (20) and (21).

Theorem 2.4 Assume that the hypotheses of Theorem 2.3 hold and that (54) is valid for some positive constant $C_0$ independent of $\alpha$ and $\varepsilon$. Then there exist a constant $C_1 > 0$, independent of $\alpha$ and $\varepsilon$, such that

\begin{equation}
\|\theta_{\alpha\varepsilon} - \theta\|_{L^2(0, T; H)} + \|\hat{q}_{\alpha\varepsilon} - \hat{q}\|_{L^\infty(0, T; H)} \leq C_1 (\alpha^{1/4} + \varepsilon^{1/4}).
\end{equation}
3. – Proof of the error estimate

This section is devoted to proof of the error estimate stated in Theorem 2.4. For any pair of positive numbers \( \alpha \) and \( \varepsilon \) let us choose an arbitrary solution \((\theta_{\alpha \varepsilon}, \chi_{\alpha \varepsilon}, q_{\alpha \varepsilon})\) of Problem \((P_{\alpha \varepsilon})\). In the sequel, the same symbol \( C \) will be employed to denote different positive constants which depends only on the data, but not on \( \alpha \) and \( \varepsilon \). Let us start observing that an integration of the equation (36) shows that

\[
 z_{\alpha \varepsilon} := \hat{f} + \theta_0 + \chi_0 - \theta_{\alpha \varepsilon} - \chi_{\alpha \varepsilon} = B \hat{q}_{\alpha \varepsilon},
\]

hence, thanks to (H5), (H8), and (40), we deduce that \( z_{\alpha \varepsilon} \) is uniformly bounded in \( L^\infty(0, T; H) \) with respect to \( \alpha \) and \( \varepsilon \). Therefore

\[
 -\int_\Omega \hat{q}_{\alpha \varepsilon}(t) \cdot \nabla v = \int_\Omega z_{\alpha \varepsilon}(t) v \quad \forall v \in V
\]

and applying Lemma 2.1 we deduce that \( \|\hat{q}_{\alpha \varepsilon}\|_{L^\infty(0, T; V)} \) is bounded independently of \( \alpha, \varepsilon \), and we see that \( \hat{q} \in L^\infty(0, T; V) \) and

\[
 \hat{q}_{\alpha \varepsilon} \rightharpoonup \hat{q} \quad \text{in} \ L^\infty(0, T; V).
\]

Now, for convenience, let us set

\[
 \Theta_{\alpha \varepsilon} := \theta_{\alpha \varepsilon} - \theta, \quad X_{\alpha \varepsilon} := \chi_{\alpha \varepsilon} - \chi, \quad \text{and} \quad \Psi_{\alpha \varepsilon} := q_{\alpha \varepsilon} - q,
\]

and let \( t \in (0, T) \). Let us integrate in time the difference of equations (36) and (44). We get, thanks to (39) and (46),

\[
 \Theta_{\alpha \varepsilon} + X_{\alpha \varepsilon} + \text{div} \hat{\Psi}_{\alpha \varepsilon} = 0 \quad \text{in} \ V', \quad \text{in} \ [0, T].
\]

Let us note that \( \hat{\Psi}_{\alpha \varepsilon} = \hat{q}_{\alpha \varepsilon} - \hat{q} \in L^\infty(0, T; V) \) and \( B \hat{\Psi}_{\alpha \varepsilon} = \text{div} \hat{\Psi}_{\alpha \varepsilon} \), thus equation (59) is in fact the following identity in \( H \):

\[
 \Theta_{\alpha \varepsilon} + X_{\alpha \varepsilon} + \text{div} \hat{\Psi}_{\alpha \varepsilon} = 0 \quad \text{in} \ H, \quad \text{in} \ [0, T].
\]

Let us multiply (60) by \( \Theta_{\alpha \varepsilon} \in L^\infty(0, T; H) \) and integrate over \((0, t) \times \Omega \). We get

\[
 \|\Theta_{\alpha \varepsilon}\|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega X_{\alpha \varepsilon} \Theta_{\alpha \varepsilon} + \int_0^t \int_\Omega \text{div} \hat{\Psi}_{\alpha \varepsilon} \Theta_{\alpha \varepsilon} = 0.
\]

Now let us subtract the equation (49) from (37). We find that

\[
 \alpha q'_{\alpha \varepsilon} + \Psi_{\alpha \varepsilon} = L \Theta_{\alpha \varepsilon} + h_\alpha - h \quad \text{in} \ V', \quad \text{a.e. in} \ (0, T).
\]

Applying equation (62) to \( \hat{\Psi}_{\alpha \varepsilon} \in L^\infty(0, T; V) \) and integrating in time we have that

\[
 \alpha \int_0^t V' \langle q'_{\alpha \varepsilon}, \hat{\Psi}_{\alpha \varepsilon} \rangle v + \frac{1}{2} \|\hat{\Psi}_{\alpha \varepsilon}(t)\|^2_H = \int_0^t \int_\Omega \Theta_{\alpha \varepsilon} \text{div} \hat{\Psi}_{\alpha \varepsilon} + \int_0^t \int_\Omega (h_\alpha - h) \cdot \hat{\Psi}_{\alpha \varepsilon}.
\]

Finally, observe that the inclusion (45) is equivalent to

\[
 H^{-1}(\chi_{\alpha \varepsilon}) \ni \theta + \theta_D \quad \text{a.e. in} \ Q.
\]

If we subtract (64) from (38), we get the inclusion

\[
 \varepsilon \chi'_{\alpha \varepsilon} + H^{-1}(\chi_{\alpha \varepsilon}) - H^{-1}(\chi) \ni \Theta_{\alpha \varepsilon} \quad \text{a.e. in} \ Q.
\]
Multiplying (65) by $\mathcal{X}_{\alpha \epsilon}$ and taking into account of the monotonicity of $H^{-1}$ we infer that
\begin{equation}
\varepsilon \int_0^t \int_{\Omega} \chi'_{\alpha \epsilon} \mathcal{X}_{\alpha \epsilon} \leq \int_0^t \int_{\Omega} \Theta_{\alpha \epsilon} \mathcal{X}_{\alpha \epsilon}.
\end{equation}

If we add equations (61), (63), and (66) and we observe that there are two cancellations, we obtain the following inequality:
\begin{equation}
\| \Theta_{\alpha \epsilon} \|^2_{L^2(0,t;H)} + \frac{1}{2} \| \hat{\Psi}_{\alpha \epsilon}(t) \|^2_{H} \\
\leq -\alpha \int_0^t \langle q_{\alpha \epsilon}'; \hat{\Psi}_{\alpha \epsilon} \rangle_V + \int_0^t \int_{\Omega} (\mathbf{h}_\alpha - \mathbf{h}) \cdot \hat{\Psi}_{\alpha \epsilon} - \varepsilon \int_0^t \int_{\Omega} \chi'_{\alpha \epsilon} \mathcal{X}_{\alpha \epsilon}.
\end{equation}

Now we are going to estimate the right hand side of (67). Concerning the first integral, observe that $q_{\alpha \epsilon} \in H^1(0,T;V') \cap L^\infty(0,T;H)$ and $\hat{\Psi}_{\alpha \epsilon} \in H^1(0,T;H) \cap L^\infty(0,T;V)$. Therefore [5, Lemma 5.1] applies and let us deduce that the function $t \mapsto \langle q_{\alpha \epsilon}(t), \hat{\Psi}_{\alpha \epsilon}(t) \rangle_V$ is absolutely continuous. Then we find a positive constant $C$ such that
\begin{equation}
-\alpha \int_0^t \langle q_{\alpha \epsilon}', \hat{\Psi}_{\alpha \epsilon} \rangle_V \\
= -\alpha \langle q_{\alpha \epsilon}(t), \hat{\Psi}_{\alpha \epsilon}(t) \rangle_V + \alpha \int_0^t \langle q_{\alpha \epsilon}, \Psi_{\alpha \epsilon} \rangle_H \\
\leq \alpha \| q_{\alpha \epsilon}(t) \|_H \| \hat{\Psi}_{\alpha \epsilon}(t) \|_H + \alpha \| q_{\alpha \epsilon} \|_{L^2(0,t;H)} \| \Psi_{\alpha \epsilon} \|_{L^2(0,t;H)} \\
\leq C \alpha^{1/2},
\end{equation}

the last inequality holding by virtue of (40), (58), and (52). The second integral in (67) can be controlled by observing that, thanks to (54),
\begin{equation}
\int_0^t \int_{\Omega} (\mathbf{h}_\alpha - \mathbf{h}) \cdot \hat{\Psi}_{\alpha \epsilon} \leq \| \mathbf{h}_\alpha - \mathbf{h} \|_{L^2(0,t;H)} \| \hat{\Psi}_{\alpha \epsilon} \|_{L^2(0,t;H)} \leq C_0 \alpha^{1/2},
\end{equation}

for some positive constant $C$. Finally, thanks to (40) and to (51) we have that
\begin{equation}
-\varepsilon \int_0^t \int_{\Omega} \chi'_{\alpha \epsilon} \mathcal{X}_{\alpha \epsilon} \leq -\varepsilon \| \chi'_{\alpha \epsilon} \|_{L^2(0,t;H)} \| \mathcal{X}_{\alpha \epsilon} \|_{L^2(0,t;H)} \\
= -\varepsilon^{1/2} \| \chi'_{\alpha \epsilon} \|_{L^2(0,t;H)} \| \mathcal{X}_{\alpha \epsilon} \|_{L^2(0,t;H)} \leq C \varepsilon^{1/2}.
\end{equation}

Hence collecting (68), (69), and (70), inequality (67) entails that
\begin{equation}
\| \theta_{\alpha \epsilon} - \theta \|_{L^2(0,T;H)}^2 + \| \tilde{q}_{\alpha \epsilon} - \tilde{q} \|_{L^\infty(0,T;H)}^2 \leq C(\alpha^{1/2} + \varepsilon^{1/2}),
\end{equation}

and (55) is proved.

REFERENCES


