The supermartingale property of the optimal portfolio process for general semimartingales

Sara Biagini*

Here we develop some new aspects in the utility maximization from terminal wealth in (very) general, incomplete, market models. In a previous article of ours (Biagini and Frittelli, ’Utility maximization in incomplete markets for unbounded processes’) we extended the existing theory to cover the case where:

- $u : \mathbb{R} \to \mathbb{R}$ is a strictly concave, regular utility function;
- the underlying semimartingale $X$ can be possibly non locally bounded

The extension relies upon a new definition of admissible strategies, for agents who are willing to take more risk. Hence we built a perfectly sensible utility maximization problem and we showed that the optimal claim $\hat{f}$ admits an integral representation as soon as the minimax measure is equivalent to $P$. Namely, $\hat{f} = (\hat{H} \cdot X)_T$.

Unfortunately, the strategy $\hat{H}$ which leads to the terminal optimal wealth may not be admissible, even in our wider sense. This phenomenon is not surprising, as it appears also in the locally bounded case.

So, we investigate on the properties of the optimal process $\hat{H} \cdot X$. We prove that $\hat{H} \cdot X$ is in fact a supermartingale (true martingale if the utility is exponential) with respect to the relevant pricing measures in our general setting, i.e. the $\sigma$-martingale measures for $X$ with finite entropy.

This result can be seen as the fourth step in the following path:

1. Six Authors’ paper 2002. When $X$ is locally bounded, the utility is exponential and a technical condition holds (the Reverse Hölder Inequality), they proved that the optimal wealth process is a true martingale wrt every local martingale measure $Q$ with finite entropy.

2. Kabanov and Stricker 2002 removed the superfluous RHI.

3. Schachermayer 2003 proved that if $\hat{Q} \sim P$, then the $\hat{H} \cdot X$ is a supermartingale under every local martingale measure with finite entropy (the ’true martingale’ property of the solution is lost when $u$ is general).

Then the supermartingale (martingale in case of exponential $u$) property of the optimal portfolio continues to hold even in the general, possibly non locally bounded, case.

*Joint work with M. Frittelli, DiMaD, Firenze.