Risk processes with delayed claims: ruin probability estimates under heavy tail conditions

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Risk processes with delayed claims

Insurance company: initial capital $u > 0$, premium rate $c > 0$, total claim amount $\{S(t)\}$, capital at time $t$

$$u + ct - S(t)$$

Delay in claim settlement (IBNR and RBNS claims): Klüppelberg and Mikosch (Bernoulli, 1995)

Poisson shot noise

$$S(t) = \sum_{n \geq 1} H(t - T_n, Z_n)1_{(0,t]}(T_n)$$

claims arrivals $\{T_n\}$, $T_0 = 0$, $\lambda > 0$; $\{Z_n\}$ iid $E$-valued rv’s idp of $\{T_n\}$; $H: \mathbb{R} \times E \rightarrow [0, \infty)$ measurable function st $H(\cdot, z) \nearrow, \forall z \in E$, $H(t, z) = 0 \ \forall t \leq 0, \ z \in E$

Interpretation: $H(t - T_n, Z_n)$ quantity paid at $t$ for claim at $T_n$, $H(\infty, Z_n)$ total claim per accident, associated Cramér-Lundberg (C-L) model
Ruin probabilities

Infinite horizon ruin probability:

$$\psi(u) = P\left(\sup_{t \geq 0} (S(t) - ct) \geq u\right)$$

Finite horizon ruin probability:

$$\psi(u, T) = P\left(\sup_{t \in [0, T]} (S(t) - ct) \geq u\right), \quad T > 0 \text{ fixed}$$

Problem:

Asymptotic behavior of ruin probabilities, as $$u \to \infty$$, under heavy tail conditions on $$H(\infty, Z)$$?
Light tail case: infinite horizon

Brémaud (JAP, 2000): Assume

\((\text{NP})\): \(\rho = \lambda E[H(\infty, Z)]/c < 1\)

\((\text{LT})\): \(E[e^{\theta H(\infty, Z)}] < \infty \ \forall \theta \in (0, \eta), \ \eta > 0\)

Then

\[
\lim_{u \to \infty} \frac{1}{u} \ln \psi(u) = -w
\]

\(w > 0\) solution of \(\lambda(E[e^{wH(\infty, Z)}] - 1) - cw = 0\)

Insensitivity: \(w\) coincides with the Lundberg parameter of the associated C-L model

LD techniques
Light tail case: finite horizon

Macci, Stabile and T. (SAJ, 2005): Assume (NP) and (LT). Then

$$\lim_{u \to \infty} \frac{1}{u} \ln \psi(u, uT) = -w(T)$$

where

$$w(T) = \begin{cases} Tf^*(1/T + c) & \text{if } T \in (0, 1/(f'(w) - c)) \\ w & \text{otherwise} \end{cases}$$

Here

$$f(\theta) = \ln E[e^{\theta H(\infty, Z)}] \quad f^*(x) = \sup_{\theta \in \mathbb{R}} (\theta x - f(\theta))$$

Insensitivity: $w(T)$ coincides with that one for the associated C-L model

LD techniques
Extreme value distributions

\[ X_1, \ldots, X_n \text{ iid rv's df } G \]

\[ M_n = \max\{X_1, \ldots, X_n\} \]

If \( \exists c_n > 0, d_n \in \mathbb{R} \), rv \( \tilde{X} \) with df \( \tilde{G} \) st

\[ c_n^{-1}(M_n - d_n) \xrightarrow{d} \tilde{X} \]

then \( G \) is said to be in the maximum domain of attraction of \( \tilde{G} \) (\( G \in MDA(\tilde{G}) \)).

We consider two (of the three) extreme value distributions:

\[ \tilde{G} = \Lambda, \quad \Lambda(x) = e^{-e^{-x}} \quad x \in \mathbb{R} \quad \text{(Gumbel)} \]

\[ \tilde{G} = \Phi_{\alpha}, \quad \Phi_{\alpha}(x) = e^{-x^{-\alpha}} \quad x > 0, \alpha > 0 \quad \text{(Fréchet)} \]
Extreme value theory and heavy tail distributions

A positive $g$ on $(0, \infty)$ is regularly varying at infinity of index $\alpha \in \mathbb{R}$ \((g \in \mathcal{R}(\alpha))\) iff \(g(x) \sim x^{\alpha}L(x)\), \(L\) slowly varying, ie \(L(tx) \sim L(x) \ \forall t > 0\). Here \(g_1 \sim g_2\) iff \(g_1/g_2 \to 1\) (ae)

It holds

\[ \overline{G} \in \mathcal{R}(-\alpha) \ \text{iff} \ G \in \text{MDA}(\Phi_{\alpha}), \ \alpha > 0 \]

Here \(\overline{G} = 1 - G\)

A df \(G\) is subexponential \((G \in S)\) if

\[ \text{supp}(G) = (0, \infty) \ \text{and} \ \overline{G}^{*2} \sim 2\overline{G} \]

\(S = \mathcal{R} \cup (S \setminus \mathcal{R})\), it’s further classified using EVT: Goldie and Resnick (AdvAP, 1988): if \(G \in S\) and smoothness cdts on \(G\) then \(G \in \text{MDA}(\Phi_{\alpha})\) or \(G \in \text{MDA}(\Lambda)\)
Further assumptions on the model, and notation

We consider shot shapes of the multiplicative form:

\[ H(t, z) = F(t)z \quad \text{for some df } F \]

Condition \( \textbf{(NP)} \) reads

\[ \rho = \lambda \frac{E[Z]}{c} < 1 \]

and is always assumed throughout this talk.

\( B \) denotes the df of \( Z \), and \( B_0 \) its integrated tail df:

\[ B_0(u) = \frac{1}{E[Z]} \int_u^\infty B(x) \, dx \quad u > 0 \]

Mean excess function of \( Z \):

\[ e(u) = E[Z - u \mid Z > u] \]
Regularly varying case: infinite horizon

**Theorem**

If $\overline{B} \in \mathcal{R}(-\alpha - 1)$, $\alpha > 0$, then

\[ \psi(u) \sim \frac{\rho}{1 - \rho} \overline{B}_0(u) \]

(\textbf{AE})

The result holds if $\{T_n\}_{n \geq 1}$, $T_0 = 0$, renewal process

Insensitivity: $\psi(u)$ is ae to the ruin probability of the C-L model
\textbf{Theorem}

(a) If $B_0 \in S$ and $F$ compact support then \textbf{(AE)} holds

(b) If $B \in \text{MDA}(\Lambda)$, $B_0 \in S$, $e \sim g$, $g$ eventually $\nearrow$, and for some $\gamma > 0$:

\begin{enumerate}[(SD1)]
\item $uF(u^{1/\gamma}) = o(e(u))$
\item $u^{1/\gamma} = o(e(u))$
\end{enumerate}

Then \textbf{(AE)} holds

Again, the result holds if $\{T_n\}_{n \geq 1}$, $T_0 = 0$, renewalal process
Examples

Assumptions of Theorem(b) satisfied by heavy tail distributions of practical interest:

- **Weibull distribution**: \( \overline{B}(u) = e^{-u^\alpha}, \ u \geq 0, \ \alpha \in (0,1) \). Then \( B \in MDA(\Lambda), \ B_0 \in S \) and

\[
e(u) \sim g(u) = \frac{u^{1-\alpha}}{\alpha}
\]

Cdots (SD1), (SD2) hold if \( \exists \gamma > 1/(1-\alpha) \) st \( F(u) = o(u^{-\gamma \alpha}) \). In particular, if \( F \) is light tail or \( F \in \mathcal{R}(-\beta), \ \beta > \alpha/(1-\alpha) \)

- **Lognormal distribution**: \( \Phi(\cdot) \) df standard normal, take \( B(u) = \Phi((\ln u - \omega)/\sigma), \ u > 0, \ \omega \in \mathbb{R}, \ \sigma > 0 \). Then \( B \in MDA(\Lambda), \ B_0 \in S \) and

\[
e(u) \sim g(u) = \frac{\sigma^2 u}{\ln u - \omega}
\]

Cdt (SD1) equivalent to \( \overline{F}(u) = o(1/\ln u) \), irrespective of \( \gamma \); (SD2) holds if \( \gamma > 1 \)
Heavy tail case: finite horizon

Theorem

(a) If $\bar{B} \in \mathcal{R}(-\alpha - 1)$, $\alpha > 0$, then

$$\psi(u, uT) \sim (1 - (1 + (1 - \rho)T)^{-\alpha})\psi(u)$$

(b) Assume conditions of previous Theorem(b) with $\gamma > 1$ and $g$ regularly varying. Then

$$\psi(u, e(u)T) \sim (1 - e^{-(1-\rho)T})\psi(u).$$

Condition (b) satisfied for all the examples considered earlier

Meaning: conditional on ruin occurring, time to ruin divided by $u$ converges in distribution to Pareto under (a); time to ruin divided by $e(u)$ converges in distribution to Exponential under (b)
Regularly varying case and infinite horizon: Sketch of the Proof

Upper bound: Use that the associated C-L dominates the model with delayed claims

Lower bound:
Idea: Fix $a > 0$, replace continuous claim flow of each event $T_n$ until $T_n + a$ by a claim of size $F(a)Z_n$ then

$$\psi(u) \geq \Psi_a((u + ca)/F(a)) \quad \text{for all } a, u > 0$$

$\Psi_a$ infinite horizon ruin probability of

$$C_a(t) = \sum_{n \geq 1} Z_n 1_{(0,t]}(T_n) - (c/F(a))t$$

Conclusion: classical estimate for C-L model, regular variation and limit as $a \to \infty$
More general shot shapes?

*Theorem* (T. 2008)

Assume

(a) Either $H(t, z) = F(t)z$ or $\sup_{z \in E} |H(t, z) - H(\infty, z)| \to 0$

(b) $P(H(\infty, Z_1) \geq x) = \exp(-x^\alpha L(x))$, $x > 0$, $\alpha \in (0, 1)$, $L$ slowly varying (semiexponential)

(c) $c > \lambda \mathbb{E}[H(\infty, Z_1)]$

Then

$$\ln \psi(u) \sim -u^\alpha L(u)$$