Regression-based algorithms for life insurance contracts with surrender guarantees

Anna Rita Bacinello
Dipartimento di Matematica Applicata ‘B. de Finetti’ - University of Trieste

Enrico Biffis
Finance Group, Tanaka Business School, Imperial College, London

Pietro Millossovich
Dipartimento di Matematica Applicata ‘B. de Finetti’ - University of Trieste

Options in life insurance

Options embedded in life insurance contracts:

- European style with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, . . . );
Options in life insurance

Options embedded in life insurance contracts:

▷ **European style** with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, . . . );

▷ **American style**: the policyholder has the right to make some well-specified actions before the natural termination of the contract ⇒ early termination feature.
Options in life insurance

Options embedded in life insurance contracts:

- **European style** with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, . . . );

- **American style**: the policyholder has the right to make some well-specified actions before the natural termination of the contract ⇒ early termination feature.

- Most common American option is the **surrender option**: the policyholder has the right to early terminate the contract and receive a cash amount, called **surrender value**
Options in life insurance

Options embedded in life insurance contracts:

- **European style** with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, . . . );

- **American style**: the policyholder has the right to make some well-specified actions before the natural termination of the contract ⇒ early termination feature.

- Most common American option is the **surrender option**: the policyholder has the right to early terminate the contract and receive a cash amount, called surrender value ⇒ (non-standard) American put option on the residual contract with the surrender value as exercise price.
Options in life insurance

Options embedded in life insurance contracts:

- **European style** with possibly random maturity ⇒ Titanic option [Milevsky and Posner - JRI(01)] (minimum guarantees, bonus options, conversion options, ...);

- **American style**: the policyholder has the right to make some well-specified actions before the natural termination of the contract ⇒ early termination feature.

- Most common American option is the **surrender option**: the policyholder has the right to early terminate the contract and receive a cash amount, called surrender value ⇒ (non-standard) American put option on the residual contract with the surrender value as exercise price.

- If mortality risk can be diversified away (by pooling), then a Titanic option can be reduced to a portfolio of European options with different maturities; this does not apply to American options ⇒ valuation problem.
Approaches to valuation

- **BINOMIAL TREES (AND EXTENSIONS):** e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)], [Vannucci - wp(03)], ...
Approaches to valuation

- **BINOMIAL TREES (AND EXTENSIONS):** e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)], [Vannucci - wp(03)], ...;

- **FINITE DIFFERENCE METHODS:** e.g. [Jensen et al. - GPRIT(01)], [Tanskanen and Lukkarinen - IME(03)], [Shen and Xu - IME(05)], [Moore and Young - NAAJ(03)];
Approaches to valuation

▶ **Binomial Trees (and Extensions):** e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)], [Vannucci - wp(03)], …;

▶ **Finite Difference Methods:** e.g. [Jensen et al. - GPRIT(01)], [Tanskanen and Lukkarinen - IME(03)], [Shen and Xu - IME(05)], [Moore and Young - NAAJ(03)];

▶ **Monte Carlo Simulation:** e.g. [Andreatta and Corradin wp(03)], [Baione et al. IMFI(06)], [Bacinello MAF(08)].
Approaches to valuation

▷ **BINOMIAL TREES (AND EXTENSIONS):** e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)], [Vannucci - wp(03)], . . .

▷ **FINITE DIFFERENCE METHODS:** e.g. [Jensen et al. - GPRIT(01)], [Tanskanen and Lukkarinen - IME(03)], [Shen and Xu - IME(05)], [Moore and Young - NAAJ(03)];

▷ **MONTE CARLO SIMULATION:** e.g. [Andreatta and Corradin wp(03)], [Baione et al. IMFI(06)], [Bacinello MAF(08)].

▷ Complexity of the problem involved ⇒ oversimplified assumptions:

   ★ no (or not realistic) mortality risk modelling;

   ★ Restrictive hypotheses on processes of concern (constant interest rates, GBM, . . .).
Approaches to valuation

▶ *BINOMIAL TREES (AND EXTENSIONS)*: e.g. [Grosen and Jorgensen - IME(00)], [Bacinello - JRI(03), NAAJ(03), IME(05)], [Vannucci - wp(03)], ...;

▶ *FINITE DIFFERENCE METHODS*: e.g. [Jensen et al. - GPRIT(01)], [Tanskanen and Lukkarinen - IME(03)], [Shen and Xu - IME(05)], [Moore and Young - NAAJ(03)];

▶ *MONTE CARLO SIMULATION*: e.g. [Andreatta and Corradin wp(03)], [Baione et al. IMFI(06)], [Bacinello MAF(08)].

▶ Complexity of the problem involved ⇒ oversimplified assumptions:

  ★ no (or not realistic) mortality risk modelling;
  ★ Restrictive hypotheses on processes of concern (constant interest rates, GBM, ...).

▶ Monte Carlo simulation combined with LS regression (*LSMC*, [Carrière - IME(96)], [Longstaff and Schwarz - RFS(01)]) allows to overcome such drawbacks.
We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.
Target

▶ We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

▶ Mortality enters the LSMC algorithm as any other variable;
We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

Mortality enters the LSMC algorithm as any other variable;

- surrender only in case of survival;
- underlying variable depends on mortality.
We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

Mortality enters the LSMC algorithm as any other variable;

- surrender only in case of survival;
- underlying variable depends on mortality.

Aim at fair valuation in a frictionless and arbitrage-free market ⇒ consistent with IASB proposal;
Target

▷ We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

▷ Mortality enters the LSMC algorithm as any other variable;
  - surrender only in case of survival;
  - underlying variable depends on mortality.

▷ Aim at fair valuation in a frictionless and arbitrage-free market ⇒ consistent with IASB proposal;

▷ market incompleteness ⇒ choice of a pricing measure;
Target

- We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

- **Mortality** enters the LSMC algorithm as any other variable;
  - ★ surrender only in case of survival;
  - ★ underlying variable depends on mortality.

- Aim at **fair valuation** in a frictionless and arbitrage-free market ⇒ consistent with IASB proposal;

- market incompleteness ⇒ choice of a pricing measure;

- choice of pricing measure ⇒ price of early termination option (?);
We exploit the flexibility of LSMC to show how a general life insurance contract embedding a surrender option can be valued even under realistic assumptions.

- **Mortality** enters the LSMC algorithm as any other variable;
  - surrender only in case of survival;
  - underlying variable depends on mortality.

- Aim at **fair valuation** in a frictionless and arbitrage-free market \(\Rightarrow\) consistent with IASB proposal;

- market incompleteness \(\Rightarrow\) choice of a pricing measure;

- choice of pricing measure \(\Rightarrow\) price of early termination option (\(?\));

- numerical example:
  - unit-linked endowment insurance with **terminal** or **cliquet** guarantees;
  - interest rates: CIR
    - reference portfolio: GBM+SV+J
    - mortality: time dependent coefficients square root+J.
Valuation framework

Given is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, P)\).
Valuation framework

▷ Given is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, P)\).

▷ Policyholder time of death (or residual lifetime) \(\tau\) is a \(\mathcal{G}\)-stopping time (s.t.).
Valuation framework

▷ Given is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, P)\).

▷ Policyholder time of death (or residual lifetime) \(\tau\) is a \(\mathcal{G}\)-stopping time (s.t.).

▷ Insurance contract (without surrender):

  * cumulated death benefit: \(D^d_t = B^d_{\tau} 1\{\tau \leq t\}\);
  * cumulated survival benefit: \(D^s_t = \int_0^t 1\{\tau > u\} dB^s_u\);
  * total cumulated benefit: \(D_t = D^d_t + D^s_t\).

Most common contracts are included in the above set-up for appropriate choices of processes \(B^d\) and \(B^s\).
Valuation framework

▷ Given is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, P)\).

▷ Policyholder time of death (or residual lifetime) \(\tau\) is a \(\mathcal{G}\)-stopping time (s.t.).

▷ Insurance contract (without surrender):
  
  * cumulated death benefit: \(D^d_t = B^d_\tau 1_{\{\tau \leq t\}}\);
  
  * cumulated survival benefit: \(D^s_t = \int_0^t 1_{\{\tau > u\}} dB^s_u\);
  
  * total cumulated benefit: \(D_t = D^d_t + D^s_t\).

  Most common contracts are included in the above set-up for appropriate choices of processes \(B^d\) and \(B^s\).

▷ Insurance contract with surrender; for any exercise policy \(\theta\) (\(\mathcal{G}\)-s.t.):
  
  * cumulated surrender benefit: \(D^w_t(\theta) = B^w_\theta 1_{\{\theta \leq t, \theta < \tau\}}\);
Valuation framework

- Given is a filtered probability space \((\Omega, \mathcal{F}, \mathcal{G}, P)\).
- Policyholder time of death (or residual lifetime) \(\tau\) is a \(\mathcal{G}\)-stopping time (s.t.).
- Insurance contract (without surrender):
  - cumulated death benefit: \(D^d_t = B^d_t 1\{\tau \leq t\}\);
  - cumulated survival benefit: \(D^s_t = \int_0^t 1\{\tau > u\} dB^s_u\);
  - total cumulated benefit: \(D_t = D^d_t + D^s_t\).

Most common contracts are included in the above set-up for appropriate choices of processes \(B^d\) and \(B^s\).

- Insurance contract with surrender; for any exercise policy \(\theta\) (\(\mathcal{G}\)-s.t.):
  - cumulated surrender benefit: \(D^w_t(\theta) = B^w_\theta 1\{\theta \leq t, \theta < \tau\}\);
  - total cumulated benefit: \(D^\wedge_t + D^w_t(\theta)\)
Valuation Framework

- Fix a risk neutral probability \( Q (\sim P) \) under which discounted (at the risk-free rate) cumulated gain for any security is a \( Q \)-martingale.
Valuation Framework

- Fix a risk neutral probability $Q (∼ P)$ under which discounted (at the risk-free rate) cumulated gain for any security is a $Q$-martingale.

- Very convenient if (see [Milevsky and Promislow - IME(01)], [Biffis - IME(05)], [Dahl - IME(05)], ...)
  
  \[ G = F ∨ H \] where $H$ is generated by $τ$;
Fix a risk neutral probability $Q \sim P$ under which discounted (at the risk-free rate) cumulated gain for any security is a $Q$-martingale.

Very convenient if (see [Milevsky and Promislow - IME(01)], [Biffis - IME(05)], [Dahl - IME(05)], ...)

- $G = F \vee H$ where $H$ is generated by $\tau$;
- $\tau$ is $G$-Cox with $F$-predictable force of mortality $(\mu_t)$ ($\Rightarrow$ easy to simulate);
Fix a risk neutral probability $Q \sim P$ under which discounted (at the risk-free rate) cumulated gain for any security is a $Q$-martingale.

Very convenient if (see [Milevsky and Promislow - IME(01)], [Biffis - IME(05)], [Dahl - IME(05)], ...):

- $G = F \vee H$ where $H$ is generated by $\tau$;
- $\tau$ is $G$-Cox with $F$-predictable force of mortality $(\mu_t)$ ($\Rightarrow$ easy to simulate);
- any other process of interest (e.g. $B^s$, $B^d$ or $B^w$) is $F$-adapted (or predictable).
... Valuation Framework

▷ Fix a risk neutral probability $Q (\sim P)$ under which discounted (at the risk-free rate) cumulated gain for any security is a $Q$-martingale.

▷ Very convenient if (see [Milevsky and Promislow - IME(01)], [Biffis - IME(05)], [Dahl - IME(05)], ... )

• $\mathcal{G} = \mathcal{F} \vee \mathcal{H}$ where $\mathcal{H}$ is generated by $\tau$;
• $\tau$ is $\mathcal{G}$-Cox with $\mathcal{F}$-predictable force of mortality $(\mu_t)$ ($\Rightarrow$ easy to simulate);
• any other process of interest (e.g. $B^s$, $B^d$ or $B^w$) is $\mathcal{F}$-adapted (or predictable).

▷ Value of the contract with surrender option (for fixed $\theta$): $V_0^w(\theta)$; the value of the contract is given by the optimal stopping problem

$$V_0^{w*} = \sup_{\theta \in \mathcal{T}} V_0^w(\theta)$$

where $\mathcal{T} = \text{set of } \mathcal{G}$-stopping times.
Fix a risk neutral probability $Q$ ($\sim P$) under which discounted (at the risk-free rate) cumulated gain for any security is a $Q$-martingale.

Very convenient if (see [Milevsky and Promislow - IME(01)], [Biffis - IME(05)], [Dahl - IME(05)], ...)

- $\mathcal{G} = \mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is generated by $\tau$;
- $\tau$ is $\mathcal{G}$-Cox with $\mathbb{F}$-predictable force of mortality $(\mu_t)$ ($\Rightarrow$ easy to simulate);
- any other process of interest (e.g. $B^s$, $B^d$ or $B^w$) is $\mathbb{F}$-adapted (or predictable).

Value of the contract with surrender option (for fixed $\theta$): $V_0^w(\theta)$; the value of the contract is given by the optimal stopping problem

$$V_0^{w*} = \sup_{\theta \in T} V_0^w(\theta)$$

where $T = \text{set of } \mathcal{G}$-stopping times.

one can replace $\mathcal{G}$-s.t. with $\mathbb{F}$-s.t. or s.t. bounded by $\tau$. 

... Valuation Framework
Unbundling of the contract:

\[ V_0^{w*} = V_0 + W_0^* \]

where \( V_0 = \) value of the contract without surrender and \( W_0^* = \) value of the surrender option (right to receive \( B^w \) and give up \( V \)).
LSMC algorithm

▷ **Unbundling** of the contract:

\[ V_0^{\ast} = V_0 + W_0^{\ast} \]

where \( V_0 \) = value of the contract without surrender and \( W_0^{\ast} \) = value of the surrender option (right to receive \( B^w \) and give up \( V \)).

▷ In order to compute \( V_0^{\ast} \) with backward dynamic programming, the LSMC requires

* discretization in the time dimension;
LSMC algorithm

▷ **Unbundling** of the contract:

\[ V_{0}^{w*} = V_{0} + W_{0}^{*} \]

where \( V_{0} \) = value of the contract without surrender and \( W_{0}^{*} \) = value of the surrender option (right to receive \( B^w \) and give up \( V \)).

▷ In order to compute \( V_{0}^{*} \) with backward dynamic programming, the LSMC requires

- **discretization** in the time dimension;
- **simulation** of all random processes;
LSMC algorithm

▷ **Unbundling** of the contract:

\[ V_{0}^* = V_0 + W_{0}^* \]

where \( V_0 \) = value of the contract without surrender and \( W_{0}^* \) = value of the surrender option (right to receive \( B^w \) and give up \( V \)).

▷ In order to compute \( V_{0}^* \) with backward dynamic programming, the LSMC requires

* **discretization** in the time dimension;
* **simulation** of all random processes;

* Approximation of the **continuation value** by regression against function of **state variables**

⇒ choice of **basis functions**;
**LSMC algorithm**

- **Unbundling** of the contract:
  
  $$V_{0}^{w*} = V_{0} + W_{0}^{*}$$

  where $V_{0} = \text{value of the contract without surrender}$ and $W_{0}^{*} = \text{value of the surrender option (right to receive $B^{w}$ and give up $V$)}$.

- In order to compute $V_{0}^{*}$ with backward dynamic programming, the LSMC requires:
  
  - **discretization** in the time dimension;
  
  - **simulation** of all random processes;
  
  - Approximation of the **continuation value** by regression against function of state variables
    
    $\Rightarrow$ choice of **basis functions**;

- Convergence of the whole scheme is guaranteed if state variables are **Markov**, see [Clément et al. - FS(02)].

- Mixed results on trade-off between basis functions and numbers of simulations and on method robustness ([Moreno and Navas - RDR(02)], [Glasserman and Yu - AAP(04)], [Stentoft - RDR(04)], [Gobet et al - AAP(05)]).
…LSMC algorithm

▷ Method I: apply the algorithm directly to

\[ V_0^w(\theta) = E^Q \left[ \int_0^\theta \frac{d(D_u + D_u^w(\theta))}{S_u^0} \right], \]

where \( S^0 \) is the money market account.
...LSMC algorithm

▷ Method I: apply the algorithm directly to

\[ V_0^w(\theta) = E^Q \left[ \int_0^\theta \frac{d(D_u + D_u^w(\theta))}{S_u^0} \right], \]

where \( S_0^0 \) is the money market account.

★ need to simulate **times of death**; the backward algorithm starts from these times.

★ At any time, only trajectories in which the insured is still alive enter the approximation.
...LSMC algorithm

▷ Method I: apply the algorithm directly to

\[
V_0^w(\theta) = E^Q \left[ \int_0^\theta \frac{d(D_u + D_u^w(\theta))}{S_u^0} \right],
\]

where \( S^0 \) is the money market account.

★ need to simulate times of death; the backward algorithm starts from these times.

★ At any time, only trajectories in which the insured is still alive enter the approximation.

▷ Method II: exploit the Cox setting ⇒ replace indicators with probabilities, i.e. discount sums at risk-adjusted rate \( r + \mu \):
LSMC algorithm

▷ Method I: apply the algorithm directly to

\[
V^w_0(\theta) = E^Q \left[ \int_0^\theta \frac{d(D_u + D^w_u(\theta))}{S^0_u} \right],
\]

where \(S^0\) is the money market account.

★ need to simulate times of death; the backward algorithm starts from these times.

★ At any time, only trajectories in which the insured is still alive enter the approximation.

▷ Method II: exploit the Cox setting \(\Rightarrow\) replace indicators with probabilities, i.e. discount sums at risk-adjusted rate \(r + \mu\):

\[
V^w_0(\theta) = E^Q \left[ \int_0^\theta \frac{d(\hat{D}_u + \hat{D}^w_u)}{\hat{S}^0_u} \right],
\]

where \(d\hat{D}_u = dB^s_u + B^d_u \mu_u du\), \(d\hat{D}^w_u = B^w_u dL_u(\theta)\) with \(L_u = 1_{\theta \leq u}\), and \(\hat{S}^0\) is the adjusted money-market account \(\Rightarrow\) contract without mortality.
Numerical examples

Focus on **single premium** unit-linked endowment insurance with maturity $T > 0$, individual aged $x$ at time 0 ([Bacinello et al. - JCAM(08)]).
Numerical examples

▷ Focus on **single premium** unit-linked endowment insurance with maturity $T > 0$, individual aged $x$ at time 0 ([Bacinello et al. - JCAM(08)]).

▷ Price process of reference portfolio is $(S_t)$. 
Numerical examples

Focus on **single premium** unit-linked endowment insurance with maturity $T > 0$, individual aged $x$ at time 0 ([Bacinello et al. - JCAM(08)]).

Price process of reference portfolio is $(S_t)$.

(Cumulated) benefits:

$$B_t^s = F_T^g 1_{t \geq T} \quad B_t^d = F_t^g 1_{t < T} \quad B_t^w = F_t^h 1_{t < T},$$

with either **Terminal** or **Cliquet** guarantee:
Focus on **single premium** unit-linked endowment insurance with maturity $T > 0$, individual aged $x$ at time 0 ([Bacinello et al. - JCAM(08)]).

Price process of reference portfolio is $(S_t)$.

(Cumulated) benefits:

$$B_t^s = F_T^g 1_{t \geq T}, \quad B_t^d = F_t^g 1_{t < T}, \quad B_t^w = F_t^h 1_{t < T},$$

with either **Terminal** or **Cliquet** guarantee:

**Terminal guarantee:**

$$F_t^l = \max\{S_t, S_0 e^{kt}\}, \quad l = g, w.$$
Numerical examples

Focus on single premium unit-linked endowment insurance with maturity $T > 0$, individual aged $x$ at time 0 ([Bacinello et al. - JCAM(08)]).

Price process of reference portfolio is $(S_t)$.

(Cumulated) benefits:

$$B_t^s = F_T^g 1_{t \geq T}, \quad B_t^d = F_t^g 1_{t < T}, \quad B_t^w = F_t^h 1_{t < T},$$

with either Terminal or Cliquet guarantee:

**Terminal guarantee:**

$$F_t^l = \max\{S_t, S_0 e^{kt}\}, \ l = g, w.$$

**Cliquet guarantee:**

$$F_t = F_t^l = S_0 \prod_{u=1}^{[t]} \max \left\{ \eta \left( \frac{S_u}{S_{u-1}} - 1 \right) + 1, e^{kt} \right\}, \ l = g, w.$$
Financial and demographic uncertainty are independent and described by:

- Interest rates (CIR): $\frac{dr_t}{\sqrt{r_t}} = \zeta_r (\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dW^r_t$. 
Financial and demographic uncertainty are independent and described by:

- **Interest rates (CIR):** \( dr_t = \zeta_r (\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_r^r. \)

- **Reference portfolio ([Bakshi et al. - JoF(97)]):** \( S = e^Y, \) with
  
  \[
  dY_t = \left( r_t - \frac{1}{2} Z_t - \lambda_J \mu_J \right) dt + \sqrt{Z_t} \left( \rho_{SZ} dW_t^Z + \rho_{Sr} dW_t^r \right) + \sqrt{1 - \rho_{SZ}^2 - \rho_{Sr}^2} dW_t^S + dJ_t
  \]

  \[
  dZ_t = \zeta_Z (\theta_Z - Z_t) dt + \sigma_Z \sqrt{Z_t} dW_t^Z
  \]

where \( W = (W^r, W^Z, W^S) \) is a standard B.m. in \( \mathbb{R}^3 \) independent of the compound Poisson \( J \) (arrival intensity \( \lambda_J \), Lognormal(\( \mu_J, \sigma_J \)) jumps).
Financial and demographic uncertainty are independent and described by:

- Interest rates (CIR): \( dr_t = \zeta_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t}dW_t^r \).

- Reference portfolio ([Bakshi et al. - JoF(97)]): \( S = e^Y \), with

\[
\begin{align*}
  dY_t &= \left( r_t - \frac{1}{2}Z_t - \lambda_J \mu_J \right) dt + \sqrt{Z_t} \left( \rho_{SZ} dW_t^Z + \rho_{Sr} dW_t^r ight) \\
  &\quad + \sqrt{1 - \rho_{SZ}^2 - \rho_{Sr}^2} dW_t^S + dJ_t \\
  dZ_t &= \zeta_Z(\theta_Z - Z_t)dt + \sigma_Z \sqrt{Z_t}dW_t^Z
\end{align*}
\]

where \( W = (W^r, W^Z, W^S) \) is a standard B.m. in \( \mathbb{R}^3 \) independent of the compound Poisson \( J \) (arrival intensity \( \lambda_J \), Lognormal(\( \mu_J, \sigma_J \)) jumps).

- Stochastic mortality: left continuous version of

\[
\begin{align*}
  d\mu_t &= \zeta_\mu(m(t) - \mu_t)dt + \sigma_\mu \sqrt{\mu_t}dW_t^\mu + dK_t
\end{align*}
\]

where \( m \) is a deterministic force of mortality, \( W^\mu \) is a standard B.m. independent of the compound Poisson \( K \) (arrival intensity \( \lambda_K \) and \( \exp(\gamma_K) \) jumps).
Figure 1: Solid line: sample path of $\mu$. Dashed line: Weibull intensity $m$. 
...Numerical example

▷ $T = 15, x = 40.$

▷ Average of 140 sets of 19000 simulations for terminal guarantee, 100 sets of 30000 simulations for cliquet guarantee.

▷ Forward discretization step = 1500, backward discretization step = 30.
...Numerical example

▷ $T = 15$, $x = 40$.

▷ Average of 140 sets of 19000 simulations for terminal guarantee, 100 sets of 30000 simulations for cliquet guarantee.

▷ Forward discretization step = 1500, backward discretization step = 30.

▷ Financial model:

- $r_0 = 0.05$, $\zeta_r = 0.6$, $\theta_r = 0.05$, $\sigma_r = 0.03$;
- $Z_0 = 0.04$, $\zeta_Z = 1.5$, $\theta_Z = 0.04$, $\sigma_Z = 0.4$;
- $S_0 = 100$, $\rho_{ZS} = -0.7$, $\rho_{rS} = 0$, $\lambda_J = 0.5$, $\mu_J = 0$, $\sigma_J = 0.07$. 
Numerical example

- \( T = 15, \ x = 40. \)
- Average of 140 sets of 19000 simulations for terminal guarantee, 100 sets of 30000 simulations for cliquet guarantee.
- Forward discretization step = 1500, backward discretization step = 30.
- Financial model:
  - \( r_0 = 0.05, \ \zeta_r = 0.6, \ \theta_r = 0.05, \ \sigma_r = 0.03; \)
  - \( Z_0 = 0.04, \ \zeta_Z = 1.5, \ \theta_Z = 0.04, \ \sigma_Z = 0.4; \)
  - \( S_0 = 100, \ \rho_{ZS} = -0.7, \ \rho_{rS} = 0, \ \lambda_J = 0.5, \ \mu_J = 0, \ \sigma_J = 0.07. \)
- Demographic model: \( m \) Weibull fitted against a SIM2001; \( \mu_0 = m(0), \ \zeta_\mu = 0.5, \ \sigma_\mu = 0.03, \ \lambda_K = 0.1, \ \gamma_K = 0.01. \)
- State variables: \( \mu, r, S, Z \) (+ \( F \) for the cliquet guarantee). Basis functions: polynomials in 4 (5) variables of order 3.
- Terminal guarantee: \( k_g = k; \) cliquet guarantee: \( k_g = k_w = k. \)
## Terminal Guarantee

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\kappa_w$</th>
<th>E1 (s.e.)</th>
<th>A1 (s.e.)</th>
<th>O1</th>
<th>E2 (s.e.)</th>
<th>A2 (s.e.)</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0%</td>
<td>107.185 (0.047)</td>
<td>113.556 (0.031)</td>
<td>6.372</td>
<td>107.224 (0.046)</td>
<td>113.577 (0.030)</td>
<td>6.353</td>
</tr>
<tr>
<td></td>
<td>2%</td>
<td></td>
<td>117.223 (0.031)</td>
<td>10.038</td>
<td></td>
<td>117.237 (0.031)</td>
<td>10.013</td>
</tr>
<tr>
<td></td>
<td>4%</td>
<td></td>
<td>123.687 (0.031)</td>
<td>16.503</td>
<td></td>
<td>123.696 (0.031)</td>
<td>16.472</td>
</tr>
<tr>
<td></td>
<td>6%</td>
<td></td>
<td>137.130 (0.031)</td>
<td>29.945</td>
<td></td>
<td>137.262 (0.030)</td>
<td>30.038</td>
</tr>
<tr>
<td>2%</td>
<td>0%</td>
<td>112.675 (0.045)</td>
<td>115.381 (0.033)</td>
<td>2.706</td>
<td>112.698 (0.044)</td>
<td>115.324 (0.033)</td>
<td>2.626</td>
</tr>
<tr>
<td></td>
<td>2%</td>
<td></td>
<td>117.551 (0.031)</td>
<td>4.876</td>
<td></td>
<td>117.524 (0.031)</td>
<td>4.825</td>
</tr>
<tr>
<td></td>
<td>4%</td>
<td></td>
<td>123.727 (0.031)</td>
<td>11.052</td>
<td></td>
<td>123.738 (0.031)</td>
<td>11.040</td>
</tr>
<tr>
<td></td>
<td>6%</td>
<td></td>
<td>137.327 (0.030)</td>
<td>24.652</td>
<td></td>
<td>137.404 (0.030)</td>
<td>24.706</td>
</tr>
<tr>
<td>4%</td>
<td>0%</td>
<td>122.901 (0.041)</td>
<td>123.087 (0.033)</td>
<td>0.176</td>
<td>122.904 (0.040)</td>
<td>122.904 (0.033)</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>2%</td>
<td></td>
<td>123.291 (0.033)</td>
<td>0.390</td>
<td></td>
<td>123.130 (0.033)</td>
<td>0.226</td>
</tr>
<tr>
<td></td>
<td>4%</td>
<td></td>
<td>124.507 (0.032)</td>
<td>1.606</td>
<td></td>
<td>124.418 (0.032)</td>
<td>1.514</td>
</tr>
<tr>
<td></td>
<td>6%</td>
<td></td>
<td>137.710 (0.030)</td>
<td>14.809</td>
<td></td>
<td>137.630 (0.030)</td>
<td>14.726</td>
</tr>
</tbody>
</table>

Table 1: Terminal guarantee
Terminal Guarantee

Figure 2: Terminal guarantees, Algorithm 1: value of the American contract.
## Cliquet Guarantee

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\eta$</th>
<th>$E_1$ (s.e.)</th>
<th>$A_1$ (s.e.)</th>
<th>$O_1$</th>
<th>$E_2$ (s.e.)</th>
<th>$A_2$ (s.e.)</th>
<th>$O_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>20%</td>
<td>65.938 (0.004)</td>
<td>97.205 (0.002)</td>
<td>31.267</td>
<td>66.239 (0.004)</td>
<td>97.216 (0.001)</td>
<td>30.977</td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>89.945 (0.009)</td>
<td>99.496 (0.003)</td>
<td>9.551</td>
<td>90.222 (0.009)</td>
<td>99.651 (0.004)</td>
<td>9.429</td>
</tr>
<tr>
<td></td>
<td>60%</td>
<td>121.902 (0.019)</td>
<td>122.067 (0.018)</td>
<td>0.165</td>
<td>122.136 (0.019)</td>
<td>122.251 (0.019)</td>
<td>0.115</td>
</tr>
<tr>
<td>2%</td>
<td>20%</td>
<td>69.915 (0.004)</td>
<td>97.598 (0.001)</td>
<td>27.683</td>
<td>70.213 (0.004)</td>
<td>97.609 (0.001)</td>
<td>27.396</td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>94.938 (0.009)</td>
<td>100.080 (0.004)</td>
<td>5.142</td>
<td>95.210 (0.009)</td>
<td>100.585 (0.005)</td>
<td>5.376</td>
</tr>
<tr>
<td></td>
<td>60%</td>
<td>128.411 (0.019)</td>
<td>128.470 (0.018)</td>
<td>0.059</td>
<td>128.636 (0.019)</td>
<td>128.698 (0.019)</td>
<td>0.062</td>
</tr>
<tr>
<td>4%</td>
<td>20%</td>
<td>75.343 (0.004)</td>
<td>98.096 (0.001)</td>
<td>22.753</td>
<td>75.635 (0.004)</td>
<td>98.109 (0.001)</td>
<td>22.474</td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>100.982 (0.009)</td>
<td>101.959 (0.007)</td>
<td>0.977</td>
<td>101.246 (0.009)</td>
<td>103.107 (0.008)</td>
<td>1.861</td>
</tr>
<tr>
<td></td>
<td>60%</td>
<td>135.952 (0.019)</td>
<td>135.952 (0.019)</td>
<td>0.000</td>
<td>136.165 (0.019)</td>
<td>136.198 (0.019)</td>
<td>0.033</td>
</tr>
</tbody>
</table>

Table 2: Cliquet guarantee
Figure 3: Cliquet guarantees, Algorithm 1: value of the American contract.