Optimal Liquidation of Derivative Portfolios

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The Problem

- Consider a risk averse agent with a portfolio of options. She wants to maximise expected utility of total revenue achieved from liquidation of the portfolio.
- Agent cannot trade underlying asset so market is incomplete (if complete market, identical options would be exercised simultaneously).
- How the agent can divide up the claim important in incomplete market - we assume claim is infinitely divisible.
The Portfolio

• The agent’s initial portfolio can be considered as a measure \( \rho \) on the space \( \mathcal{K} \subseteq \mathbb{R}_+ \) of possible strikes

If \( \rho_t \) is the measure of un-exercised options at time \( t \) then \( (\rho_t)_{t \geq 0} \) is a decreasing family of measures with \( \rho_0 = \rho \) and \( \rho_t \geq 0 \).

• Denote by \( \Theta_t \) the total number of remaining options,

\[
\Theta_t = \int_{\mathcal{K}} \rho_t(dk) \quad \text{and} \quad \Theta_0 = \theta_0 = \int_{\mathcal{K}} \rho(dk)
\]

Eg. Portfolio of identical calls of strike \( K \): \( \rho_0(dk) = \theta_0 \delta(k - K) \), where \( \delta \) is the Dirac function

Eg. Portfolio of calls with different strikes:

\[
\rho_0(dk) = \sum_i \theta^{(i)} \delta(k - K^{(i)}) \quad \text{where} \quad \sum_i \theta^{(i)} = \theta_0
\]
Price Dynamics

- Asset value $X$ follows time-homogeneous diffusion

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt \quad X_0 = x$$

such that $X$ is transient to a lower value $x$ (typically 0). Denote scale function of $X$ by $S$ which we normalise so that $S(x) = 0$.

- Eg. Lognormal dynamics: $dX_t = X_t[\sigma dB_t + \mu dt]$ with $\mu < \sigma^2/2$. Then $x = 0$ and $S(x) = x^\beta$ with $\beta = 1 - 2\mu/\sigma^2$
Optimisation Problem

- Options are American perpetual and payoff from exercising a unit of options with strike \( k \) is \( C(x, \theta, k) \)
- Total revenue \( R \) from liquidation is

\[
R = \int_0^\infty \int_K C(X_t, \Theta_t, k)(-d\rho_t(k))
\]

- The agent with initial wealth \( w \) maximises

\[
\mathbb{E}U(w + R)
\]

for concave (and continuously differentiable) utility function \( U \) over adapted, decreasing families of measures \( \rho_t \)
An Example with Price Impact

- Suppose $X$ is lognormal, and set $Y_t = X_t e^{-p(\theta_0 - \Theta_t)}$ so that

$$
\frac{dY_t}{Y_t} = \sigma dB_t + \mu dt + pd\Theta_t
$$

Take $C(X_t, \Theta_t, k) = (X_t e^{-p(\theta_0 - \Theta_t)} - k)^+ = (Y_t - k)^+$

Here, $X_t$ is ‘fundamental’ value of the share, $Y_t$ is the trading price of the share.

The parameter $p$ describes the (permanent) price impact of liquidation.
Applications

- Real options
- Executive stock options
- Liquidation of stock or option portfolios with price impact

Literature

Lemma 1: Suppose $C(x, \theta, k) = (G(x, \theta) - k)^+$ where $G$ is non-decreasing in $\theta$. For any strategy for which options with a high strike are exercised before options with a low strike, there is a modified version for which options are exercised in increasing strike order which raises at least as much revenue.

Write the strike of the $\theta$-to-go option as $J(\theta)$. Then $C(x, \theta, J(\theta))$ is just $C(x, \theta)$ and, given $\Theta_t$, $\rho_t$ can be reconstructed as

$$
\rho_t(A) = \int_0^{\Theta_t} d\theta I\{J(\theta) \in A\}
$$

where $I$ is the indicator function.
Optimal Boundary

- Approach: solve for value function for an arbitrary boundary and use calculus of variations to determine the *optimal* boundary.
  Verification lemma to show solution is a true optimum
- Time-homogeneity of problem – exercise strategy must take the form of a set of thresholds: \( \Theta_t = H(S_t) \) where \( S_t = \max_{s \leq t} X_s \) and \( H \) is a non-increasing function
  \( H \) may depend on \( w_0 \) the initial wealth of the agent. Denote by \( h = h(\theta) = h(\theta; w_0) \) the inverse of \( H \)
- If \( \theta_0 > H(x_0) \) then the agent exercises the \( \theta_0 - H(x_0) \) options with lowest strike immediately
Figure 1: A generic threshold $h(\phi)$.
**Theorem 2** For strategies of threshold type and for $\theta_0 \leq H(x_0)$ we have

$$\mathbb{E}[U(w+R)] = U(w) + S(x) \int_{0}^{\theta_0} \frac{C(h(\theta), \theta)}{S(h(\theta))} U'(w + \int_{0}^{\theta_0} C(h(\phi), \phi) d\phi) d\theta$$

**Proof:** Total revenue from exercise is

$$R = -\int_{t=0}^{\infty} C'(X_t, \Theta_t) d\Theta_t = \int_{0}^{\theta_0} I_{\{S \geq h(\theta)\}} C(h(\theta), \theta) d\theta$$

where $S = S_\infty = \max_t X_t$. Conditional on $S$,

$$R = R(S) = \int_{H(S)}^{\theta_0} C(h(\theta), \theta) d\theta$$

is deterministic.
Then $\mathbb{E}[U(w + R)]$ is

$$\int_x^\infty \mathbb{P}(S \in ds)U(w + R(s))$$
$$= \left[-U(w + R(s))\mathbb{P}(S \geq s)\right]_x^\infty + \int_x^\infty \mathbb{P}(S \geq s)U'(w + R(s))R'(s)ds$$
$$= U(w) - \int_x^\infty ds \frac{S(x)}{S(s)} \frac{dH}{ds} C(s, H(s))U'(w + R(s))$$
$$= U(w) + S(x) \int_0^{\theta_0} \frac{C(h(\theta), \theta)}{S(h(\theta))} U'(w + R(h(\theta)))d\theta$$
Fix $x_0, w_0$ and $\theta_0$. The aim is to choose a function $h^*(\phi) = h^*(\phi; w_0, \theta_0)$ to maximise

$$
\int_{0,\theta_0} C(h(\theta), \theta) \frac{U'}{S(h(\theta))} \left( w_0 + \int_\theta^{\theta_0} C(h(\phi), \phi) d\phi \right) d\theta
$$

- Need to find $h^*(0)$. For a general $h$ set $\bar{h} = h(0)$, and for small $\theta$ consider

$$
\frac{1}{\theta} \int_0^\theta \frac{C(h(\psi), \psi)}{S(h(\psi))} U' \left( w + \int_\psi^\theta C(h(\phi), \phi) d\phi \right) d\psi
$$

By l’Hôpital’s rule, provided $C$ is continuous, this tends to

$$
\frac{C(\bar{h}, 0)}{S(\bar{h})} U'(w)
$$

- Effectively $h^*(0)$ is fixed by considering the problem for a risk-neutral agent
Given $C$ write $c$ for the inverse $c = C^{-1}$ so that if $z = C(x, \phi)$ then $x = c(z, \phi)$

**Theorem 3** The first-order optimal $h$ satisfies

$$h'(\phi) = -\frac{c_\phi - A(h, \phi; w_0, \theta_0)C^2 c_z + 2C_\phi c_z + CC_\phi c_{zz} + Cc_{z\phi}}{[2C_x c_z + B(S, h(\phi))Cc_z + CC_x c_{zz}]}$$

(1)

where

$$A(h, \phi; w, \theta) = \frac{U''(w + \int_\phi^\theta C(h(\psi), \psi)d\psi)}{U'(w + \int_\phi^\theta C(h(\psi), \psi)d\psi)}$$

$$B(S, h(\phi)) = \frac{S''(h(\phi))}{S'(h(\phi))} - 2 \frac{S'(h(\phi))}{S(h(\phi))}$$

and (1) is evaluated at $x = h(\phi)$ and $z = C(h(\phi), \phi)$
Proof:

Set $D(\phi) = -\int_0^{\phi_0} C(h(\psi), \psi) d\psi$. Then, $h(\phi) = c(D'(\phi), \phi)$ and

$$
\int_{0, \theta_0} d\phi D'(\phi) D(D'(\phi), \phi) U'(w_0 - D(\phi))
$$

where $D(z, \phi) = 1/S(c(z, \phi))$. Choosing the optimal $h$ is equivalent to choosing the optimal $D$. The optimal $D$ solves

$$
D'(\phi) U'(w - D(\phi)) \frac{\partial D}{\partial \phi}(D'(\phi), \phi) + \frac{d}{d\phi} \left[ U'(w - D(\phi)) D'(\phi)^2 \frac{\partial D}{\partial D'}(D'(\phi), \phi) \right] = 0
$$

Corollary 4 If $C(x, \theta) = C(x)$ is independent of $\theta$ (in particular if the options are identical, and there is no price impact) then $h^*$ solves

$$
\frac{C(h(\phi))^2 S'(h(\phi))}{C''(h(\phi)) S(h(\phi))^2} U' \left( w_0 + \int_0^{\phi_0} d\psi C(h(\psi)) \right) = \text{constant}
$$
A Portfolio of Options with different strikes

- Suppose the agent has a portfolio of perpetual American call options such that she starts with a measure $\rho_0(\text{dk})$ of options with strike $k$. We write $C(x, \theta) = (x - J(\theta))^+$ for a decreasing function $J(\theta)$.
- Suppose the agent has exponential utility $U(w) = -e^{-\gamma w}/\gamma$.
- Asset price $X_t$ has lognormal dynamics with scale function $S(x) = x^\beta$.
- Assuming $J$ is differentiable, the Theorem gives $h$ solves

\[
h'(\phi) = -\frac{h(\phi)[\gamma(h(\phi) - J(\phi))^2 - J'(\phi)]}{2h(\phi) - (h(\phi) - J(\phi))(1 + \beta)}.
\]
More realistic is an agent with \( \theta_1 \) options with strike \( k_1 \) and \( \theta_2 - \theta_1 \) options with strike \( k_2 < k_1 \), then \( J(\phi) = k_1 \) for \( \phi \leq \theta_1 \) and \( J(\phi) = k_2 \) for \( \theta_1 < \phi \leq \theta_2 \).

For \( \phi < \theta_1 \) the optimal \( h \) solves

\[
h'(\phi) = -\frac{\gamma(h(\phi) - k_1)^2 h(\phi)}{k_1(1 + \beta) + (1 - \beta)h(\phi)}
\]

which can be solved by considering the inverse function \( H = h^{-1} \), so that \( dH/dz = -(k_1(1 + \beta) + (1 - \beta)z)/ (\gamma z(z - k_1)^2) \).

For \( \beta > 1 \), \( h(0) = k_1 \beta/(\beta - 1) < \infty \) and

\[
H(x) = \frac{2}{\gamma(x - k_1)} - \frac{2(\beta - 1)}{\gamma k_1} + \frac{(1 + \beta)}{\gamma k_1} \ln \left( \frac{\beta(x - k_1)}{x} \right)
\]

For \( \beta \leq 1 \), \( h(0) = \infty \) and

\[
H(x) = \frac{2}{\gamma(x - k_1)} + \frac{(1 + \beta)}{\gamma k_1} \ln \left( \frac{x - k_1}{x} \right)
\]
Figure 2: Thresholds for agent with $\theta_1 = 10$ options with strike $k_1 = 1.5$ and $\theta_2 - \theta_1 = 15$ options with strike $k_2 = 1$. Also shown (dashed line) are $h_1$ and $h_2$ which satisfy $h_2 \leq h \leq h_1$. Other parameters are $\beta = 2$, $\gamma = 1$. 
A Model with Price Impact

• Suppose $X$ is lognormal, and set $Y_t = G(X_t, \Theta_t)$ where $G(x, \theta) = xe^{-p(\theta_0 - \theta)}$. We have

$$dY_t/Y_t = \sigma dB_t + \mu dt + pd\Theta_t$$

• The agent has a portfolio of call options, so take

$C(x, \theta) = (xe^{-p(\theta_0 - \theta)} - k)^+$

• Suppose the agent has exponential utility $U(w) = -e^{-\gamma w}/\gamma$

• Here $c(z, \theta) = (z + k)e^{p(\theta - \theta_0)}$, $D(z, \theta) = (z + k)^{-\beta}e^{-\beta p(\theta_0 - \theta)}$, so if $D(\phi) = -\int_{\phi_0}^{\theta_0} C(h(\psi), \psi)d\psi = -\int_{\phi_0}^{\theta_0} (h(\psi)e^{-p(\theta_0 - \psi)} - k)d\psi$ then the problem is to maximise

$$e^{-\gamma w} \int_0^{\theta_0} d\phi D'(\phi)(D'(\phi) + k)^{-\beta}e^{\beta p(\phi - \theta_0)}e^{\gamma D(\phi)}.$$
Set

\[ E(\phi) = D(\phi) + \frac{\beta p}{\gamma} (\phi - \theta_0) \]

to remove \( \phi \) dependence, then as before, the optimal \( h \) is such that

\[ e^{\gamma E(\phi)} \left( E'(\phi) + k - \frac{\beta p}{\gamma} \right)^{-(\beta+1)} \left[ E'(\phi)^2 - \frac{p}{\gamma} (1+\beta)E'(\phi) - \frac{p}{\gamma} \left( k - \frac{\beta p}{\gamma} \right) \right] \]

is constant.

• Write \( g(\psi) = e^{-p(\theta_0-\psi)} h(\psi) \) and abbreviate \( p/\gamma \) to \( \xi \). Then we have

\[ g'(\theta) = \frac{-\gamma g \left( g^2 + g(\xi(\beta - 1) - 2k) + k(k - \beta\xi) \right)}{g(1 - \beta) + (\beta + 1)k} \]

with initial condition \( g(0) = e^{-p\theta_0} \bar{h} \) where \( \bar{h} = \arg\max h^{-\beta} (he^{-p\theta_0} - k) \).
• For $k \neq 0$, the large $\theta$ limit of $g$ is

$$
g(\infty) := \lim_{\theta \to \infty} g(\theta) = k + \frac{\xi}{2} \left[ \left( (\beta - 1)^2 + \frac{4k}{\xi} \right)^{1/2} - (\beta - 1) \right] > k
$$

so even with arbitrarily large initial holdings, possible agent never exercises any options

• For $p > 0$, it can be shown that $g$ converges exponentially fast to the limiting value, whereas when $p = 0$ the convergence is at rate $\theta^{-1}$
Figure 3: Exercise boundaries for options with strike $k = 1$. Other parameters are $\beta = 2$ and $\gamma = 1$. The rightmost boundary uses price impact parameter $p = 0.05$ and for these parameters, $g(\infty) = 1.2$. The leftmost boundary has no price impact and hence $g(\infty) = k = 1$. Both boundaries have $g(0) = k \frac{\beta}{\beta - 1} = 2$. 

• If \( k = 0 \) (stocks) then

\[
g' = -g\gamma \left( \frac{g}{1 - \beta} - \xi \right).
\]

If \( \beta \geq 1 \), the problem is degenerate and all shares sold instantly

• If \( 0 < \beta < 1 \) then

\[
g(\theta) = \xi(1 - \beta)/(1 - e^{-p\theta}) = h_0(\theta)p\theta/(1 - e^{-p\theta}) > h_0(\theta)
\]

where \( h_0(\theta) = (1 - \beta)/(\gamma \theta) \) is the solution in the absence of price impact
• In the limit $\xi \downarrow 0$ we find $g(\infty) = k + (\xi k)^{1/2} + O(\xi)$

• In the limit as $\xi \uparrow \infty$ we find

\[
\begin{align*}
g(\infty) &= \xi(1 - \beta) + k \frac{(2 - \beta)}{(1 - \beta)} + O(1/\xi); \quad \beta < 1 \\
g(\infty) &= \frac{k \beta}{(\beta - 1)} + O(1/\xi); \quad \beta > 1
\end{align*}
\]
Agent with CRRA utility

- We assume that $X$ is lognormal, and there is no price impact, $G(x, \theta) = x$, and we consider a stock portfolio so $C(x, \theta) = x$
- Suppose $U(w) = w^{1-\alpha}/(1 - \alpha)$ with $\alpha \in (0, \infty) \setminus \{1\}$
- By the Corollary, $h$ satisfies

$$h(\phi)^{1-\beta} \left( w_0 + \int_\phi^{\theta_0} h(\psi) d\psi \right)^{-\alpha} = \text{constant} = \left( \frac{q\alpha}{1 - \beta} \right)^{-\alpha},$$

for some constant $q$, subject to the fact that $h(0)$ maximises $\tilde{h}^{1-\beta}$
- Again, if $\beta > 1$, it is optimal to sell all stock immediately, so we consider only $0 < \beta < 1$, where $h(0)$ is infinite
• We can solve for \( h \) to give

\[
h(\theta) = h(\theta; w_0, \theta_0) = \frac{w_0}{\theta_0} (\eta^{-1} - 1) \left( \frac{\theta_0}{\theta} \right)^{1/\eta}
\]

if \( x < h(\theta_0; w_0, \theta_0) \)

• If \( x > h(\theta_0; w_0, \theta_0) \) then optimal behaviour is to sell a strictly positive amount of stock instantly at time zero to take you to

\[
\theta_1 = (w_0 + \theta_0 x)(1 - \eta)/x
\]