Portfolios and Risk Premia for the Long Run

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Goal:
A Tractable Framework for Portfolio Choice and Derivatives Pricing.

Model:
Stochastic Investment Opportunities with Several Assets.

Main Result:
Long Run Portfolios and Risk Premia.

Implications:
Static Fund Separation. Horizon effect.

Large Deviations:
Connection with Donsker-Varadhan results.

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Outline

- Model: Stochastic Investment Opportunities with Several Assets.
- Main Result: Long Run Portfolios and Risk Premia.
- Implications: Static Fund Separation. Horizon effect.
- Large Deviations: Connection with Donsker-Varadhan results.
Asset Prices and State Variables

- Market with risk-free rate and $n$ risky assets.
  Investment opportunities driven by $k$ state variables.

$$\frac{dS^i_t}{S^i_t} = r(Y_t)dt + dR^i_t \quad 1 \leq i \leq n$$

$$dR^i_t = \mu_i(Y_t)dt + \sum_{j=1}^{n} \sigma_{ij}(Y_t)dZ^j_t \quad 1 \leq i \leq n$$

$$dY^i_t = b_i(Y_t)dt + \sum_{j=1}^{k} a_{ij}(Y_t)dW^j_t \quad 1 \leq i \leq k$$

$$d\langle Z^i, W^j \rangle_t = \rho_{ij}(Y_t)dt \quad 1 \leq i \leq n, 1 \leq j \leq k$$

- $Z, W$ Brownian Motions.
- $\Sigma(y) = (\sigma \sigma')(y)$, $\Upsilon(y) = (\sigma \rho a')(y)$, $A(y) = (aa')(y)$. 
(In)Completeness

- $\gamma'\Sigma^{-1}\gamma$: covariance of hedgeable state shocks. Measures degree of market completeness.

- $A = \gamma'\Sigma^{-1}\gamma$: complete market. State variables perfectly hedgeable, hence replicable.

- $\gamma = 0$: fully incomplete market. State shocks orthogonal to returns.

- Otherwise state variable partially hedgeable.

- One state: $\gamma'\Sigma^{-1}\gamma/a^2 = \rho'\rho$. Equivalent to $R^2$ of regression of state shocks on returns.
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Payoffs

- Find optimal portfolio today for a long horizon $T$.
- $\pi_t = (\pi^1_t, \ldots, \pi^n_t)$ proportions of wealth in risky assets.
- Portfolio value $X_t^\pi$ evolves according to:
  $$\frac{dX_t^\pi}{X_t^\pi} = rdt + \pi_t dR_t$$
- Maximize expected power utility $U(x) = \frac{x^p}{p}$, $p < 1$, $p \neq 0$.
- Power utility motivated by Turnpike Theorems.
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Power utility motivated by Turnpike Theorems.
Stochastic Discount Factors

All stochastic discount factors of the form:

\[ M_T^\eta = \frac{1}{S_0^T} \mathcal{E} \left( - \int_0^T (\mu' \Sigma^{-1} + \eta' \gamma' \Sigma^{-1}) \sigma dZ + \int_0^T \eta' adW \right)_T \]

for some adapted integrable process \((\eta_t)_{t \geq 0}\).

- \(\eta\): risk premia of state-variable shocks.
- Assumption: there exists a stochastic discount factor.
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Duality Bound

$q = \frac{p}{p-1}$. If $X, M \geq 0$ satisfy $E[XM] \leq 1$:

$$\frac{1}{p} E[X^p] \leq \frac{1}{p} E[M^q]^{1-p}$$

- Analogue of Hansen-Jagannathan bound for power utility.
- Any payoffs lives below any stochastic discount factor.
- If a payoff and a stochastic discount factor live together, the payoff is maximal, and the discount factor is minimal.
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Certainty Equivalent

Any pair of policies \((\hat{\pi}, \hat{\eta})\) satisfies in a finite horizon \(T\):

\[
\frac{1}{p} E_p^y \left[ (X_T^{\hat{\pi}})^p \right] \leq u_T(y) \leq \frac{1}{p} E_p^y \left[ (M_T^\hat{\eta})^q \right]^{1-p}
\]  

(1)

Certainty Equivalent loss \(l_T\) measures performance: increase in risk-free rate that covers utility loss:

\[
\frac{1}{p} E_p^y \left[ (e^{l_T T} X_T^{\hat{\pi}})^p \right] = u_T(y)
\]  

(2)

(2) in (1) yields upper bound:

\[
l_T \leq \frac{1}{p} \left( \frac{1}{T} \log E_p^y \left[ (M_T^\hat{\eta})^q \right]^{1-p} - \frac{1}{T} \log E_p^y \left[ (X_T^{\hat{\pi}})^p \right] \right)
\]  

(3)
Certainty Equivalent

- Any pair or policies \((\hat{\pi}, \hat{\eta})\) satisfies in a finite horizon \(T\):

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**Long Run Optimality**

**Definition**

A pair \((\hat{\pi}, \hat{\eta}) \in C^1(E, \mathbb{R}^n) \times C^1(E, \mathbb{R}^k)\) is *Long-Run Optimal* if:

\[
\lim_{T \to \infty} \frac{1}{T} \log E^y_P \left[ (X_T^{\hat{\pi}})^p \right] = \lim_{T \to \infty} \frac{1}{T} \log E^y_P \left[ (M_T^{\hat{\eta}})^q \right]^{1-p}
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- Certainty equivalent loss vanishes for long horizons.
- Alternative interpretation: an agent with sufficiently long horizon prefers the long-run optimal portfolio to the optimal finite horizon portfolio, if the long-run portfolio has slightly lower fees.
- \(\hat{\pi}\) and \(\hat{\eta}\) depend on state alone (no \(t\)).
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Solution Method

- Differential equation delivers candidate solutions.
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- Differential equation delivers candidate solutions.
- Finite-horizon bounds.
  Characterize performance of candidate.
  Most relevant for applications.
- Long-run optimality.
  Guarantees that candidate prevails in the long run.
  Theoretical justification.
Finite-Horizon Bounds: Assumptions

**Theorem**

i) \( \delta = \frac{1}{1-q\rho'\rho} \) constant;

ii) for some \( \lambda \in \mathbb{R} \), the linear ODE:

\[
\frac{A}{2} \dddot{\phi} + \left( b - q\gamma'\Sigma^{-1}\mu \right) \ddot{\phi} + \frac{1}{\delta} \left( pr - \frac{q}{2} \mu'\Sigma^{-1}\mu - \lambda \right) \phi = 0
\]

admits a strictly positive solution \( \phi \in C^2(E,\mathbb{R}) \);

iii) both models:

\[
\begin{cases}
    dR_t = \mu dt + \sigma dZ_t \\
    dY_t = b dt + a dW_t
\end{cases}
\]

\[
\begin{cases}
    dR_t = \frac{1}{1-\rho} \left( \mu + \delta \gamma' \dot{\phi} \right) + \sigma d\hat{Z}_t \\
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well-posed under equivalent probabilities \( P \) and \( \hat{P} \).
Finite-Horizon Bounds

**Theorem**

The portfolio \( \hat{\pi} \) and the risk premia \( \hat{\eta} \) defined by:

\[
\hat{\pi} = \frac{1}{1 - p} \Sigma^{-1} \left( \mu + \gamma \delta \dot{\phi} \right) \quad \hat{\eta} = \delta \frac{\dot{\phi}}{\phi}
\]

satisfy the equalities:

\[
E^y_P \left[ (X_T^{\hat{\pi}})^p \right] = e^{\lambda T} \phi(y)^\delta E^y_P \left[ \phi(Y_T)^{-\delta} \right]
\]
\[
E^y_P \left[ (M_T^{\hat{\eta}})^q \right]^{1-p} = e^{\lambda T} \phi(y)^\delta E^y_P \left[ \phi(Y_T)^{-\frac{\delta}{1-p}} \right]^{1-p}
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- \( \hat{\pi} \) and \( \hat{\eta} \) as candidate long-run solutions.
- Long Run and transitory components (Hansen and Scheinkman, 2006).
Finite-Horizon Bounds

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$$E_P^Y \left[ (X_T^\hat{\pi})^p \right] = e^{\lambda_T} \phi(y)^\delta E_P^Y \left[ \phi(Y_T)^{-\delta} \right]$$

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- $\hat{\pi}$ and $\hat{\eta}$ as candidate long-run solutions.
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  (Hansen and Scheinkman, 2006).
Theorem

In addition to the previous assumptions, suppose that:

i) for some $\bar{t} > 0$, the random variables $(Y_t)_{t \geq \bar{t}}$ are $\hat{P}$-tight;

ii) $\sup_{y \in E} F(y) < \infty$, where $F \in C(E, \mathbb{R})$ defined as:

$$\begin{cases} 
pr - \lambda - \frac{q}{2} \mu' \Sigma^{-1} \mu + \frac{1}{2} \delta(\delta - 1) \phi \frac{\delta}{\phi} A \phi^{-\delta} & p < 0 \\
pr - \lambda - \frac{1}{2} q \mu' \Sigma^{-1} \mu - \frac{1}{2} q \delta^2 (1 - \rho' \rho) \phi \frac{\delta}{\phi} A \phi^{-\frac{\delta}{1-p}} & 0 < p < 1
\end{cases}$$

Then the pair $(\hat{\pi}, \hat{\eta})$ is long-run optimal.
Pricing measure

- Long-run version of $q$-optimal measure:

\[
\begin{cases}
    dR_t = \sigma d\tilde{Z}_t \\
    dY_t = \left( b - \gamma'\Sigma^{-1}\mu + (A - \gamma'\Sigma^{-1}\gamma) \frac{\delta\phi}{\phi} \right) dt + ad\tilde{W}_t
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\]

- Indifference pricing, for a long horizon.
- Changes drift for Monte Carlo simulation.
- First term: original drift.
  Second term: effect of correlation.
  Third term: effect of preferences.
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Extreme Cases

- $\gamma' \Sigma^{-1} \gamma = A$: Complete Market.
  - Pricing trivial: $\eta = 0$.
  - Portfolio nontrivial: $\pi = \frac{1}{1-p} \Sigma^{-1} \left( \mu + \gamma \delta \frac{\phi}{\phi} \right)$
  - Perfect intertemporal hedging. No unhedgeable risk.

- $\gamma = 0$: Fully Incomplete.
  - Portfolio trivial: $\pi = \frac{1}{1-p} \Sigma^{-1} \mu$.
  - Pricing nontrivial: $\eta = \delta \frac{\phi}{\phi}$
  - Intertemporal hedging infeasible within asset span.
    Unhedgeable risk premia reflect latent hedging demand.
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  - Intertemporal hedging infeasible within asset span. Unhedgeable risk premia reflect latent hedging demand.
Extreme Cases

- $\gamma' \Sigma^{-1} \gamma = A$: Complete Market.
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  - Portfolio nontrivial: $\pi = \frac{1}{1-\rho} \Sigma^{-1} \left( \mu + \gamma \delta \frac{\dot{\phi}}{\phi} \right)$
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The Myopic Probability

\[ \hat{P} \] is neither the physical probability \( P \), nor risk neutral.

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Several Assets with Predictability

Several assets, one state:

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\begin{align*}
\frac{dR_t}{dt} &= (\sigma \nu_0 + b \sigma \nu_1 Y_t) dt + \sigma dZ_t \\
\frac{dY_t}{dt} &= -b Y_t dt + dW_t \\
\frac{d\langle R, Y \rangle_t}{dt} &= \sigma \rho adt \\
\frac{r(Y_t)}{dt} &= r_0
\end{align*}
\]

\(\sigma\) matrix, \(\nu_0, \nu_1\) vectors, \(b\) scalar.

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Think of \(Y\) as the dividend yield.
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Solution

- Long-run portfolios and risk premia linear in the state:

\[
\pi(y) = \frac{1}{1 - \rho} \Sigma^{-1} (\mu(y) + v_0 \sigma \rho - v_1 y \sigma \rho)
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\[
\eta(y) = v_0 - v_1 y
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- Utility growth rate:

\[
\lambda = pr_0 - \frac{q}{2} \nu'_0 \nu_0 + \frac{1}{2} \delta^{-1} v_0^2 - qv_0 \rho' \nu_0 - \frac{1}{2} v_1
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- \( v_0 \) and \( v_1 \) constants. Depend on preferences and price dynamics.

\[
v_1 = \delta b \left( \sqrt{\Theta} - (1 + q \rho' v_1) \right)
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Static Fund Separation

- One state variable: dynamic separation into three funds \((2 + 1)\): risk-free, myopic, and hedging portfolios.
- Drawback: dynamic weights unknown.
- Long-run portfolios:
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  - Static separation into four “funds” (preference-free portfolios).
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Long-Run Optimality

Set $\nu_1 = -\kappa \rho$, for some $\kappa > 0$.
Sensitivities proportional to correlations.

Long-run optimality holds if:

$$q \rho' \rho < \frac{1}{4} + \frac{1}{2\kappa}$$

Always satisfied if at least one of the following holds:
- Sensitivity $\kappa$ sufficiently low.
- Market sufficiently incomplete.
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Calibration

- One risk asset: equity index.
  - One state variable: dividend yield.
- $\rho = -0.935$, $r = 0.14\%$, $\sigma = 4.36\%$, $\nu_0 = 0.0788$, $\kappa = 0.8944$, $b = 0.0226$.
- Long-run optimality holds for risk-aversion less than 13.4.

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Long Run vs. Myopic. Risk-aversion 5
Long-run asymptotics: tractable framework for portfolio choice and asset pricing.

- Long-run policies widely available in closed form.
  Finite-horizon solutions rarely explicit.
- Long-run portfolios for investing.
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  Then horizon does not matter if it is long.
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Donsker-Varadhan Asymptotics

- Y ergodic Feller diffusion with generator L on domain \( \mathcal{D} \).
- \( M_1(E) \) set of Borel probabilities on \( E \).
- For \( V \in C(E, \mathbb{R}) \), and under joint conditions on \( Y \) and \( V \), Donsker and Varadhan (1975, 1976, 1983) show:
  \[
  \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \left[ \exp \left( \int_0^T V(Y_t) dt \right) \right] = \sup_{\mu \in M_1(E)} \left( \int_E V d\mu - I(\mu) \right)
  \]
  rate function defined as \( I(\mu) = - \inf_{u > 0, u \in \mathcal{D}} \int_E \frac{L}{u} \frac{d\mu}{d\psi^2} d\mu \)
- For a one-dimensional diffusion with generator \( Lu = \frac{1}{2} a^2 u'' + bu' \) and invariant density \( m(y) \) the function \( I \) admits the simpler expression:
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- For any strategy $\pi$, find explicit utility rate.
- Obtain set of lower bounds of utility growth rate.
- Maximize rate lower bound over $\pi$.
- For any risk premium $\eta$, find explicit dual growth rate.
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Large Deviations Approach

- For any strategy $\pi$, find explicit utility rate.
- Obtain set of lower bounds of utility growth rate.
- Maximize rate lower bound over $\pi$.
- For any risk premium $\eta$, find explicit dual growth rate.
- Obtain set of upper bounds of utility growth rate.
- Minimize rate upper bound over $\eta$.
- Two variational problems coincide.
Primal Side Calculation

Terminal utility:

\[(X_T^\pi)^p = \exp \left( \int_0^T \left( pr + p\pi'\mu + \frac{1}{2}p(p - 1)\pi'\Sigma\pi \right) dt \right) \mathcal{E} \left( \int_0^T p\pi'\sigma dZ_t \right) \]

Expected utility under change of measure:

\[E \left[ (X_T^\pi)^p \right] = E_{P_\pi} \left[ \exp \left( \int_0^T (pr + p\pi'\mu + \frac{1}{2}p(p - 1)\pi'\Sigma\pi) dt \right) \right] \]

Donsker-Varadhan asymptotics:

\[
\lim_{T \to \infty} \frac{1}{T} \log E \left[ (X_T^\pi)^p \right] = \\
\sup_{\psi: \int_E \psi^2 m_\pi = 1} \int_E \left( pr + p\pi'\mu + \frac{1}{2}p(p - 1)\pi'\Sigma\pi - \frac{A}{2} \left( \frac{\psi}{\psi} \right)^2 \right) \psi^2 m_\pi
\]
More Calculations

- Change of variable: $\psi^2 m_\pi = \phi^2 m_0$.

$$\int_E \phi^2 m_0 \left( pr - \frac{1}{2} ( a_{\phi}^2 - q \rho' \nu )^2 + p \pi' \sigma \left( 1 - q \rho \rho' \right) \nu + a_{\phi} \dot{\phi} \rho \right)$$

$$+ \frac{1}{2} p (p - 1) \pi' \sigma \left( 1 - q \rho \rho' \right) \sigma' \pi$$

- Quadratic function of $\pi$. Maximizer is candidate optimal:

$$\hat{\pi} = \frac{1}{1 - p} \Sigma^{-1} \left( \mu + \delta \gamma \phi \right)$$

- Maximum is the Long Run Risk Return tradeoff:

$$\sup_{\phi: \int_E \phi^2 m_0 = 1} \int_E \left( pr - \frac{1}{2} q \mu' \Sigma^{-1} \mu - \frac{\delta}{2} A_{\phi} \phi^2 \right) \phi^2 m_0$$
More Calculations

- Change of variable: \( \psi^2 m_\pi = \phi^2 m_0 \).

\[
\int_E \phi^2 m_0 \left( pr - \frac{1}{2} \left( a_\phi \frac{\dot{\phi}}{\phi} - q\rho'\nu \right)^2 + p\pi'\sigma \left( (1 - q\rho\rho')\nu + a_\phi \frac{\dot{\phi}}{\phi}\rho \right) 
+ \frac{1}{2} p(p - 1)\pi'\sigma \left( 1 - q\rho\rho' \right)\sigma'\pi \right)
\]

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More Calculations

- Change of variable: \( \psi^2 m_\pi = \phi^2 m_0 \).

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\]
\[
\left. + \frac{1}{2} p(p - 1)\pi' \sigma \left( 1 - q\rho \rho' \right) \sigma' \pi \right) \]

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Long Run Risk-Return Tradeoff

Suppose there is an invariant density $m$ for the process:

$$dY_t = (b - q\gamma'\Sigma^{-1}\mu)dt + adW_t$$

ODE is Euler-Lagrange equation for:

$$\max \int_E \phi^2 m = \int_E \left( pr - \frac{1}{2} q\mu'\Sigma^{-1}\mu - \frac{1}{2} \delta A \left( \frac{\phi}{\phi} \right)^2 \right) \phi^2 dm$$

If $pr - \frac{1}{2} q\mu'\Sigma^{-1}\mu$ constant, maximum achieved at $\phi \equiv 1$.

Both portfolios and risk-premia are trivial.

Covers $p \to 0$ (logarithmic utility) or $r, \mu'\Sigma^{-1}\mu$ constant.
Long Run Risk-Return Tradeoff

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- If $pr - \frac{1}{2} q\mu'\Sigma^{-1}\mu$ constant, maximum achieved at $\phi \equiv 1$.
  - Both portfolios and risk-premia are trivial.
  - Covers $\rho \to 0$ (logarithmic utility) or $r, \mu'\Sigma^{-1}\mu$ constant.
Suppose there is an invariant density $m$ for the process:

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ODE is Euler-Lagrange equation for:

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Related Topics


Well Posedness

Law of process \((R, W)\) determined by:

- Risk-free rate \(r(y)\), drifts \(\mu(y)\) and \(b(y)\).
- Covariances \(\Sigma(y) = (\sigma \sigma')(y)\), \(\Upsilon(y) = (\sigma \rho \alpha')(y)\), \(A(y) = (aa')(y)\).

Assumption

- \((\Omega, \mathcal{F}) = C([0, \infty), \mathbb{R}^{n+k})\) with Borel \(\sigma\)-algebra.
- \(E \subset \mathbb{R}^k\) open connected set.
- \(b \in C^1(E, \mathbb{R}^k), \mu \in C^1(E, \mathbb{R}^n), A \in C^2(E, \mathbb{R}^{k \times k}), \Sigma \in C^2(E, \mathbb{R}^{n \times n}), \Upsilon \in C^2(E, \mathbb{R}^{n \times k}).\)
- \(A(y), \Sigma(y)\) nonsingular for all \(y \in E\).
- For all \(y \in E\), there exists a unique probability \(P^y\) on \((\Omega, \mathcal{F})\) such that \((R, Y)\) satisfies system \((R_0, Y_0) = (0, y)\).
- Solution global and internal: \(P^y(Y_t \in E \text{ for all } t \geq 0) = 1\).
Implicit Solution

- Value function:

$$u(y, T) = \max_\pi \frac{1}{p} E_P^\pi [(X_T^\pi)^p]$$

- Implicit solution (Merton 1971):

$$\pi = \frac{1}{1 - p} \left( \Sigma^{-1} \mu + \Sigma^{-1} \gamma \frac{u_y}{u} \right)$$

- Dynamic \((k + 2)\)-fund separation
  - risk-free asset
  - myopic portfolio \(\Sigma^{-1} \mu\)
  - intertemporal hedging portfolio \(\Sigma^{-1} \gamma\)

- Myopic weight constant (passive strategy)
  Hedging weight time-varying (active strategy).
Example

- One-asset, one-state model:

  \[
  dR_t = Y_t \, dt + dZ_t \\
  dY_t = -\lambda Y_t \, dt + dW_t
  \]

- Closed-form solution for \( p < 0 \):

  \[
  u(y, T) = \frac{1}{p} \frac{\sqrt{\alpha e^{\frac{\lambda}{2} T}}}{\sqrt{\sinh(\alpha \lambda T) + \alpha \cosh(\alpha \lambda T)}} \, e^{-q \frac{y^2}{2\lambda} \frac{\sinh(\alpha \lambda T)}{\sinh(\alpha \lambda T) + \alpha \cosh(\alpha \lambda T)}}
  \]

  where \( y = Y_0 \), \( q = p/(p - 1) \) and \( \alpha = \sqrt{1 + q/\lambda^2} \).
Long Run Limit

- Formulas simplify dramatically as $T \to \infty$.
- Value function:
  \[ u(T) \sim e^{\frac{\lambda}{2}(1-\alpha)T + o(T)} \]
- Intertemporal hedging independent of $t$, and linear in the state:
  \[ \frac{u_y}{u} = -\frac{q}{1 + \alpha} y \]
- Mean-reverting drift \textit{asymptotically} equivalent to constant drift $\mu = \frac{\lambda}{q}(\alpha - 1)$:
  \[ dR_t = \mu dt + dZ_t \]
General Case

- Quasilinear PDE in $(v, \lambda)$

\[
\frac{1}{2} \text{tr} (AD^2 v) + \frac{1}{2} \nabla v' (A - q \gamma' \Sigma^{-1} \gamma) \nabla v \\
+ \nabla v' \left(b - q \gamma' \Sigma^{-1} \mu\right) + pr - \frac{1}{2} q \mu' \Sigma^{-1} \mu = \lambda
\]

- Policies:

\[
\hat{\pi} = \frac{1}{1 - p} \Sigma^{-1} (\mu + \gamma \nabla v) \quad \hat{\eta} = \nabla v
\]

- Finite-horizon bounds:

\[
E_P^Y \left[ (X_T^{\hat{\pi}})^p \right] = e^{\lambda T + v(y)} E_P^Y \left[ e^{-v(Y_T)} \right] \\
E_P^Y \left[ (M_T^{\hat{\eta}})^q \right]^{1 - p} = e^{\lambda T + v(y)} E_P^Y \left[ e^{-\frac{1}{1 - p} v(Y_T)} \right]^{1 - p}
\]

- Quadratic solution $v(y) = y' Ay + By + C$ for many models.

- Includes linear diffusion: $r, \mu, b$ affine in $y$, $\Sigma, A, \gamma$ constant.
## Long Run Optimality

### Proposition

Long-run optimality holds if $\delta < 4$.

If $\delta \geq 4$, long-run optimality fails. In particular:

i) if $\delta > 4$, there exists a finite $T$ such that $\frac{1}{p}E[(X^T)^p] = -\infty$.

ii) if $\delta = 4$ and $\xi = 0$, the certainty equivalent loss converges to $-\frac{\gamma}{2p}$.

iii) if $\delta = 4$ and $\xi \neq 0$, the certainty equivalent loss diverges to $\infty$.

- Long-run optimality fails for nearly complete market, and highly risk-averse investor.
- Departure from optimality near $T$ becomes intolerable.
Finite Horizons as Long-Horizon Expansions

- Assume finite horizon solutions solve HJB equations (Duffie, Fleming, Soner, Zariphopoulou 1997, Pham 2002).
- Linear PDE via power transformation (Zariphopoulou 2001).
- Separation of variables. Expand value function as a series of eigenvectors with respect to invariant density.

\[ u_T(y) = \sum_{n=1}^{\infty} e^{\frac{\lambda_n}{\delta} T} \phi_n(y) m(y) \]

- In the long run, only first eigenvector survives in \( \frac{u_y}{u} \).