Large deviation estimates of the crossing probability for pinned Gaussian processes

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Based on the papers:


The starting problem

INGREDIENTS:

- $X = (X_t)_{t \geq 0}$ a stochastic process on $\mathbb{R}^d$ (a diffusion process)
- a (continuous) Gaussian process

- $U, L : [0, +\infty) \to \mathbb{R} \cup \{\pm \infty\}$ with $L(t) \leq U(t), \forall t$
  (meaning: $L$: “lower barrier”, $U$: “upper barrier”)

- $\tau$: the hitting time of $X$ on the barrier(s)

PROBLEM:

Evaluate expectations of functionals involving $\tau$, e.g. a function of the process $X$ to be killed when the boundaries are reached. For example:

$$\mathbb{E} [ f(\tau, X_\tau) 1_{\tau \leq T} + g(T, X_T) 1_{\tau > T} ]$$

HERE:

Monte Carlo procedures for the numerical evaluation.
For example, if \( X \) is the underlying asset price process, the quantity

\[
\mathbb{E}\left[ f(\tau, X_\tau)1_{\tau \leq T} + g(T, X_T)1_{\tau > T}\right]
\]

is strictly connected to

- the price of barrier options (also Parisian options, options on the max/min…):
  - if \( f \equiv 0 \) and \( g(t, x) = e^{-rt}(x - K)_+ \) ⇒ price of a knock-out call option
  - if \( f(t, x) = e^{-rt}(x - K)_+ \) and \( g \equiv 0 \) ⇒ price of a knock-in call option

- the price of defaultable securities (bonds, options etc.), i.e. contingent claims subject to default and/or interest rate risk, assuming the existence of a bankruptcy process (e.g. the value of the firm delivering the security) giving the default when it becomes smaller than a given threshold.

⇒ In order to set up an accurate Monte Carlo method one is forced to have good approximations of the exit time.
A good approximation of the exit time can be done if a good approximation of the exit probability in a small time period can be provided.

THE CRUDE PROCEDURE

Let $X^\varepsilon$ be a discretization of the process $X$ over $[0,T]$ with step $\varepsilon$:

$$\varepsilon = T/N, \quad t_k = k\varepsilon \quad \text{and} \quad X^\varepsilon_{t_k} \simeq X_{t_k}, \quad k = 0, 1, \ldots, N$$

Then

$$\tau \simeq \tau^\varepsilon = \inf \left\{ t_k \geq 0 : X^\varepsilon_{t_k} \text{ exits} \right\}$$

Under suitable hypothesis, $\tau^\varepsilon$ suitably converges to $\tau$ as $\varepsilon \to 0$ BUT slowly, that is $\tau^\varepsilon$ is a (very!) rough estimate for $\tau$

The following figures, referring to $X=$Brownian motion, show why.
THE CORRECTED PROCEDURE

Step $i$: one has $X_0^\varepsilon = x_0, \ldots, X_{t_i-1}^\varepsilon = x_{i-1}$ and then simulates $X_{t_i}^\varepsilon = x_i$. Set

$$p_i^\varepsilon(x_1, \ldots, x_i) = \mathbb{P}\left( \begin{array}{c} X \text{ reaches the boundary during } [t_{i-1}, t_i] \equiv [t_{i-1}, t_{i-1} + \varepsilon] \\ \text{past} \quad \text{future} \end{array} \right) \begin{cases} \begin{aligned} X_0^\varepsilon &= x_0, \ldots, X_{t_i-1}^\varepsilon = x_{i-1} \\ \text{AND} \quad X_{t_i}^\varepsilon &= x_i \end{aligned} \end{cases}$$

$\Rightarrow$ $p_i^\varepsilon(x_1, \ldots, x_i)$ is the hitting probability of the bridge process in a time interval of length $\varepsilon$

Now:

- with probability $= p_i^\varepsilon(x_1, \ldots, x_i)$: one sets $\tau^\varepsilon = t_i$, and
- with probability $= 1 - p_i^\varepsilon(x_1, \ldots, x_i)$: the simulation is carried on
**PROBLEM:**
compute $p_i^\varepsilon(x_1, \ldots, x_i)$

**UNFORTUNATELY:**
$p_i^\varepsilon(x_1, \ldots, x_i)$ is known only in some (very) special cases, both for the process $X$ and the barriers $L, U$

**BUT:**
if an approximation $\hat{p}_i^\varepsilon(x_1, \ldots, x_i)$ was known then the above scheme ("corrected procedure") could be performed with $p_i^\varepsilon(x_1, \ldots, x_i)$ replaced by $\hat{p}_i^\varepsilon(x_1, \ldots, x_i)$

**HERE:**
large deviation estimates $\hat{p}_i^\varepsilon(x_1, \ldots, x_i)$ for $p_i^\varepsilon(x_1, \ldots, x_i)$
Models for $X$

Results for $X$ evolving as:

I. a diffusion process: [BC02], [CP02], [CI02], [BCI99]

II. a Gaussian process (e.g. fractional Brownian motion, iterated Gaussian integrals, etc.): [CP08]

REMARK

Because of the Markov property, in case I. one has

$$p_i^\varepsilon(x_1, \ldots, x_i) \equiv p_i^\varepsilon(x_{i-1}, x_i)$$

while in case II., this is not true.
A family of processes \( \{Y^\varepsilon\}_\varepsilon \) satisfies a large deviation principle on \( C([0, 1]) \) with inverse speed \( \gamma_\varepsilon^2 \) and rate function \( I \) if:

- \( \lim_{\varepsilon \to 0} \gamma_\varepsilon^2 = 0; \)

- \( I : C([0, 1]) \to [0, +\infty] \) is lower semicontinuous and \( \{I \leq a\} \) is a compact set in \( C([0, 1]) \), for any \( a; \)

- for any closed set \( \Gamma \) and open set \( O \) in \( C([0, 1]) \),

\[
\limsup_{\varepsilon \to 0} \gamma_\varepsilon^2 \log \mathbb{P}(Y^\varepsilon \in \Gamma) \leq - \inf_{\xi \in \Gamma} I(\xi)
\]

\[
\liminf_{\varepsilon \to 0} \gamma_\varepsilon^2 \log \mathbb{P}(Y^\varepsilon \in O) \geq - \inf_{\xi \in O} I(\xi)
\]
If $A = \{\text{paths which hit the boundary}\}$ and a large deviation principle holds, then morally

$$\log \mathbb{P}(Y^\varepsilon \in A) \sim -I(A)/\gamma^2_\varepsilon, \quad \text{as } \varepsilon \to 0$$

or equivalently

$$\mathbb{P}(Y^\varepsilon \in A) = C_\varepsilon(A) e^{-I(A)/\gamma^2_\varepsilon}, \quad \text{as } \varepsilon \to 0, \quad \text{with } \lim_{\varepsilon \to 0} \gamma_\varepsilon \log C_\varepsilon(A) = 0$$

If the asymptotic behavior of $C_\varepsilon(A)$ is known as well, one says to have a **sharp large deviation estimate** for $\mathbb{P}(Y^\varepsilon \in A)$.

**REMARK**

Large deviations are widely studied and there are many references in the literature on this topics, concerning a big variety of contexts. On the contrary, sharp estimates in large deviations are absolutely not classified: it remains an open problem and only some few cases have been already studied.
LINK TO THE CORRECTED PROCEDURE

The process $Y^\varepsilon$ over $[0,1]$ is the one having the same law of the bridge process:

$$
(Y^\varepsilon_t)_{t\in[0,1]} \overset{\mathcal{L}}{=} \left((X^\varepsilon_{t_i+\varepsilon t})_{t\in[0,1]} \mid X^\varepsilon_0 = x_0, \ldots, X^\varepsilon_{t_i-1} = x_{i-1}, X^\varepsilon_{t_i-1+\varepsilon} = x_i \right)
$$

so that

$$p^\varepsilon_i(x_1, \ldots, x_i) = \mathbb{P}(Y^\varepsilon \in A), \quad A = \{\text{paths which cross the boundary}\}
$$

Therefore, for practical purposes,

$$\hat{p}^\varepsilon_i(x_1, \ldots, x_i) = C^\varepsilon(A) \exp \left(-\frac{I(A)}{\gamma^2} \right) \quad \text{or} \quad \tilde{p}^\varepsilon_i(x_1, \ldots, x_i) = \exp \left(-\frac{I(A)}{\gamma^2} \right)
$$

in the case sharp large deviation estimates can or cannot be achieved respectively.

HERE

We present the more recent results concerning Gaussian processes.
Gaussian processes

Let $X$ be a continuous Gaussian process (cfr. [CP08]).

1. Set $X^i$ the (Gaussian) process giving the behavior of $X$ over $[t_{i-1}, t_{i-1} + \varepsilon]$ conditional to the past:

$$
(X^i_t)_{t \in [t_{i-1}, t_{i-1} + \varepsilon]} \overset{\mathcal{L}}{=} 
\left\{ (X_t)_{t \in [t_{i-1}, t_{i-1} + \varepsilon]} \mid X_0 = x_0, \ldots, X_{t_{i-1}} = x_{i-1} \right\}
\Rightarrow \text{large deviations for } \left\{ (X^i_{t_{i-1} + \varepsilon s})_{s \in [0,1]} \right\}_\varepsilon \text{ on } C([0,1])
$$

2. Set $Y^i$ the (Gaussian) process giving the behavior of $X$ over $[t_{i-1}, t_{i-1} + \varepsilon]$ conditional to the past and the future observation a time $t_i = t_{i-1} + \varepsilon$:

$$
(Y^i_t)_{t \in [t_{i-1}, t_{i-1} + \varepsilon]} \overset{\mathcal{L}}{=} 
\left\{ (X_t)_{t \in [t_{i-1}, t_{i-1} + \varepsilon]} \mid X_0 = x_0, \ldots, X_{t_{i-1}} = x_{i-1}, X_{t_{i-1} + \varepsilon} = x_i \right\}
\overset{\mathcal{L}}{=} 
\left\{ (X^i_t)_{t \in [t_{i-1}, t_{i-1} + \varepsilon]} \mid X^i_{t_{i-1} + \varepsilon} = x_i \right\}
\Rightarrow \text{large deviations for } \left\{ (Y^i_{t_{i-1} + \varepsilon s})_{s \in [0,1]} \right\}_\varepsilon \text{ on } C([0,1])
$$

REMARK: “sharp” large deviations: work in progress!
Gaussian processes

Some preliminaries.

A [centered] **continuous Gaussian process** $X$ on $C([0,1])$ is fully determined by its (continuous) **covariance function** $k(t,s) = \mathbb{E}(X_t X_s)$, which in turn identifies the associated **reproducing kernel Hilbert space (rkHs)** $\mathcal{H}$.

Set: $\mu(A) = \mathbb{P}(X \in A)$, $A = \text{Borel set in } C([0,1])$; $\mathcal{M}[0,1] = \text{signed measures on } [0,1]$.

**DEF 1.** $\Gamma = \{Y \in L^2(\mu) : Y(\cdot) = \langle \lambda, \cdot \rangle, \lambda \in \mathcal{M}[0,1] \} \subset \{Y : C([0,1]) \to \mathbb{R}, Y \text{ Gaussian} \}$

- $Y_1, Y_2 \in \Gamma$, $(Y_1, Y_2)_{L^2(\mu)} = \int_{[0,1]^2} k(t,s)\lambda_1(dt)\lambda_2(ds)$.

**DEF 2.** $H = \overline{\Gamma} \| \cdot \|_{L^2(\mu)} \subset L^2(\mu)$

- $H$ is a Hilbert space with $(Y_1, Y_2)_H = (Y_1, Y_2)_{L^2(\mu)}$.

**DEF 3.** $S : H \to C([0,1])$, $Y \mapsto (SY)_t \overset{\text{def}}{=} \int x_t Y(x) \mu(dx) = \mathbb{E}(X_t Y) \overset{Y \in \Gamma}{=} \int_0^1 k(t,s)\lambda_Y(ds)$

- $S$ linear, injective, continuous: $\forall x \in SH \subset C([0,1]) \exists! Y \in H : x = SY$

**DEF 4.** The rkHs is $\mathcal{H} = SH = \{x : x_t = (SY)_t, Y \in H \}$

- $\mathcal{H}$ is a Hilbert space with $(x_1, x_2)_{\mathcal{H}} \equiv (S^{-1}x_1, S^{-1}x_2)_H = (S^{-1}x_1, S^{-1}x_2)_{L^2(\mu)}$.

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[Cramèr transform] Set:

- \( \hat{\mu}(\lambda) = \log \mathbb{E}(e^{\langle \lambda, X \rangle}) \equiv \frac{1}{2} \int_{[0,1]^2} k(t,s)\lambda(dt)\lambda(ds), \ \lambda \in \mathcal{M}[0,1] \)

- \( I(x) = \sup_{\lambda \in \mathcal{M}[0,1]} \left( \langle \lambda, x \rangle - \hat{\mu}(\lambda) \right) \)

Then, \( I \) is a rate function and \( I(x) = \begin{cases} 
\frac{1}{2} \|x\|_\mathcal{H}^2 & \text{if } x \in \mathcal{H} \\
+\infty & \text{otherwise}
\end{cases} \)

Gartner-Ellis Theorem + Gaussian processes theory \( \Rightarrow \) a large deviation principle holds for the family \( \{U^{\varepsilon}\}_\varepsilon \) of centered continuous Gaussian processes, with inverse speed \( \gamma^2_\varepsilon \) and rate function \( I \) as above, iff for any \( \lambda \in \mathcal{M}[0,1] \),

\[
\lim_{\varepsilon \to 0} \frac{\text{Var}(\langle \lambda, U^{\varepsilon} \rangle)}{\gamma^2_\varepsilon} = \int_{[0,1]^2} k(t,s)\lambda(dt)\lambda(ds)
\]

Morally, \( U^{\varepsilon} \overset{\mathcal{L}}{\sim} \gamma_\varepsilon U \), as \( \varepsilon \to 0 \), with \( U \) Gaussian process with covariance function \( k \).
Gaussian processes

Our case.

**Step 1: the \(i\)-fold conditional process:** as \(j = 1, \ldots, i\), set

\[
(X^j_t)_{t \in [t_{j-1}, t_j]} \overset{\mathcal{L}}{=} \left( (X_t)_{t \in [t_{j-1}, t_j]} \bigg| X_0 = x_0, \ldots, X_{t_{j-1}} = x_{j-1} \right).
\]

Then,

\[
X^j_t \overset{\mathcal{L}}{=} X^{j-1}_t - \alpha_j(t)(X^{j-1}_{t_j} - x_j), \quad t > 0
\]

where \(X^0 \equiv X\), \(k_0 \equiv k\) and \(\alpha_j(t) = \frac{k_{j-1}(t, t_j)}{k_{j-1}(t_j, t_j)}\), \(k_{j-1}(t, s) = \text{Cov}(X^{j-1}_t, X^{j-1}_s)\)

**Assumption A** \(\exists \gamma_\varepsilon \text{ s.t. } \lim_{\varepsilon \to 0} \gamma_\varepsilon = 0\) and

[A1] \(\exists \bar{k}(t, s) = \lim_{\varepsilon \to 0} \frac{\text{Cov}(X_{t_i + \varepsilon t} - X_{t_i}, X_{t_i + \varepsilon s} - t_i)}{\gamma_\varepsilon^2}, \text{ uniformly as } t, s \in [0, 1]\)

[A2] for any \(T > 0\), \(\exists \bar{\rho}(t) = \lim_{\varepsilon \to 0} \frac{\text{Cov}(X_{t_i + \varepsilon t} - X_{t_i}, X_T)}{\gamma_\varepsilon}, \text{ uniformly as } t \in [0, 1]\)
Theorem 1. Under Assumption A, \( \{X^i_{t_{i-1}+\varepsilon t}\}_\varepsilon \) satisfies a large deviation principle on \( C([0,1]) \) with inverse speed \( \gamma^2_\varepsilon \) and rate function

\[
J_i(h) = \begin{cases} 
\frac{1}{2} \| h - x_{i-1} \|_{\mathcal{H}_i} & \text{if } h_0 = x_{i-1} \text{ and } h - x_{i-1} \in \mathcal{H}_i \\
+\infty & \text{otherwise}
\end{cases}
\]

where \( \mathcal{H}_i \) is the rkHs associated to the (continuous) covariance function

\[
\bar{k}_i(t,s) = k(t,s) - \sum_{n=1}^{i} k_{n-1}(t_n, t_n) \bar{\alpha}_n(t) \bar{\alpha}_n(s)
\]

with \( \bar{\alpha}_n(t) = \lim_{\varepsilon \to 0} \frac{\alpha_n(t_i + \varepsilon t) - \alpha_n(t_i)}{\gamma_\varepsilon} = \frac{\bar{\rho}_{n-1}(t, t_n)}{k_{n-1}(t_n, t_n)}. \)
IDEA OF THE PROOF

\[ X^i_{t_{i-1}+\varepsilon t} = \left( X^i_{t_{i-1}+\varepsilon t} - \mathbb{E}(X^i_{t_{i-1}+\varepsilon t}) \right) + \mathbb{E}(X^i_{t_{i-1}+\varepsilon t}) \]

\[ \parallel U^\varepsilon_t \parallel \]

centered Gaussian process

unif. over [0, 1]

and as \( \lambda \in \mathcal{M}[0, 1] \),

\[ \frac{\text{Var}(\langle \lambda, U^\varepsilon \rangle)}{\gamma^2_\varepsilon} = \int_{[0,1]^2} \frac{\text{Cov}(X^i_{t_{i-1}+\varepsilon t} - X^i_{t_{i-1}+\varepsilon s}, X^i_{t_{i-1}+\varepsilon s} - X^i_{t_{i-1}})}{\gamma^2_\varepsilon} \lambda(dt)\lambda(ds) \]

Assumption A \( \Rightarrow \)

\[ \bar{k}_i(t, s) \]

unif. over [0, 1]²
LEADING EXAMPLES

EX 1. fractional Brownian motion of Hurst index $H$ ($0 < H < 1$) [shortly, fBm($H$)]

$$k(t, s) \equiv k_H(t, s) = \frac{t^{2H} + s^{2H} - |t - s|^{2H}}{2}$$

($H = 1/2$: Brownian motion; $H < 1/2$: short memory dependence property; $H > 1/2$: long memory dependence property)

$\Rightarrow$ **Theorem 1 holds with** $\gamma_\varepsilon^2 = \varepsilon^{2H}$ **and** $\bar{k}_i(t, s) \equiv k_H(t, s)$.

(one “looses” the times $t_1, \ldots, t_i$ and the past observations and only the present one $(x_{i-1})$ affects the asymptotic behavior)

EX 2. integral of a Gaussian process [shortly, iGp]: $X_t = \int_0^t Z_u du$, $Z$ Gaussian process with (continuous) covariance function $\rho$. Then

$$k(t, s) = \int_0^t du \int_0^s dv \rho(u, v)$$

$\Rightarrow$ **Theorem 1 holds with** $\gamma_\varepsilon^2 = \varepsilon^2$ **and** $\bar{k}_i(t, s) \equiv c_i ts$, where $c_i > 0$ depends on $t_1, \ldots, t_{i-1}, x_1, \ldots, x_{i-1}$. 
Step 2: the bridge of the $i$-fold conditional process: as $t \in [0, 1]$,

\[ Y_{t_{i-1}+\varepsilon}^i \overset{\mathcal{L}}{=} \left[ X_{t_{i-1}+\varepsilon}^i \mid X_{t_{i-1}+\varepsilon}^i = x_i \right] \overset{\mathcal{L}}{=} X_{t_{i-1}+\varepsilon}^i - \beta_t^\varepsilon (X_{t_{i-1}+\varepsilon}^i - x_i) \]

where \( \beta_t^\varepsilon = \frac{k_i(t_{i-1} + \varepsilon, t_{i-1} + \varepsilon)}{k_i(t_{i-1} + \varepsilon, t_{i-1} + \varepsilon)} \overset{\varepsilon \to 0}{\longrightarrow} \bar{\beta}_t = \frac{k_i(t, 1)}{k_i(1, 1)} \) under Assumption A.

For the associated large deviations, two classes of results:

a) “non degenerate” ones [ex. fractional Brownian motion]

b) “degenerate” ones [ex. integral of Gaussian processes]:

by refining Assumption A, one gets faster ("non degenerate") large deviations

b1) for $k$ regular but not too much

b2) for $k$ definitively regular
Theorem 2. Under Assumption A, \( \{Y^i_{t_{i-1}+\epsilon t}\}_\epsilon \) satisfies a large deviation principle on \( C([0,1]) \) with inverse speed \( \gamma^2_\epsilon \) and rate function

\[
J^Y_i(h) = \begin{cases} 
\frac{1}{2} \| h - \bar{m} \|_{\mathcal{H}^Y_i} & \text{if } h_0 = x_{i-1}, h_1 = x_i \text{ and } h - \bar{m} \in \mathcal{H}^Y_i \\
\infty & \text{otherwise}
\end{cases}
\]

where \( \mathcal{H}^Y_i \) is the rkHs associated to the (continuous) covariance function

\[
\bar{k}^Y_i(t,s) = k_i(t,s) - \frac{k_i(t,1)k_i(s,1)}{k_i(1,1)}
\]

and \( \bar{m}_t = x_{i-1} + \bar{\beta}_t(x_i - x_{i-1}) \).

EX 1: fBm(\( H \)). Theorem 2 holds with \( \gamma^2_\epsilon = \epsilon^{2H} \). Moreover,

\[
J^Y_i(h) = \begin{cases} 
\frac{1}{2} \left( \| h - x_{i-1} \|_{\mathcal{H}_H} - |x_i - x_{i-1}|^2 \right) & \text{if } h_0 = x_{i-1}, h_1 = x_i \text{ and } h - x_{i-1} \in \mathcal{H}_H \\
\infty & \text{otherwise}
\end{cases}
\]

\( \mathcal{H}_H \) being the rkHs associated to a fBm(\( H \)).
**EX 2: iGp.** Theorem 2 holds with $\gamma_{\varepsilon}^2 = \varepsilon^2$ and $\bar{k}^Y_i(t, s) \equiv 0$, i.e.

$$J^Y_i(h) = \begin{cases} 
0 & \text{if } h = \bar{m} \\
+\infty & \text{otherwise}
\end{cases}$$

⇒ degenerate result, wherever $k \in C^2$: the large deviation speed has to be faster in order to have significant estimates!

Notice that Assumption A under smooth $k$ gives:

[A1] \( \text{Cov}(X_{t_i+\varepsilon t} - X_{t_i}, X_{t_i+\varepsilon s} - X_{t_i}) = \varepsilon^2(\bar{k}(t, s) + o(1)), \) with $\bar{k}(t, s) + o(1) = \partial^2_{ts} k(t, s) + o_1(1)$

[A2] \( \text{Cov}(X_{t_i+\varepsilon t} - X_{t_i}, X_T) = \varepsilon(\rho(t, T) + o(1)), \) with $\rho(t, T) + o(1) = \partial_t k(t, T) + o_2(1)$

⇒ two classes of results according to different behaviors for $o_1(1)$ and $o_2(1)$
Assumption B  For some $\alpha \in (0, 1]$,

B1  $o_1(1) = \bar{\varphi}(t, s)\varepsilon^\alpha + R^1(1)$, with $R^1 = o(\varepsilon^\alpha)$ uniformly over $[0, 1]^2$

B2  $o_2(1) = \bar{\psi}(t)\varepsilon^\alpha + R^2(t)$, with $R^2 = o(\varepsilon^\alpha)$ uniformly over $[0, 1]$

Theorem 3. Under Assumption B, $\{Y_{t_{i-1}+\varepsilon t}\}_{\varepsilon}$ satisfies a large deviation principle on $C([0, 1])$ with inverse speed $\varepsilon^{2+\alpha}$ and rate function as in Theorem 1 with

$$\bar{k}^Y_{i}(t, s) = \bar{\varphi}_i(t, s) + ts\bar{\varphi}_i(1, 1) - t\bar{\varphi}_i(1, s) - s\bar{\varphi}_i(t, 1)$$

with $\bar{\varphi}_i(t, s)$ recursively written starting from $\bar{\varphi}$ and $\bar{\psi}$.

**BUT**

Theorem 3 gives again “degenerate” estimates wherever $k \in C^3$!
Assumption C For some $\alpha \in (0, 1]$,

C1 $o_1(1) = b t s (t + s) \varepsilon + \bar{\phi}(t, s) \varepsilon^{1+\alpha} + R_\varepsilon^1(t, s)$, with $R_\varepsilon^1 = o(\varepsilon^{1+\alpha})$ uniformly over $[0, 1]^2$

C2 $o_2(1) = d t^2 \varepsilon + \bar{\psi}(t) \varepsilon^{1+\alpha} + R_\varepsilon^2(t)$, with $R_\varepsilon^2 = o(\varepsilon^{1+\alpha})$ uniformly over $[0, 1]$

**Theorem 4.** Under Assumption C, $\{Y^i_{t-1+\varepsilon t}\}$ satisfies a large deviation principle on $C([0, 1])$ with inverse speed $\varepsilon^{3+\alpha}$ and rate function as in Theorem 1 with

$$k_Y^i(t, s) = \begin{cases} 
\bar{\phi}_i(t, s) + ts\bar{\phi}_i(1, 1) - t\bar{\phi}_i(1, s) - s\bar{\phi}_i(t, 1) & \text{if } \alpha < 1 \\
b_i^2(ts^2 + t^2s - t^2s^2 - st) + \\
\bar{\phi}_i(t, s) + ts\bar{\phi}_i(1, 1) - t\bar{\phi}_i(1, s) - s\bar{\phi}_i(t, 1) & \text{if } \alpha = 1
\end{cases}$$

with $\bar{\phi}_i(t, s)$ and $b_i$ recursively written starting from $\bar{\phi}, \bar{\psi}, b, d$. 

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**EX: iGp** Take $X = \int_0^t Z_u du$ where $Z = \text{fBm}(H)$

- if $H \leq 1/2$: Assumption B holds and Theorem 3 can be applied
- if $H > 1/2$: Assumption C holds and Theorem 4 can be applied

In summary,

**a large deviation principle holds for the bridge of the $i$-fold conditional process, with inverse speed $\varepsilon^{2+2H}$ and asymptotic covariance function**

$$
\bar{k}_i^Y(t, s) \equiv \bar{k}_H^Y(t, s) = \Phi_H(t, s) + ts\Phi_H(1, 1) - t\Phi_H(1, s) - s\Phi_H(t, 1)
$$

with

$$
\Phi_H(t, s) = \begin{cases} 
\frac{(|t - s|^{2H+2} - t^{2H+2} - s^{2H+2})}{2(2H + 1)(2H + 2)} & H \neq 1/2 \\
\frac{(t \wedge s)^3}{3} + \frac{(t \wedge s)^2}{2}|t - s| & H = 1/2.
\end{cases}
$$
Gaussian processes

THE EXIT PROBLEM

Recall: \( p_\varepsilon^i(x_1, \ldots, x_i) = \mathbb{P}\left( X \text{ reaches the boundary during } [t_{i-1}, t_i] \equiv [t_{i-1}, t_{i-1} + \varepsilon] \bigg| X_0 = x_0, \ldots, X_{t_{i-1}} = x_{i-1} \text{ AND } X_{t_i} = x_i \right) \), so that

\[
p_\varepsilon^i(x_1, \ldots, x_i) = \mathbb{P}(Y_{t_{i-1}+\varepsilon}^i \text{ reaches the boundary during } [0, 1])
\]

**Theorem 5** Suppose Assumption A holds, in addition to Assumption B or C, so that \( \{Y_{t_{i-1}+\varepsilon}^i\}_\varepsilon \) satisfies a (non degenerate) large deviation principle on \( C([0, 1]) \) with inverse speed \( \eta_\varepsilon^2 \) and asymptotic covariance function \( \bar{k}_Y^i \). If \( L \) and \( U \) are continuous,

\[
\log p_\varepsilon^i(x_1, \ldots, x_i) \simeq -E_i/\eta_\varepsilon^2
\]

where \( E_i = E_i^L, E_i^U, E_i^{L,U} \) in the lower, upper and double barrier case respectively, with
\[ E^L_i = \inf_{t \in [0,1]} \frac{\left((x_{i-1} - L_t)(1 - \beta_t) + \beta_t (x_i - L_t)\right)^2}{2 \hat{k}^Y_i(t,t)} \quad x_{i-1}, x_i > L_{t_{i-1}} \]

\[ E^U_i = \inf_{t \in [0,1]} \frac{\left((U_t - x_{i-1})(1 - \beta_t) + \beta_t (U_t - x_i)\right)^2}{2 \hat{k}^Y_i(t,t)} \quad x_{i-1}, x_i < U_{t_{i-1}} \]

\[ E^{L,U}_i = \min \left( E^L_i, E^U_i \right) \quad x_{i-1}, x_i \in (L_{t_{i-1}}, U_{t_{i-1}}) \]
Numerical results

fractional Brownian Motion \([fBm(H), \ H \in (0,1)]\)

A fBm\((H)\) is a continuous Gaussian process with covariance function

\[
k_H(t, s) = \frac{t^{2H} + s^{2H} - |t - s|^{2H}}{2}
\]

\((H = 1/2: \text{Brownian motion}; \ H < 1/2: \text{short memory dependence property}; \ H > 1/2: \text{long memory dependence property})\)

**REMARK**

Use of fBm\((H)\) in Finance: still not clear, even if it would be important to model non Markovianity properties for financial markets. But see Cheridito (Bernoulli 2001 and Finance and Stochastics 2003) for interesting semimartingale properties (for a linear combination of a fBM(1/2) and a fBm\((H)\) with \(H > 3/4\), independent each other).
The table shows the Monte Carlo estimated probability of crossing the upper barrier $U = 1$ up to time 1, for varying values of the Hurst index $H$. In brackets, the associated 95% confidence interval (true value available only for $H = 1/2$: exit probability $= 0.31732$).

<table>
<thead>
<tr>
<th>Method</th>
<th>Step</th>
<th>$H = 0.3$</th>
<th>$H = 0.5$</th>
<th>$H = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>corrected</td>
<td>0.01</td>
<td>0.60876 (0.60573, 0.61178)</td>
<td>0.31820 (0.31531, 0.32109)</td>
<td>0.20564 (0.20313, 0.20814)</td>
</tr>
<tr>
<td>corrected</td>
<td>0.002</td>
<td>0.61841 (0.61540, 0.62142)</td>
<td>0.31790 (0.31691, 0.32269)</td>
<td>0.20274 (0.20025, 0.20523)</td>
</tr>
<tr>
<td>crude</td>
<td>0.01</td>
<td>0.47909 (0.47599, 0.48219)</td>
<td>0.28918 (0.28637, 0.29199)</td>
<td>0.19884 (0.19637, 0.20131)</td>
</tr>
<tr>
<td>crude</td>
<td>0.002</td>
<td>0.54114 (0.53805, 0.544230)</td>
<td>0.30496 (0.30211, 0.30781)</td>
<td>0.20222 (0.19973, 0.20471)</td>
</tr>
<tr>
<td>crude</td>
<td>0.001</td>
<td>0.56082 (0.55774, 0.56390)</td>
<td>0.30878 (0.30592, 0.31164)</td>
<td>0.20251 (0.20002, 0.20500)</td>
</tr>
</tbody>
</table>
THANKS!