Optimization under uncertainty: modeling and solution methods

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Lecture 6: Risk measurement/management


1 Risk aversion

2 Risk measures

3 Ex post vs. ex ante
You are offered a lottery, linked to the flip of a fair coin. If the outcome is head, you win €10; in the case of tail, you lose €5. Do you accept?

You are offered a lottery, linked to the flip of a fair coin. If the outcome is head, you win €10 million; in the case of tail, you lose €5 million. Do you accept?

What if you could repeat the game 1000 times, and settle the score only at the end?
In the microeconomics of uncertainty, the basic tool to represent risk aversion is a utility function $u(\cdot)$.

If $\pi(x, \omega)$ is the random monetary outcome of decision $x$, the model assumes that the expected utility is maximized:

$$\max_{x \in X} E_\omega [u(\pi(x, \omega))] \quad (1)$$

Risk aversion is related to the concavity of $u(\cdot)$; if this is just the identity function, we are risk neutral and care only about expected values.

For instance, let us compare

1. a lottery with equally likely outcomes $\mu - \delta$ and $\mu + \delta$;
2. a certain (sure) amount $\mu$.

A risk averse decision maker would not like the mean preserving spread $\delta$. 
Due to the concavity of $u(\cdot)$, for the risky lottery we have

$$E[u(\pi)] = \frac{1}{2} u(\mu - \delta) + \frac{1}{2} u(\mu - \delta) \leq u(\mu) = u(E[\pi])$$

This can be generalized to generic probabilities $\lambda$ and $1 - \lambda$, and to an arbitrary number of outcomes.
Utility functions

Several utility functions have been proposed in the literature.

Unfortunately, despite their theoretical appeal, utility functions have a few shortcomings:

- It is difficult to specify a utility function for a single decision maker, even though there are procedures for its elicitation.
- It is difficult to specify a utility function for a group of decision makers.
- When you come up with a decision, it is difficult to justify it with your boss, since utility measures attitude towards risk, but not the risk itself.
- The theoretical foundation of the approach is based on a set of axioms about preferences; some of these axioms have been questioned.
- They are not quite consistent with observed behavior.
Mean-risk models

Since utility functions may not be easy to use, a more practical approach is based on the definition of risk measures.

Formally, a risk measure is a function $\rho(X)$ mapping a random variable to a non-negative number.

Rather than just maximizing expected profit, we may trade off expected values and risk measures.

One possibility is to aggregate the two components into a single objective:

$$\max_{x \in \mathcal{X}} \ E_\omega[\pi(x, \omega)] - \lambda \rho[\pi(x, \omega)]$$

where $\lambda$ is a coefficient related to risk aversion.
Mean-risk models

Alternatively, we may define a risk budget $\alpha$ and maximize expected profit:

$$\max_{x \in X} E_\omega [\pi(x, \omega)]$$

s.t. $\rho[\pi(x, \omega)] \leq \alpha$ (3)

Finally, we may define an expected profit target $\beta$ and minimize risk:

$$\min_{x \in X} \rho[\pi(x, \omega)]$$

s.t. $E_\omega [\pi(x, \omega)] \geq \beta$ (4)

The choice of one of the three frameworks above depends on the difficulty in solving the problem and in selecting an appropriate value for the relevant parameter.

See (Ruszczyński and Shapiro, 2003) for a discussion of mean-risk stochastic programming models.
The prototypical mean-risk model is the Markowitz portfolio optimization model. We have a set of $n$ risky assets with random return $R_i$, $i = 1, \ldots, n$, expected return $\mu_i$, and covariance matrix $\Sigma = [\sigma_{ij}]$.

Given a portfolio represented by a vector $w$ of weights, its expected return and variance are:

$$
\mu_p = \sum_{i=1}^{n} \mu_i w_i = \mu^T w, \quad \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \sigma_{ij} w_j = w^T \Sigma w
$$

Then, we solve the QP:

$$
\min w^T \Sigma w \quad \text{s.t.} \quad \mu^T w = r_T, \quad e^T w = 1, \quad w \geq 0
$$

where $r_T$ is a target expected return, $e$ is a vector of ones, and the non-negativity constraints rule out short-selling.
Note that we use standard deviation as a risk measure, but minimize variance for computational convenience.

By changing the target return, we may sweep the portfolio efficient frontier:
Standard deviation is an intuitive risk measure, as it measures the variability of the outcome.

The idea can also be applied outside the financial domain.

However, it has some pitfalls, as we do not distinguish the good and the bad tail of the distribution.

When we want to account for skewness, we may consider alternative risk measures, such as:

- expected shortfall,
- value at risk (V@R),
- conditional value at risk (CV@R).
Expected shortfall with respect to a profit target $\beta$ is defined as

$$E_\omega[\max\{0, \beta - \pi(x, \omega)\}] = E_\omega[\beta - \pi(x, \omega)]^+.$$ 

If we have a set of scenarios with probability $p^s$ and let $\pi^s(x)$ be the profit under scenario $s$, we may consider the objective function

$$\max \sum_s p^s \pi^s(x) - \lambda \sum_s p^s [\beta - \pi^s(x)]^+$$

where $\lambda$ is a risk aversion parameter.

This is easy to linearize by the introduction of auxiliary shortfall variables $w^s$:

$$\max \sum_s p^s \pi^s(x) - \lambda \sum_s p^s w^s$$

s.t. $w^s \geq \beta - \pi^s(x)$, $w^s \geq 0$, $\forall s$

$x \in X$
A widely used measure in financial engineering is Value-at-Risk (V@R). V@R is a quantile-based risk measure, i.e., it is a quantile of the distribution of loss over a chosen time horizon, at some probability level.

Roughly speaking, if V@R with confidence level 99% over one day is $x$, we are “99% sure” that we will not lose more than $x$ in one day.

V@R is widely used in banking and it has the advantage of intuitive appeal, as well as the ability to merge different kinds of risk in a sensible way; see (Jorion, 2007).

However, such a measure suffers from severe drawbacks:

1. it does not fully account for what may happen in the bad tail of the distribution,
2. it is not a coherent measure of risk,
3. it is hard to optimize within scenario-based optimization, because the resulting model need not be convex.
What is the riskier loss distribution? Does V@R at 95% tell the difference?

To appreciate the difference, we should take a conditional expectation of loss on the bad tail. This leads to the idea of Conditional V@R.
In (Artzner et al., 1999) a set of properties characterizing a coherent measure of risk.

These properties include subadditivity:

$$\rho(X + Y) \leq \rho(X) + \rho(Y),$$

for any random variables $X$ and $Y$, i.e., diversification cannot increase risk.

Exercise: Consider two zero coupon bonds, with face value 100, which may independently default with probability 96%. Prove that V@R at 95% for each position is 0, whereas V@R at 95% for the joint portfolio is 100.
Conditional Value at Risk (CV@R) is related to a conditional tail expectation.

Roughly speaking, it is the expected value of loss, conditional on being on the bad tail of the distribution; in other words, it is the expected loss if we are beyond the corresponding Value at Risk (V@R), for a confidence level $1 - \alpha$.

Despite its apparent complexity, CV@R is nicer to optimize than V@R, and we often obtain convex, possibly linear programming problems.

Furthermore, it is a coherent risk measure.

NOTE. We are cutting a few corners here. Quantiles are trivial for continuous distributions with nice CDFs; however, when dealing with discrete distributions some care is needed. See (Rockafellar and Uryasev, 2002) for the related technicalities.
Let $f(x, \xi)$ be a loss function, depending on decision variable $x$ and random variable $\xi$, and consider the function $F_\alpha(x, \zeta)$ defined as

$$F_\alpha(x, \zeta) = \zeta + \frac{1}{1 - \alpha} \int [f(x, \xi) - \zeta]^+ g_\xi(y) dy,$$

where $g_\xi(y)$ is the PDF of $\xi$.

CVaR minimization is accomplished by the minimization of $F_\alpha(x, \zeta)$ with respect to its arguments (Rockafellar and Uryasev, 2000).

In a stochastic programming model with discrete scenarios, where $f(x, \xi^s)$ the loss in scenario $s$, we solve:

$$\min \quad \zeta + \frac{1}{1 - \alpha} \sum_s p^s z^s$$

$$\text{s.t.} \quad z^s \geq f(x, \xi^s) - \zeta, \quad \forall s$$

$$\zeta \in \mathbb{R}, \quad z^s \geq 0, \quad \forall s$$

subject to the additional constraints depending on the specific model.
Consider the following decision tree (consisting of chance and decision nodes). What is your choice at node $N_1$?
And what would you choose in this case?
In the following decision tree, before the random outcome at chance node $N_1$ is realized, you have to state your choice at node $N_2$. 
The above example, described in (Bell, 1985), points out an inconsistency in standard models of rational decisions.

The typical contradiction observed in decision makers may be explained in terms of emotions related to disappointment and regret.

Regret-based optimization models have been proposed as a robust decision making tool (Kouvelis, Yu, 1996).

Risk management policies should be evaluated ex ante, but sometimes managers are evaluated ex post. Regret is relevant in such a case.
If we knew which scenario $\omega_s$ will be realized, we would just solve the scenario problem $\max_{x \in \mathcal{X}} \pi(x, \omega_s)$ with optimal solution $x^*_s$ and profit $\pi^*_s = \pi(x^*_s, \omega_s)$.

In practice, since we cannot wait and see, we solve the *here and now* problem, whose solution is $x^*_h$. The profit, if scenario $s$ occurs, will be $\hat{\pi}_s = \pi(x^*_h, \omega_s)$.

The *regret*, with respect to the optimal scenario $s$ solution, is:

$$R_s = \pi^*_s - \hat{\pi}_s.$$  

It is possible to optimize mean regret, maximum regret, or risk measures based on regret,