

# Mathematical models for soil consolidation problems: a state of the art report

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## Introduction

The object of the consolidation theory is, in a broad sense, any movement of the soil that yields a non-negligible effect at the macroscale. This definition includes prediction of settlement rate of structures interacting with the soil, subsidence, oil production, diffusion of pollutants, transient phenomena occurring in earthquakes and wave propagation.

Since the pioneering work of Terzaghi (1923), there has been a growing interest in understanding the physical basis and the related mathematical modelling of the consolidation theory. As outlined in the recent book by de Boer (2000), where the reader can find an introduction to the subject in a historical perspective, this interest arises from both theoretical basis, the mechanics of porous media, as well as from engineering applications. By the way, we note that the consolidation is the most ancient field of application of a more general science, the porous media theory, that is currently applied in a surprisingly large number of different fields. As an example, in biomechanics (Gu et al., 1999) the bone can be represented as a porous medium infiltrated by blood and in composite material manufacturing the injection moulding process is in essence the infiltration of a non-newtonian fluid into a solid porous matrix (see Ambrosi&Preziosi, 2000).

The class of problems we are interested in is characterized by the motion of solids and fluids at different velocities. The global mechanical behavior of the fluid-solid system is then dictated by both the mechanical properties of the single constituents and the by interaction occurring between them, due to their relative displacement. The aim is to describe the property of the solid-fluid system at a spatial scale much larger than the one characterizing the velocity fields. Therefore, in its essence, the problem is relevant in all scientific areas where a "coupling between the evolution of a porous deformable medium and the motion of fluids in the connected porosity" occurs.

The aim of this chapter is to provide a summary of the main results of the porous media theory that can be obtained rigorously moving from the principles of continuum mechanics. As a by product, we make an effort to put in an unique framework equations and notations that are used in the mathematical as well engineering literature, stressing the assumptions that lead to the different models. The presentation is mainly intended for an engineering audience, so that the mathematical formalism is kept at a minimum.

This chapter is divided into five sections. Section 1 outlines the general framework of the modelling of porous materials, i.e. the "averaging approaches", emphasizing the related basic assumptions. Section 2 discusses the "effective stress" concept. Section 3 deals with kinematics and balance equations at a macroscopic level according to the *mixture theory*, and derives a consistent Lagrangian model for both the soil skeleton and the fluid. Section 4 formulates the initial boundary value problem of typical geotechnical applications, such as consolidation of a stratum between draining and impervious boundaries with stress or velocity conditions boundary conditions: in this context Biot's theory is illustrated. Finally, Section 5 discusses aspects related to the constitutive equations, which is of crucial importance when dealing with reliable quantitative predictions.

# 1 Modeling porous media

In the current literature the mechanics of a porous medium is typically described by two different approaches. The *averaging approach* starts from the equations describing the motion of all the phases constituting the porous medium separately: each of them is considered in its own domain as a single body, while interactions are just accounted for as a contribution arising at the internal boundary. The link between this description and the macroscopic behavior is obtained by the averaging procedure. Conversely, the *macroscopic approach* starts from an *a priori* assumption of homogenized phases and is based on the axioms of the theory of mixtures supplemented by the concept of volume fraction.

It is to be noted that the final goal of both these approaches is the description of predictable and repeatable macroscopic features of the dynamics of the porous medium. In fact, even though it might be at some extent desirable to achieve a detailed knowledge of the flow of the fluid in the pores and the displacement of the solid particles at each point, this appears to be an impossible task: we can not know the detailed geometry at the microscopic scale and, more important, this geometry will be different from specimen to specimen, i.e. there is a lack of repeatability. For these reasons it is relevant to focus the attention on some macroscopic features independent of the specific configuration of the pores which can be verified by repeatable macroscopic experiments. Despite the aim is to provide a large scale description, it is not to be understood that the smaller scale can be completely disregarded: a knowledge of the microscopic mechanisms that drive the macroscopic repeatable processes is of great interest to state an effective macroscopic model.

The continuum mechanics approach starts from these basic considerations and any experimental determination of the constitutive response of a given medium yield relationships between "gross properties". Therefore, in their essence, the methods illustrated in this chapter are aimed at "the replacement of a micro-heterogeneous medium with a homogeneous one, which macroscopically behaves in the same manner". Such a process poses a fundamental question: how can one describe the macroscopic behavior of a medium which exhibits microscopic heterogeneity, on the base of microstructural information?

In the following paragraph we first discuss the conceptual passage from a microscopic to a macroscopic viewpoint through averaging procedure, while in the next one we present the description based on porous media theory. A comparison of the results obtained by these approaches enables one to clarify the meaning and the role of several terms appearing in the field equations.

## 1.1 Volume and Ensemble Averaging Approach

We start considering the real non-homogeneous structure of the porous medium, in which the scale of heterogeneity is of the same order of magnitude of the pore or grain dimension. Then we focus on fields describing the status of a single phase, defined only at the points occupied by that phase. Because, as already mentioned, this level of detail is not needed for our purposes, we use an averaging technique in order to get a macroscopic description (see Slattery, 1981; Bear and Bachmat, 1991; Lewis and Schrefler, 1998). In this respect the crucial concept is the Representative Elementary Volume (REV). This idea introduced for the first time by Lorentz in 1952, in his book on the theory of electrons. Later Bear (1972) discussed the same concept in relation to porous media and further rigorous contributions were given by Nemat-Nasser and Hori (1993) and Drugan and Willis (1996). We introduce the definition of REV in a rather heuristic manner: "It is a volume that, at a macroscopic scale, is small enough to be treated as a point of the heterogeneous medium, and, on the reverse, at a microscale, is large enough to contain a significant number of single heterogeneities". The interested reader can find a comprehensive discussion on this subject in the paper of Markov (2000).

From the above definition it follows that the position of a REV is identified, within the domain which mathematically defines the medium of interest, by the vector position  $\mathbf{x}$ . Any field  $q$  of interest at the micro-level, i.e. within the REV, depends on the local coordinate  $\mathbf{r}$  spanning the REV:

$$q = q(\mathbf{x}, \mathbf{r}, t). \quad (1.1)$$

The dependence expressed by (1.1) of any internal field from both the macro-coordinate  $\mathbf{x}$  and the micro-coordinate  $\mathbf{r}$  implies that the internal field can behave in different ways within different REVs of the medium. The relation between macro and micro quantities is obtained through the volume averaging with respect to the micro-coordinate  $\mathbf{r}$ , which allows to define the so-called phase average and intrinsic phase average of any field  $q$ :

$$\overline{q_\alpha(\mathbf{x}, t)} =: \frac{1}{V} \int_v H_\alpha(\mathbf{r}, t) q(\mathbf{r}, t) d\mathbf{r}, \quad (1.2)$$

$$\overline{q(\mathbf{x}, t)}^\alpha =: \frac{1}{V_\alpha} \int_v H_\alpha(\mathbf{r}, t) q(\mathbf{r}, t) d\mathbf{r}, \quad (1.3)$$

where  $v$  is the REV of volume  $V$  centered on  $\mathbf{x}$  and  $H_\alpha$  is an indicator function for the  $\alpha$ -th constituent, its value being 1 if, at time  $t$ , the constituent is in  $r$ , and 0 otherwise. According to this definition, the volume of the constituent within a REV is given by

$$V_\alpha(\mathbf{x}, t) =: \int_v H_\alpha(\mathbf{r}, t) d\mathbf{r}, \quad (1.4)$$

and the corresponding volume fraction  $n_\alpha$  is

$$n_\alpha(\mathbf{x}, t) =: \frac{V_\alpha}{V} = \frac{1}{V} \int_v H_\alpha(\mathbf{r}, t) d\mathbf{r}. \quad (1.5)$$

As a consequence, we define  $\rho_\alpha^R =: \overline{\rho}^\alpha$  material density and  $\rho_\alpha = \overline{\rho} = n_\alpha \rho_\alpha^R$  is the partial density. From the above definitions, it follows that the material incompressibility is not equivalent to the bulk incompressibility of each component: as  $\rho_\alpha = n_\alpha \rho_\alpha^R$  the partial density of the  $\alpha$ -th component in the mixture can change due to changes in the volume fraction even though  $\rho_\alpha^R$  is constant.

Once the REV average property has been obtained, in order to define the properties of the whole porous medium we should in principle perform experiments with  $N$  different REVs and obtain the so-called *ensemble average*

$$\overline{q_i^*} = \frac{1}{N} (\overline{q_{i,v_1}} + \overline{q_{i,v_2}} + \dots + \overline{q_{i,v_N}}) \quad (1.6)$$

Provided that the microscale length of a typical inhomogeneity is small enough when compared with the volume of the medium, it can be expected that the ensemble and the volume averages coincide. This implies the introduction of the hypothesis of ergodicity, i.e. a macroscopic or statistical homogeneity of the porous medium. Roughly speaking, this hypothesis means that the macroscopic properties of all the REV are the same and the REV has the same properties of the porous medium as a whole. For this reason in the following we make use of the volume averaging approach. Moreover, in some cases we also introduce the hypothesis that the medium is statistically isotropic, i.e. the macroscopic properties are independent of the orientation of the medium in the space.

## 1.2 Balance Equations

In order to derive macroscopic field equations, a general strategy can be pursued starting from the microscopic balance equations of any extensive quantity in a continuum and obtaining the corresponding macroscopic balance equation by averaging rules. In this section we follow the approach of Bear and Bachmat (1991).

Let  $q$  be the volume density of any extensive phase quantity  $Q$  which is assumed to be a differentiable function of time and space within the phase domain. The general balance microscopic equation takes the form

$$\frac{\partial q}{\partial t} + \nabla \cdot (q \mathbf{V}^Q) = \rho \Gamma^Q, \quad (1.7)$$

where  $\rho$  is the mass density of the phase,  $\Gamma^Q$  is the rate of internal production of  $Q$  per unit mass and  $\mathbf{V}^Q$  is the velocity attached to  $Q$ . If we introduce the volume averaged velocity  $\mathbf{V}$  we can define the diffusive flux  $\mathbf{j}^Q$ :

$$\mathbf{j}^Q =: q(\mathbf{V}^Q - \mathbf{V}), \quad (1.8)$$

and the balance equation (1.7) can be rewritten in the following form,

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{V} + \mathbf{j}^Q) = \rho\Gamma^Q. \quad (1.9)$$

Now in Eq. (1.9) one can distinguish the contribution of the advective flux from the diffusive one. Taking the volume average of (1.9), and by using the definitions (1.3) and (1.5) one gets

$$n_\alpha \frac{\partial \overline{q}}{\partial t} + n_\alpha \overline{\nabla \cdot (q\mathbf{V} + \mathbf{j}^Q)} = n_\alpha \overline{\rho\Gamma^Q}. \quad (1.10)$$

By applying standard averaging rules to both sides of (1.10) (for more details, see Bear and Bachmat, 1991), the general macroscopic differential balance equation is obtained

$$\frac{\partial}{\partial t}(n_\alpha \overline{q}) + \nabla \cdot (n_\alpha \overline{q\mathbf{V}} + \overline{\mathbf{j}^Q}) + \frac{1}{V} \int_{S_\alpha} [q(\mathbf{V} - \mathbf{v}^{S_\alpha}) + \mathbf{j}^Q] \cdot \mathbf{n} dS = n_\alpha \overline{\rho\Gamma^Q}. \quad (1.11)$$

The surface  $S_\alpha$ , moving with velocity  $\mathbf{v}^{S_\alpha}$ , separates the phase under consideration from the other phases. The term containing the surface integral represents the flux of the extensive quantity under consideration through  $S_\alpha$ . It is worth noticing that a boundary condition on the interface becomes, thanks the averaging process, a source term in the macroscopic equation.

### 1.3 Mass balance equation of a phase

Now we specialize the definitions above to the mass of the  $\alpha$ -th component (i.e.  $Q = m$ ). The corresponding density is  $\rho$  and we assume that  $\Gamma^m = 0$ . We chose the surface  $S_\alpha$  as a material surface for the phase under consideration, so that there is no mass flux across it. In addition we decompose the average advective flux  $\overline{\rho\mathbf{V}^\alpha}$  into a macroscopic advective flux  $\overline{\rho^\alpha\mathbf{V}^\alpha}$  and a dispersive flux which the difference between the former and the latter. If the diffusive ( $\mathbf{j}^Q$ ) and dispersive fluxes can be neglected when compared to the advective one, the general macroscopic balance equation (1.11) specifies to:

$$\frac{\partial}{\partial t}(n_\alpha \overline{\rho^\alpha}) + \nabla \cdot (n_\alpha \overline{\rho^\alpha\mathbf{V}^\alpha}) = 0. \quad (1.12)$$

### 1.4 Momentum balance equation of a phase

Following the interpretation of Bear and Bachmat, by similar arguments the extensive quantity *linear momentum* (i.e.  $\mathbf{Q} = m\mathbf{V}^m$ ) has density  $q = \rho\mathbf{V}^m$ , the rate of momentum production is the body force per unit mass  $\mathbf{b}$  and the diffusive flux of linear momentum is the stress tensor  $\mathbb{T}$ , where  $\mathbf{V}^m$  is the mass weighted velocity defined by

$$\rho\mathbf{V}^m(\mathbf{x}, t) =: \overline{\rho^R\mathbf{v}(\mathbf{r}, t)}. \quad (1.13)$$

Under the assumption that the momentum dispersive flux can be neglected when compared with the advective one and that the interface surface is material for the component at hand, equation (1.11) gives the macroscopic momentum balance

$$n_\alpha \overline{\rho^\alpha} \frac{D\overline{\mathbf{V}^m}}{Dt} - \nabla \cdot (n_\alpha \overline{\mathbb{T}^\alpha}) = \frac{1}{V} \int_{S_\alpha} \mathbb{T} \cdot \mathbf{n} dS + n_\alpha \overline{\rho^\alpha\mathbf{b}^\alpha}. \quad (1.14)$$

where

$$\frac{D}{Dt} =: \frac{\partial}{\partial t} + (\mathbf{V}^m \cdot \nabla) \quad (1.15)$$

The surface integral term at the r.h.s. of Equation (1.14) represents the interaction force between the phases across the microscopic interface that separates them. In general, such an interaction occurs between wetting and non-wetting fluid phases, as well as between solid and fluid. Usually the assumption of continuity of traction at the interface between the solid and the fluids  $S_{s,f}$  is made, so that no jump exists across this microscopic interface, i.e.:

$$[[\mathbb{T}]]_{s,f} \cdot \mathbf{n} = 0. \quad (1.16)$$

On the contrary it is known that the jump at the interface between two immiscible fluids  $S_{f,f}$  is balanced by the surface tension  $\sigma$ :

$$\sigma = [[\mathbb{T}]]_{f,f} \cdot \mathbf{n}. \quad (1.17)$$

Summing up equations (1.14) for all phases and assuming that a quasi-stationary process occurs (so that the inertial terms can be neglected) the equilibrium of the porous medium as a whole writes

$$\nabla \cdot \bar{\mathbb{T}} + \rho \mathbf{b} + \bar{\sigma} = 0, \quad (1.18)$$

where  $\bar{\mathbb{T}}$  is the total stress and

$$\bar{\sigma} =: \frac{1}{V} \int_{f,f} [[\mathbb{T}]]_{f,f} \cdot \mathbf{n} dS. \quad (1.19)$$

**Remark.** When considering the specific problem of two phases (solid and fluid), it is convenient to assume as a basic kinematic variable the solid displacement  $\mathbf{u}^s$  and to introduce the motion of the fluid relative to the solid, the relative velocity being  $\mathbf{w}^f = \dot{\mathbf{x}}^f - \dot{\mathbf{x}}^s$ . With such an assumption, balance equations for the two phases are

$$\rho_s \ddot{\mathbf{u}}^s - \nabla \cdot \mathbb{T}^s = \rho_s \mathbf{b} - \hat{\mathbf{p}}, \quad (1.20)$$

$$\rho_f \frac{\partial}{\partial t} (\dot{\mathbf{u}}^s + \mathbf{w}^f) + \rho_f \mathbf{w}^f \cdot \nabla (\dot{\mathbf{u}}^s + \mathbf{w}^f) - \nabla \cdot \mathbb{T}^f = \rho_f \mathbf{b} + \hat{\mathbf{p}}, \quad (1.21)$$

where

$$\mathbf{w}(\mathbf{x}, t) =: \mathbf{v}^f(\mathbf{x}, t) - \mathbf{v}^s(\mathbf{x}, t). \quad (1.22)$$

The term  $\hat{\mathbf{p}}$  represents the local interaction per unit volume between the phases.

## 2 The effective stress

The interaction between the soil skeleton and the pore water has been first described by Terzaghi (1923) with the introduction of the *principle of effective stress*. Such a principle states that the total stress acting in a solid–liquid mixture can be decomposed into two additive contributions. The former acts both in the water and in the solid in every direction and is called *neutral stress* (or pore water pressure), the latter represents an excess over the neutral one, it has its seat exclusively in the solid phase of the soil and is called *effective stress*. In a more formal presentation of this principle, Terzaghi (1936) also stated that "porous materials (such as sand, clay and concrete) react to a change of pore pressure as if they were incompressible and as if their internal friction were equal to zero. All the measurable effects of a change of stress, as a compression, distortion and a change of shearing resistance are exclusively due to changes in the effective stress".

Note that Terzaghi's definition of effective stress does not follow from any theoretical investigation of basic porous medium balance equations, but it is introduced in a rather heuristic manner in terms of cause and measurable effects.

For sake of simplicity, in the following we refer to a two–phases medium, i.e. a solid skeleton saturated by water. This is the case of major interest in soil mechanics, but obviously the methodological approach can be extended to more than two phases. Moreover, we consider a saturated medium:

$$n_s + n_f = 1. \quad (2.1)$$

In particular, summing up Equations (1.12) for two incompressible constituents it turns out that the following kinematic constraint has to be satisfied:

$$\nabla \cdot \left( n_f \bar{\mathbf{V}}^w + (1 - n_f) \bar{\mathbf{V}}^s \right) = 0. \quad (2.2)$$

In order to clarify the meaning of the effective stress in the present framework, we consider first the case of a single fluid phase that saturates the void space. In this case, the principle of the effective stress reads:

$$\bar{\mathbb{T}} = (1 - n_f) \bar{\mathbb{T}}^s - n_f \bar{p}^f \mathbb{I}, \quad (2.3)$$

According to the definition of effective stress, this equation is usually written in Soil Mechanics in the form

$$\bar{\mathbb{T}} = \bar{\mathbb{T}}' - \bar{p}^f \mathbb{I}, \quad (2.4)$$

where  $\bar{\mathbb{T}}'$  is then defined by

$$\bar{\mathbb{T}}' = (1 - n_f) \left( \bar{\mathbb{T}}^s + \bar{p}^f \mathbb{I} \right). \quad (2.5)$$

At this stage it is relevant to note that the partial stress tensor assumes in the saturated mixtures theory the more general expression (see Section 4)

$$\mathbb{T}_\alpha = -n_\alpha p \mathbb{I} + \mathbb{T}_\alpha^E \quad (2.6)$$

where the partial stress for each phase is splitted into two contributions: the pressure  $p$  and an extra term linked to the deformations of the  $\alpha$ -th phase corresponding to the effective stress, or to the fluid dissipation. The interest in comparing (2.5) and (2.6) arises from the fact that the latter expression is derived on thermodynamical basis, requiring that the entropy inequality of the overall medium is satisfied in any admissible process (see Ehlers, 1993; Diebels and Ehlers, 1996; Wilmanski 1996, 2000). Note that, in this framework,  $p$  is the pressure of the *mixture* as a whole, that is the internal reaction force necessary for the accomplishment of the kinematic constraint (2.2) and there is no apparent reason for identifying it as the pressure of the water in the pores.

Wilmanski (2000) has proved that such a constraint is thermodynamically admissible only if the constitutive equations depend on  $\nabla n_f$ . If, in addition, the partial stress tensors are supposed to depend linearly on the relative velocity, then the constitutive relations take the following form:

$$\mathbb{T}_s = -(1 - n_f) p \mathbb{I} + \mathbb{T}_s^E(n_f, \mathbb{F}_s), \quad (2.7)$$

$$\mathbb{T}^f = -n_f p \mathbb{I}, \quad (2.8)$$

where  $\mathbb{F}_s$  is the tensor gradient of deformation of the soil skeleton and the material coefficients (that is the parameters characterizing the mechanical behavior of the material) depend on the porosity  $n_f$ .

When considering the three phasic flow (solid with two immiscible fluids), equation (2.3) can be rewritten in the form

$$\bar{\mathbb{T}} = (1 - n_f) \bar{\mathbb{T}}^s - S_w \bar{p}^w \mathbb{I} - S_n \bar{p}^n \mathbb{I}. \quad (2.9)$$

where  $S_w$  and  $S_n$  are the volume fractions of the wettin and non-wetting phase referred to the porosity  $n_f$  (i.e.  $S_w + S_n = n_f$ ). Accordingly, an average fluid pressure can be defined as

$$\bar{p}^f =: \frac{S_w \bar{p}^w + S_n \bar{p}^n}{n_f} \quad (2.10)$$

and the effective stress tensor again formally satisfied the relationship (2.4). A relevant consequence of (2.10) is that when considering unsaturated soils, the air is the non-wetting fluid and for pressure values in excess to the atmospheric one the following expression is obtained for the effective stress

$$\bar{\mathbb{T}}' = \bar{\mathbb{T}} - \frac{S_w}{n_f} \bar{p}^w \mathbb{I} \quad (2.11)$$

The agreement of Eq. (2.11) with experimental results is still a matter of investigation, also considering that in the literature different expressions have been suggested for the coefficient that multiplies the pore pressure.

The interaction force  $\hat{\mathbf{p}}$  typically involves a Fickian contribution, accounting for diffusion, plus a Darcian one, so that the interaction force is usually modelled as:

$$\hat{\mathbf{p}} = p \nabla n_f - (n_f)^2 \mu \mathbb{K}^{-1} \mathbf{w}^f. \quad (2.12)$$

where  $\mu$  is the dynamic viscosity of the fluid and  $\mathbb{K}(\mathbf{x})$  is the permeability tensor (possibly dependent on  $t$  for large deformations). so that the balance equation for the soil skeleton can be written in the following form, more usual in Soil Mechanics

$$\rho_s \ddot{\mathbf{u}}^s - \operatorname{div} \left( \mathbb{T}_E^S - p \mathbb{I} \right) = (1 - n_f) (\rho_{sR} - \rho_{fR}) \mathbf{b} + \rho_{fR} \mathbf{g}. \quad (2.13)$$

?????? COME LA SI OTTIENE?????? The momentum transfer at the solid-fluid interface is usually much larger than the inertial force and the viscous resistance inside the fluid. When neglecting the inertial terms the momentum equation for the fluid phase simplifies to

$$\rho_f n_f (\mathbf{v}^f - \mathbf{v}^s) = -\frac{\mathbb{K}}{\mu} \nabla p + \rho_f \mathbf{g}. \quad (2.14)$$

Equation (2.14) is commonly referred as Darcy for deformable porous media.

### 3 Lagrangian description of porous media

As already mentioned in the introduction, the macroscopic approach starts from an *a priori* assumption of homogenized phases and is based on the axioms of the theory of mixtures, supplemented with the concept of volume fraction. Fundamentals of the theory of mixture date back to the works of Truesdell (1957), Truesdell and Toupin (1960), Atkin and Craine (1976), Bedford and Drumheller (1983), Bowen (1976, 1980, 1982), de Boer and Ehlers (1986), Ehlers (1989, 1993, 1996), Coussy (1995), Müller (1968), Rajagopal and Tao (1995), Wilmanski (1996, 1998). The paper by de Boer (1996) is particularly relevant for its review character.

The basic assumption of the theory of mixture is that the individual components of the porous medium are statistically distributed over the control space, i.e. each spatial point  $\mathbf{x}$  of the control space is simultaneously co-occupied by particles of all components. As a consequence, mathematical functions describing both geometrical and physical properties of each constituent are field functions defined all over the control space. In addition, principles of mixtures theory to be used in balance equations are clearly stated by Truesdell in "Rational Thermodynamics" (1984):

- All properties of the mixture must be mathematical consequences of properties of the constituents.
- So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.
- The motion of the mixture is governed by the same equations as for a single body.

#### 3.1 Kinematics

Usually, the kinematics of the fluid phase is described by an Eulerian approach, whereas for the kinematics of the solid structure reference is made a Lagrangian frame of reference. In order to overcome shortcoming deriving from this mixed description, recently Coussy (1989, 1995), Bourgeois and Dormieux (1996) and Wilmanski (1996) have suggested to introduce a unified Lagrangian description, by assuming the soil skeleton as a material reference volume.

For each phase we define the motion function

$$\mathbf{x} = \chi_\alpha(\mathbf{X}_\alpha, t), \quad (3.1)$$

and a corresponding velocity and acceleration field

$$\dot{\mathbf{x}}_\alpha = \frac{\partial}{\partial t} \chi_\alpha(\mathbf{X}_\alpha, t), \quad (3.2)$$

$$\ddot{\mathbf{x}}_\alpha = \frac{\partial^2}{\partial t^2} \chi_\alpha(\mathbf{X}_\alpha, t). \quad (3.3)$$

Referring to the solid matrix and dropping the index  $\alpha = s$  for simplicity, the deformation gradient tensor, defined by

$$\mathbb{F}_s =: \frac{\partial \chi_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{e}_j, \quad (3.4)$$

maps the material vector  $d\mathbf{X}$  onto its current configuration  $d\mathbf{x}$

$$d\mathbf{x} = \mathbb{F}_s d\mathbf{X}. \quad (3.5)$$

Similarly, the infinitesimal initial volume  $dV_0$  is transformed into the corresponding area in the current configuration thanks to the relation

$$dV = J dV_0, \quad (3.6)$$

where

$$J =: \det \mathbb{F}_s > 0. \quad (3.7)$$

The determinant of  $\mathbb{F}$  is restricted to take positive values, so that the motion function of each phase is invertible. A material area  $d\mathbf{A}$ , oriented by its normal  $\mathbf{N}$  (i.e.  $d\mathbf{A} = \mathbf{N} dA$ ), is transformed into the current configuration according to the rule

$$\mathbf{n} da = J \mathbb{F}_s^{-T} \cdot \mathbf{N} dA \quad (3.8)$$

where  $\mathbb{F}_s^{-T}$  indicates the transpose of the inverse of  $\mathbb{F}_s$ .

### 3.2 Mass balance

In the Lagrangian description, the conservation of mass of the solid component is identically satisfied,

$$\frac{\partial}{\partial t} [J \rho_s (1 - n_f)] = 0, \quad (3.9)$$

where  $\rho_s =: \rho_s^R$ . The balance of mass for the fluid phase over a material control volume  $V$  fixed on the solid matrix and bounded by the surface  $S$  writes, in integral form,

$$\frac{\partial}{\partial t} \int_V \rho_f \frac{e}{1+e} dV + \int_S \rho_f \frac{e}{1+e} (\mathbf{v}^w - \mathbf{v}^s) \cdot \mathbf{n} da = 0, \quad (3.10)$$

where  $\rho_f =: \rho_f^R$ ,  $\mathbf{v}^w$  and  $\mathbf{v}^s$  are the velocity of the liquid and solid constituent, respectively, and  $e$  is the void ratio defined as  $e =: n_f / (1 - n_f)$ .

By recalling (3.8), transforming the volume integral at the l.h.s. of (3.10) into an integral over the reference volume and by using the Gauss theorem in order to transform the surface integral into an integral over the reference volume, one gets (if the void ratio in the reference configuration is constant)

$$\frac{\partial e}{\partial t} + \operatorname{div} [e \mathbb{F}^{-1} (\mathbf{v}^w - \mathbf{v}^s)] = 0, \quad (3.11)$$

where the symbol "div" indicates the divergence operator in Lagrangian coordinates.

According to Biot (1977) and Coussy (1995), it is sometimes convenient to introduce the change of fluid mass referred to the initial volume  $m$ . Accordingly, the quantity  $m dV_0$  indicates the difference of the fluid mass passing from the initial to the current configuration,

$$m =: J \rho_f n_f - \rho_{w0} n_f^0. \quad (3.12)$$

By defining the Eulerian vector

$$\mathbf{w} =: \rho_f n_f (\mathbf{v}^w - \mathbf{v}^s), \quad (3.13)$$

representing the mass flux of water relative to the solid skeleton, the mass balance can also be expressed as

$$\frac{\partial m}{\partial t} + \operatorname{div} (J \mathbb{F}^{-1} \mathbf{w}) = 0. \quad (3.14)$$

### 3.3 Momentum balance equation

When denoting by  $\rho$ ,  $\mathbf{a}$ ,  $\mathbf{a}^w$  the density of the porous medium, the acceleration of the soil particles and the acceleration of the fluid, respectively. The Eulerian form of first Cauchy's law of motion for the porous medium as a whole writes

$$\int_V [\rho \mathbf{a} + n \rho_f (\mathbf{a}^w - \mathbf{a})] dV = \int_V \rho \mathbf{b} dV + \int_S \mathbb{T} dS, \quad (3.15)$$

where  $V$  is a control volume fixed on the solid skeleton and  $S$  its surface. As the relationship (3.15) holds for any control volume, for regular enough fields the local form of the momentum equation is obtained thanks to the Gauss theorem:

$$\nabla \cdot \mathbb{T} + \rho (\mathbf{b} - \mathbf{a}) - n \rho_f (\mathbf{a}^w - \mathbf{a}) = 0. \quad (3.16)$$

In order to obtain the corresponding Lagrangian formulation, we introduce the second Piola-Kirchhoff stress tensor

$$\tilde{\mathbb{T}} =: J \mathbb{F}^{-1} \mathbb{T} \mathbb{F}^{-T}. \quad (3.17)$$

The relationship (3.8) enables to write the transport formula

$$\mathbb{F} \tilde{\mathbb{T}} \cdot \mathbf{N} dA = \mathbb{T} \cdot \mathbf{n} da, \quad (3.18)$$

so that the (3.15) reads

$$\int_{A_0} \mathbb{F} \tilde{\mathbb{T}} \cdot \mathbf{N} dA + \int_{V_0} [(\rho_0 + m) (\mathbf{b} - \mathbf{a}) - (\rho_{w0} n_0 + m) (\mathbf{a}^w - \mathbf{a})] dV = 0, \quad (3.19)$$

where  $\rho_0$ ,  $A_0$ ,  $V_0$  are the density, surface and volume in the reference configuration. The local Lagrangian form is

$$\operatorname{div} (\mathbb{F} \tilde{\mathbb{T}}) + (\rho_0 + m) (\mathbf{b} - \mathbf{a}) - (n_0 \rho_{w0} + m) (\mathbf{a}^w - \mathbf{a}) = 0. \quad (3.20)$$

In the Lagrangian framework, the Darcy's law (2.14) takes the generalized expression

$$\rho_f n_f (\mathbf{v}^w - \mathbf{v}^s) = -\frac{\mathbb{K}}{\mu} \left( \mathbb{F}^{-T} \operatorname{grad} p \right) + \rho_f \mathbf{g}, \quad (3.21)$$

where  $\mathbb{K}$  is the permeability tensor,  $g$  is the gravity acceleration,  $\mu$  is the dynamic viscosity of the fluid and "grad" represents the gradient operator in Lagrangian coordinates.

## 4 Consolidation theories

The first attempt to describe the consolidation of a deformable porous medium with pores completely filled by water is due to Terzaghi (1923, 1925). He introduced the concept of effective stress in a strictly one-dimensional framework. Later Biot (1941) generalized the theory to the three-dimensional case in a framework consistent with the basic principles of continuum mechanics. His work deals with small strains and elastic behavior of the soil skeleton, and it is briefly recalled in this section as a basis to outline fields of further developments.

## 4.1 Biot's theory

Within the general framework illustrated in the previous sections, the Biot's formulation reads as a macroscopic one. In particular, the behavior of the soil skeleton is described by global deformation characteristics, which include all local deformations (i.e., rolling and sliding of particles).

The momentum balance equation of the fluid phase is expressed by the Darcy's law (2.14). By combining it with the saturation condition (2.2) one gets

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) = \frac{K}{\rho_f} \nabla p, \quad (4.1)$$

where

- $\mathbf{u}$  is the displacement vector of the soil skeleton;
- $p$  is the pore pressure in excess to the initial equilibrium value;
- $K$  is the (constant) hydraulic conductivity of the porous medium.

The total stress tensor  $\mathbb{T}$  can be decomposed into the effective stress tensor and the pore pressure according to Terzaghi's equation

$$\mathbb{T} = \mathbb{T}' - p\mathbb{I}. \quad (4.2)$$

Assuming an elastic behavior for the soil skeleton for small deformations, the equilibrium equations can be expressed in terms of displacement components (Navier equations). Introducing the tensor of the linear elasticity

$$\epsilon =: \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda \nabla \mathbf{u}, \quad (4.3)$$

for the effective stress, one obtains

$$(\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} - \nabla p = \rho \mathbf{b}. \quad (4.4)$$

Equations (4.1) and (4.4) provide a system of four differential equations for the unknowns  $(\mathbf{u}, p)$ . Without loss of generality,  $\mathbf{u}$  and  $p$  can be taken to be null at  $t = 0$ . The boundary conditions can be specified as

- prescribed pore pressure:  $p = h$
- prescribed flux:  $\frac{K}{\rho_f g} \nabla p \cdot \mathbf{n} = f$ ;
- prescribed traction:  $\mathbb{T} \cdot \mathbf{n} = \mathbf{t}$ ;
- prescribed displacement:  $\mathbf{u} = \mathbf{a}$ .

**Remark** Exact solutions of the consolidation problem are limited to simple boundary conditions under the assumption of elastic behavior of the soil skeleton. A general treatment of the problem, involving a more realistic soil behavior, coupled with the diffusion process of the fluid phase calls for numerical simulation.

## 4.2 One-dimensional finite deformation theory

There are two major areas of developments in which departures from Terzaghi and Biot's theories occur: finite deformations and non linear soil response. In this paragraph we deal with the former aspect, with special reference to the one-dimensional case.

Consider a soil stratum of infinite extent in the  $OXY$  plane and transversely isotropic about the  $OZ$  axis. The upper surface is supposed to be a free draining boundary, so that

$$p(t, z = h) = 0. \quad (4.5)$$

The lower boundary is supposed to be impermeable and the corresponding condition is

$$\frac{\partial p}{\partial z}(t, z = 0) = 0. \quad (4.6)$$

At time  $t = 0$  the upper boundary is uniformly loaded:

$$\mathbf{t}(t, z = h) = -\Delta q \mathbf{e}_z. \quad (4.7)$$

Because of the one-dimensional nature of the problem, the deformation gradient reduces to

$$\mathbb{F} = \mathbb{I} + \frac{\partial u_z}{\partial Z} \mathbf{e}_z \otimes \mathbf{e}_z, \quad (4.8)$$

where  $u_z$  is the displacement in the  $z = Z$  direction, so that

$$\det \mathbb{F} = 1 + \frac{\partial u_z}{\partial Z}. \quad (4.9)$$

The unique non-trivial component of the second Piola-Kirchhoff tensor is related to the Cauchy tensor according to Eq.(3.17)

$$\tilde{T}_{ZZ} = T_{ZZ} \left( 1 + \frac{\partial u_z}{\partial Z} \right)^{-1}. \quad (4.10)$$

After integration in time, the Lagrangian mass balance equation for the solid phase (3.9) reduces to

$$\det \mathbb{F} = \left( 1 + \frac{\partial u_z}{\partial Z} \right) = \frac{1 + e}{1 + e_0}, \quad (4.11)$$

where  $e_0$  is the void ratio in the reference configuration. The mass balance equation for the fluid phase in Lagrangian coordinates (3.11) reads

$$\frac{\partial e}{\partial t} + (1 + e_0) \frac{\partial}{\partial Z} \left[ \frac{e}{1 + e} (v_z^w - v_z^s) \right] = 0. \quad (4.12)$$

By recalling the definition of  $m$  (3.12) and neglecting the inertial terms, the balance of linear momentum of the porous medium (3.20) reads

$$\frac{\partial \tilde{T}_{ZZ}}{\partial Z} + g \frac{e \rho_f + \rho_s}{1 + e_0} = 0. \quad (4.13)$$

The Lagrangian formulation of the Darcy's law is given by

$$\rho_f \frac{e}{1 + e} (v_z^w - v_z^s) = -\frac{K}{\mu} \left[ \frac{1 + e_0}{1 + e} \frac{\partial p}{\partial Z} \right] - \rho_f g. \quad (4.14)$$

Plugging Equation (4.14) into (4.12) and by using the definition of effective stress into (4.13), one obtains the Lagrangian equation of one-dimensional consolidation:

$$\frac{\partial e}{\partial t} + (1 + e_0) \frac{\partial}{\partial Z} \left[ \frac{K}{\mu} \left( \frac{\rho_s - \rho_f}{1 + e} g + \frac{\partial T'_{ZZ}}{\partial Z} \frac{1 + e_0}{1 + e} \right) \right] = 0. \quad (4.15)$$

????????? CONTROLLARE QUESTO CONTO ????????? The equation above represents the most general formulation of the problem (see Gibson et al.(1967), Lancellotta and Preziosi (1997) and Pane (1985)). The one-dimensional models currently used in the literature can be interpreted as a special case of Eq. (4.15).

As shown by Carter et al. (1979), the solution of Eq. (4.15) under the assumption of an elastic soil behavior characterized by the Young modulus  $E'$  and Poisson's ratio  $\nu'$  depends on the following dimensionless parameters only:

$$\frac{\Delta q}{E'}, \quad \frac{\rho_f g H}{E'}, \quad \nu', \quad e_0, \quad \frac{\rho_s}{\rho_f}. \quad (4.16)$$

The Terzaghi solution is obtained when both the parameters  $\delta q/E'$  and  $\rho_f g H/E'$  approach zero, thus indicating that the behavior of deep soft layers can be significantly different from the one predicted by Terzaghi's theory.

## 5 Constitutive equations

According to Truesdell and Noll (1965) three fundamental postulates are assumed to be valid for any constitutive theory:

- Principle of determinism for stress: the stress in a body is determined by its motion history.
- Principle of local action: in determining the stress at a given particle, the motion outside an arbitrary neighborhood of the particle can be disregarded.
- Principle of material frame-indifference or principle of material objectivity: constitutive equations must be invariant under changes of frame of reference.

In the following we concentrate on the latter principle, because, despite its effectiveness in limiting the generality of the constitutive equations of elasticity as well as the functionals appearing in the constitutive equations of simple materials with memory, its use is not yet common in the engineering literature.

A mathematical statement of this principle is the following: if a constitutive equation is satisfied for a process with a motion and a symmetric stress tensor given by

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbb{T} = \mathbb{T}(\mathbf{X}, t), \quad (5.1)$$

it must also be satisfied for any equivalent process, given by the motion and the stress tensor

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbb{Q}(t)\mathbf{x} \quad (5.2)$$

$$\mathbb{T}^* = \mathbb{Q}(t)\mathbb{T}\mathbb{Q}^T(t), \quad (5.3)$$

$$t^* = t - a, \quad (5.4)$$

for any arbitrary position vector  $\mathbf{c}(t)$ , any arbitrary real number  $a$  and any arbitrary orthogonal tensor  $\mathbb{Q}(t)$ . In this respect, functions and fields of scalar, vector and tensor are objective if both the dependent and the independent vectors and tensors transform according to Eq. (5.3), while the scalar variables are unchanged. Note that the coordinate transformation equations between various coordinate systems, in the same frame of reference are time-independent, so that the invariance under this kind of coordinate change is ensured by the requirement that the constitutive equations have to be tensor equations. But this does not ensure invariance under the time dependent change of reference frame, unless the relationships above are satisfied.

A constitutive equation allows to obtain the stress field from the knowledge of the history of the deformation of a body. In elasticity the stress is a function of deformation, so that the deformation history is immaterial and the stress depends only on the current value of deformation. Soil behavior is not elastic and, in order to account for its mechanical history, a general formulation should provide the stress increment in terms of some non-integrable function of the deformation increment

$$d\mathbb{T} = f(d\epsilon). \quad (5.5)$$

If we consider a material without internal time scale (i.e. a rate independent behavior), Eq.(5.5) can also be represented as a rate equation

$$\dot{\mathbb{T}} = f(\dot{\epsilon}). \quad (5.6)$$

This is the usual approach in plasticity and hypoplasticity. Trusdell (1965) introduced for the constitutive relationships the general form

$$\dot{\mathbb{T}} = f(\mathbb{T}, \mathbb{D}). \quad (5.7)$$

If the function  $f$  is linear in  $\mathbb{T}$  and  $\mathbb{D}$ , the behaviour is called hypoelastic: Eq. (5.7) can produce a nonlinear response, but it is inappropriate to describe plastic behavior. In order to overcome these limits (see Kolymbas, 1991), the function  $f$  is requested to be non-linear in  $\mathbb{D}$ . In particular, when  $f$  is first order homogeneous in  $\mathbb{D}$  it describes a rate independent behavior. Homogeneity in  $\mathbb{T}$  is also a prerequisite in order to describe proportional stress paths in case of proportional strain path. The above requirements constitute rules for the mathematical function (5.7).

## 5.1 Finite elasticity

When we apply the principles of determinism and local action in finite elasticity, the most general constitutive equation is of the form

$$\mathbb{T} = f(\mathbb{F}), \quad (5.8)$$

where the response function  $f$  is a non-linear tensor-valued function of the single tensor  $\mathbb{F}$ . By noting that in a change of frame the deformation gradient  $\mathbb{F}$  transforms like a vector, the response function must satisfy the following identity, required by the objectivity principle

$$f(\mathbb{Q}\mathbb{F}) = \mathbb{Q}f(\mathbb{F})\mathbb{Q}^T. \quad (5.9)$$

Therefore, the frame-indifference principle requires that the dependence on  $\mathbb{F}$  have the form of an arbitrary function with additional dependence on  $\mathbb{Q}$  as shown by Eq. (5.9). It can in addition be proved that in terms of the second Piola-Kirchhoff stress tensor, the response function depends only on one of the strain tensors (i.e.,  $\mathbb{U}$  or  $\mathbb{C}$  or  $\mathbb{E}$ ) and not on rotations. The requirements of frame indifference is relaxed if it is introduced the assumption that the motion involves a small contribution of rotation in a polar decomposition of the deformation tensor, as it is commonly done in the classical small displacement elasticity.

## 5.2 Rate equations

When the constitutive equation includes the material time derivative of the stress tensor  $\dot{\mathbb{T}}$ , the principle of objectivity is not straightforwardly satisfied for any arbitrary motion. The difficulty is that the material time derivative  $\dot{\mathbb{T}}$  is not frame-indifferent, even though  $\mathbb{T}$  is frame invariant. A possible remedy is to use the co-rotational stress rate tensor

$$\hat{\mathbb{T}} =: \dot{\mathbb{T}} - \mathbb{W}\mathbb{T} + \mathbb{T}\mathbb{W}, \quad (5.10)$$

where  $\mathbb{W}$  is the spin tensor, provided that all the other variables are frame-indifferent (as it is, for example, the rate of deformation tensor  $\mathbb{D}$ ).

## 5.3 Finite non-linear consolidation

A theoretical formulation and a numerical solution method for the consolidation of an elasto-plastic soil with finite deformation has been given by Carter et al. (1979), with constitutive rate equations satisfying the objectivity principle. Here again the general conclusion is resumed that the need to account for large deformation as well as non linear soil behavior arises for soft materials and when the imposed load is large if compared with the soil stiffness. Both these cases are of interest in Soil Mechanics, when dealing with soft clay strata.

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