# A Galerkin method for Uncertain Hyperbolic Systems: Roe solver and Entropy corrector.

**Julie Tryoen,**[1,2]

Alexandre Ern[1], Olivier Le Maître[2], Michael Ndjinga[3]

[1]Université Paris Est, CERMICS, Ecole des Ponts, France
[2]LIMSI-CNRS, Orsay, France
[3]CEA-Saclay, Gif-sur-Yvette, France

Workshop Torino
May 10-13, 2010

# Introduction

## Parametric Uncertainty Quantification

- uncertainties in input quantities (model parameters, initial and boundary conditions)
- uncertain quantities parametrized by random variables with known distribution functions

## Stochastic spectral methods

- decompose random quantities on suitable approximation bases (Ghanem and Spanos 91)
- Stochastic expansion of the solution of a model :

$$U(x,t,\xi) \approx U^{\mathrm{P}}(x,t,\xi) = \sum_{\alpha=1}^{\mathrm{P}} u_\alpha(x,t)\Psi_\alpha(\xi).$$

$u_\alpha(x,t)$ stochastic modes of the solution.

## Different computational strategies

- probabilistic collocation : stochastic modes evaluated by polynomial interpolation
- non-intrusive projection : stochastic modes evaluated by numerical integration
- stochastic Galerkin : reformulated deterministic problem for the stochastic modes

Stochastic Galerkin methods :

- rely on a weak form of the problem
- well suited for mathematical analysis
- design of adaptive methods

# State of the art

Stochastic spectral methods applied to a large variety of engineering problems (elasticity, thermal science, fluid flows, chemical/biological systems,...) governed by elliptic, parabolic ,ODE or incompressible NS models.

## Hyperbolic models

Two main difficulties :

- solutions discontinuous in the spatial as well as in the stochastic domains
- nonlinearities in the flux functions

# State of the art

## Hyperbolic models

- non-intrusive approaches : multi-element probabilistic collocation methods (Lin et al. 08)
- pseudo-intrusive methods : stochastic modes of flux computed by quadrature methods (Ge et al. 08, Poette et al. 09)
- intrusive methods : scalar linear wave equation (Gottlieb and Xiu 08), mean flux upwinding (Lin et al. 06)

# Objectives

- Develop an intrusive stochastic Galerkin method
- Investigate hyperbolicity of the Galerkin system
- Design a Roe-type solver with entropy corrector

# Outline

# Outline

## Stochastic parametrization

$\xi = (\xi_1, \ldots, \xi_{\mathrm{N}}) \sim \mathcal{U}(\Xi = [0,1]^{\mathrm{N}}) \rightarrow L^2(\Xi, p_\xi = 1)$
corresponding space of the second-order random variables with the expectation operator $\langle H \rangle = \int_\Xi H(y) p_\xi(y) dy$.

## Stochastic hyperbolic systems

We seek for $U(x,t,\xi) \in \mathbb{R}^m \otimes L^2(\Xi, p_\xi)$ solving a.s.

$$\begin{cases} \dfrac{\partial}{\partial t} U(x,t,\xi) + \dfrac{\partial}{\partial x} F(U(x,t,\xi);\xi) = 0, \\ U(t=0,x,\xi) = U_0(x,\xi). \end{cases}$$

$(x,t,\xi) \in \Omega \times [0,T] \times \Xi, \qquad \Omega = [0,1],$

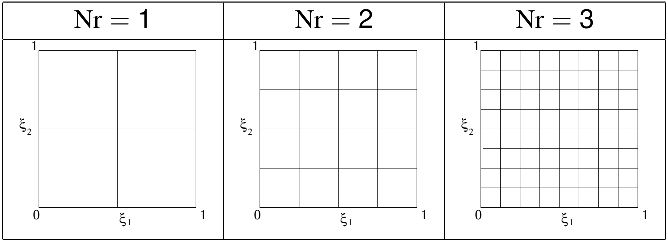
$\nabla_U F \in \mathbb{R}^{m,m} \otimes L^2(\Xi, p_\xi)$ stochastic Jacobian matrix $\mathbb{R}$-diagonalizable almost surely.

# Stochastic discretization

We approximate $U(x,t,\xi)$ in the stochastic space of fully tensorized piecewise polynomial functions $\mathcal{S}^{\mathrm{No},\mathrm{Nr}}$ :

- $\mathrm{Nr}$ : resolution level $\rightarrow$ $2^{\mathrm{N}\mathrm{Nr}}$ SE (stochastic elements),
- $\mathrm{No}$ : expansion order $\rightarrow$ $U(\xi)_{|SE} \approx U^{\mathrm{P}}(\xi)_{|SE} \in \mathbb{Q}_{\mathrm{No}}^{\mathrm{N}}[\xi]$.

<u>Case $\mathrm{N} = 2$.</u>




$$\dim \mathcal{S}^{\mathrm{No},\mathrm{Nr}} = (\mathrm{No}+1)^{\mathrm{N}} 2^{\mathrm{N}\mathrm{Nr}} := \mathrm{P}.$$

# Stochastic discretization

We approximate $U(x,t,\xi)$ in the stochastic space of fully tensorized piecewise polynomial functions $\mathcal{S}^{\mathrm{No},\mathrm{Nr}}$ :

- $\mathrm{Nr}$ : resolution level $\rightarrow 2^{\mathrm{NNr}}$ SE (stochastic elements),
- $\mathrm{No}$ : expansion order $\rightarrow U(\xi)_{|SE} \approx U^{\mathrm{P}}(\xi)_{|SE} \in \mathbb{Q}^{\mathrm{N}}_{\mathrm{No}}[\xi]$.

$$\boxed{\dim \mathcal{S}^{\mathrm{No},\mathrm{Nr}} = (\mathrm{No}+1)^{\mathrm{N}} 2^{\mathrm{NNr}} := \mathrm{P}.}$$

Select the Stochastic Element (SE) orthonormal basis $\{\Psi_\alpha\}_{\alpha=1,\dots,\mathrm{P}}$, which corresponds to local fully tensorized (rescaled) Legendre polynomial bases, s.t.

$$\boxed{span(\Psi_1,\dots,\Psi_{\mathrm{P}}) = \mathcal{S}^{\mathrm{No},\mathrm{Nr}}.}$$

# The Galerkin system

## Stochastic expansion of the solution :

$$U(x,t,\xi) \approx U^{\mathrm{P}}(x,t,\xi) = \sum_{\alpha=1}^{\mathrm{P}} u_\alpha(x,t)\Psi_\alpha(\xi).$$

$u_\alpha(x,t) \in \mathbb{R}^m$ stochastic modes of the solution.

Galerkin projection of the original stochastic problem :

$$\begin{cases} \left\langle \Psi_\alpha \dfrac{\partial U^{\mathrm{P}}}{\partial t} \right\rangle + \left\langle \Psi_\alpha \dfrac{\partial F(U^{\mathrm{P}};\cdot)}{\partial x} \right\rangle = 0, & \forall \alpha = 1,\dots,\mathrm{P}, \\ \left\langle \Psi_\alpha U^{\mathrm{P}} \right\rangle (t=0) = \left\langle \Psi_\alpha U_0(x,\cdot) \right\rangle, & \forall \alpha = 1,\dots,\mathrm{P}. \end{cases}$$

# The Galerkin system

We seek for $u(x,t) \in \mathbb{R}^{m\mathrm{P}}$ solving

$$\boxed{\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0, \quad u(x,t=0) = u^0(x),}$$

$$u(x,t) = (u_1(x,t),\dots,u_{\mathrm{P}}(x,t))^T \in \mathbb{R}^{m\mathrm{P}},$$
$$f(u(x,t)) = (f_1(u),\dots,f_{\mathrm{P}}(u))^T \in \mathbb{R}^{m\mathrm{P}},$$
$$f_\alpha(u) := \left\langle \Psi_\alpha F\left(U^{\mathrm{P}};\cdot\right)\right\rangle, \quad \alpha = 1,\dots,\mathrm{P}.$$

Is the Galerkin system hyperbolic ?

# Hyperbolicity of the Galerkin system

$$\boxed{(\nabla_u f(u))_{\alpha,\beta=1,\dots,\mathrm{P}} = \left\langle \nabla_U F(U^{\mathrm{P}};\cdot)\Psi_\alpha\Psi_\beta \right\rangle_{\alpha,\beta=1,\dots,\mathrm{P}}.}$$

Is $\nabla_u f(u) \in \mathbb{R}^{m\mathrm{P},m\mathrm{P}}$ $\mathbb{R}$-diagonalizable ?

Diagonal block structure owing to the decoupling of the problem over different stochastic elements :

$$\nabla_u f(u) = \begin{pmatrix} [\nabla_u f]^{SE=1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & [\nabla_u f]^{SE=i} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & [\nabla_u f]^{SE=2^{\mathrm{NNr}}} \end{pmatrix}.$$

$\rightarrow$ It suffices to analyze one single block $[\nabla_u f]^{SE}$ of size $m(\mathrm{No}+1)^{\mathrm{N}}$.

# Hyperbolicity of the Galerkin system

HYPERBOLICITY proven in two specific cases :

**Theorem**
*HYPERBOLICITY, if the original stochastic system is a scalar conservation law.*

**Theorem**
*HYPERBOLICITY, if the stochastic Jacobian matrix* $\nabla_U F(\cdot;\xi)$

- *is either symmetric (almost surely)*
- *or its eigenvectors are deterministic (independent of the uncertainty).*

Applications : Scalar wave equation with uncertain sound velocity. / Linear hyperbolic systems with uncertainty only on initial or boundary counditions.

# Hyperbolicity of the Galerkin system

In the general case, we consider the approximate Galerkin Jacobian matrix $\left[\overline{\nabla_u f}\right]^{SE}$ whose coefficients are obtained by approaching the coefficients of $[\nabla_u f]^{SE}$ by a Gauss quadrature

$$\left[\left(\overline{\nabla_u f}(u)\right)_{\alpha,\beta=1,\dots,(\mathrm{No}+1)^{\mathrm{N}}}\right]^{SE} = \left(\sum_{\gamma=1}^{(\mathrm{No}+1)^{\mathrm{N}}} \varpi_\gamma \nabla_U F(U^{\mathrm{P}}(\xi_\gamma);\xi_\gamma)\Psi_\alpha(\xi_\gamma)\Psi_\beta(\xi_\gamma)\right)_{\alpha,\beta=1,\dots,(\mathrm{No}+1)^{\mathrm{N}}},$$

$\{\xi_\gamma\}_{\gamma=1,\dots,(\mathrm{No}+1)^{\mathrm{N}}}$ set of the Gauss points in the SE, with associated weights $\{\varpi_\gamma\}_{\gamma=1,\dots,(\mathrm{No}+1)^{\mathrm{N}}}$.

# Hyperbolicity of the Galerkin system

Assume $\nabla_U F(U^{\mathrm{P}}(\xi);\xi) = L(\xi)\Lambda(\xi)R(\xi)$,

**Theorem**

$\left[\overline{\nabla_u f}\right]^{SE}$ *is* $\mathbb{R}$*-diagonalizable.*

*Explicit expressions of the eigenvalues*

$$\{\lambda'_\gamma\} = \{\Lambda^k(\xi_\eta)\},$$

*and of the left and right eigenvectors*

$$\{r'_\gamma\} = \{(\varpi_\eta R^k(\xi_\eta)\Psi_\beta(\xi_\eta))\},$$
$$\{l'_\gamma\} = \{(\varpi_\eta L^k(\xi_\eta)\Psi_\beta(\xi_\eta))\}.$$

$\rightarrow$ $\{\lambda'_\gamma\}$, $\{r'_\gamma\}$, and $\{l'_\gamma\}$, $\gamma = 1,\ldots,m(\mathrm{No}+1)^{\mathrm{N}}$, approximations of the eigenvalues and eigenvectors of $[\nabla_u f]^{SE}$.

# Outline

## Numerical scheme

Discretization of the Galerkin system using a FV method :

$$u_i^{n+1} = u_i^n - \frac{\Delta^n t}{\Delta x}\left(\varphi(u_i^n, u_{i+1}^n) - \varphi(u_{i-1}^n, u_i^n)\right),$$

- $\Delta x$ (uniform) spatial step,
- $\Delta^n t$ time step,
- $u_i^n \approx \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u(x, t^n) dx$,
- $\varphi(\cdot,\cdot)$ $1^{st}$ order numerical flux function :

$$\varphi(u_i^n, u_{i+1}^n) = \underbrace{\frac{f(u_i^n) + f(u_{i+1}^n)}{2}}_{\substack{\uparrow \\ \text{centered part of the flux}}} - \underbrace{|a(u_i^n, u_{i+1}^n)|}_{\substack{\uparrow \\ \text{upwind matrix}}} \frac{u_{i+1}^n - u_i^n}{2}.$$

# Roe linearized matrix

- Assume that the original stochastic problem possesses a Roe linearized matrix and a Roe state a.s.,

$$(U_L, U_R) \to A^{\text{Roe}}(U_L, U_R) = \nabla_U F(U_{LR}^{\text{Roe}}; \cdot).$$

- Given two states $u_L$ and $u_R$ of the Galerkin system,

$$\left.\begin{array}{l} u_L \to U_L^{\text{P}} \\ u_R \to U_R^{\text{P}} \end{array}\right\rangle \to U_{LR}^{\text{Roe}},$$

$$\to a^{\text{Roe}}(u_L, u_R) = \langle \nabla_U F(U_{LR}^{\text{Roe}}; \cdot)\Psi_\alpha \Psi_\beta \rangle.$$

**Theorem**

*$a^{\text{Roe}}$ is a Roe linearized matrix for the Galerkin system.*

Choice of upwinding :

$$\varphi(u_i^n, u_{i+1}^n) = \frac{f(u_i^n) + f(u_{i+1}^n)}{2} - |a^{\text{Roe}}(u_i^n, u_{i+1}^n)| \frac{u_{i+1}^n - u_i^n}{2}.$$

$\to$ Consistency of the numerical scheme

$\to$ Conservativity through shocks

## Efficient approximation of $|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)|$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = \sum_{\alpha=1}^{mP} |\lambda_\alpha| l_\alpha \otimes r_\alpha .$$

$$a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n) = diag(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}).$$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = diag(|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}|).$$

$\Rightarrow$ Procedure on each SE.

## Efficient approximation of $|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)|$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = \sum_{\alpha=1}^{m\mathrm{P}} |\lambda_\alpha| l_\alpha \otimes r_\alpha.$$

$$a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n) = diag(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}).$$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = diag(|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}|).$$

$\Rightarrow$ Procedure on each SE.

**1)** Evaluate approximate eigenvalues of $\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}$

$$\boxed{\{\lambda'_\gamma\} = \{\Lambda^k(U_{i,i+1}^{\mathrm{Roe}}; \xi_\eta)\}.}$$

## Efficient approximation of $|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)|$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = \sum_{\alpha=1}^{mP} |\lambda_\alpha| l_\alpha \otimes r_\alpha.$$

$$a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n) = diag(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}).$$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = diag(|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}|).$$

$\Rightarrow$ Procedure on each SE.

**1)** Evaluate approximate eigenvalues of $\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}$

$$\boxed{\{\lambda'_\gamma\} = \{\Lambda^k(U_{i,i+1}^{\mathrm{Roe}}; \xi_\eta)\}.}$$

**2)** Determine a local polynomial $q_{d,\{\lambda'_\gamma\}}^{SE}$ of low degree d ($d < 10$ in practice) minimizing the least-square error

$$\sum_\gamma (|\lambda'_\gamma| - q_{d,\{\lambda'_\gamma\}}^{SE}(\lambda'_\gamma))^2.$$

## Efficient approximation of $|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)|$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = \sum_{\alpha=1}^{mP} |\lambda_\alpha| l_\alpha \otimes r_\alpha.$$

$$a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n) = diag(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}).$$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| = diag(|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}|).$$

$\Rightarrow$ Procedure on each SE.

**1)** Evaluate approximate eigenvalues of $\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}$

$$\{\lambda'_\gamma\} = \{\Lambda^k(U^{\mathrm{Roe}}_{i,i+1}; \xi_\eta)\}.$$

**2)** Determine a local polynomial $q^{SE}_{d,\{\lambda'_\gamma\}}$ of low degree d ($d < 10$ in practice) minimizing the least-square error

$$\sum_\gamma (|\lambda'_\gamma| - q^{SE}_{d,\{\lambda'_\gamma\}}(\lambda'_\gamma))^2.$$

**3)** Approximation of $|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}|$

$$|\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}| \approx q^{SE}_{d,\{\lambda'_\gamma\}}(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}).$$

# The upwind scheme

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta^n t}{\Delta x}\left(\varphi(u_i^n, u_{i+1}^n) - \varphi(u_{i-1}^n, u_i^n)\right),}$$

where the numerical flux $\varphi(u_i^n, u_{i+1}^n)$ is computed in this way

$$\varphi(u_i^n, u_{i+1}^n) = \frac{f(u_i^n) + f(u_{i+1}^n)}{2} - diag\left(q_{d,\{\lambda'_\gamma\}}^{SE}\left(\left[a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)\right]^{SE}\right)\right)\frac{u_{i+1}^n - u_i^n}{2}.$$

Procedure on each SE.

CFL condition :

$$\frac{\Delta^n t}{\Delta x} = \min_{1 \leq SE \leq 2^{\mathrm{NNr}}} \frac{CFL}{\max_{LR \in \mathcal{I}, \gamma} |\lambda'_\gamma|}.$$

# Outline

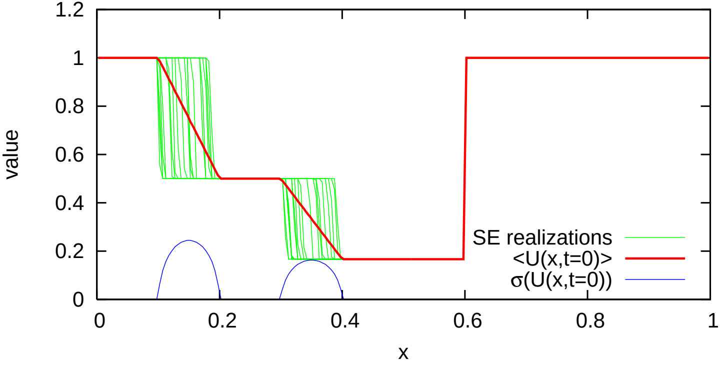
1. Galerkin projection
2. Numerical method
3. Results
   - Periodic Burgers equation
   - Euler equations
   - Entropy corrector

# Periodic Burgers equation

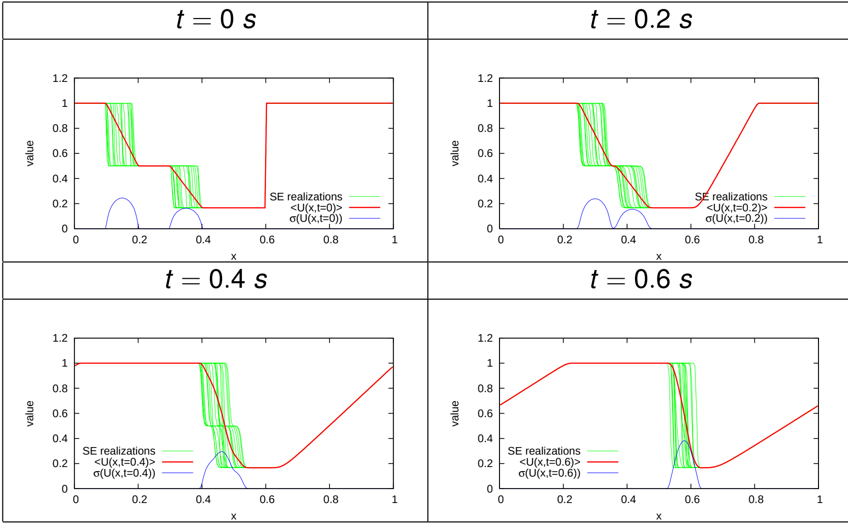
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},$$

initial random shock locations :

$$X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0,1].$$

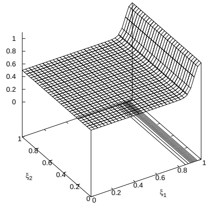
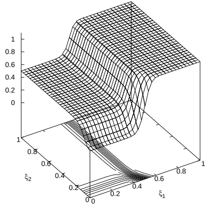
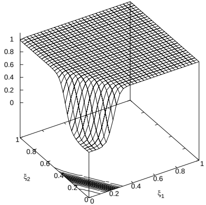


# Periodic Burgers equation




Computations with $\mathrm{No} = \mathrm{Nr} = 3$.

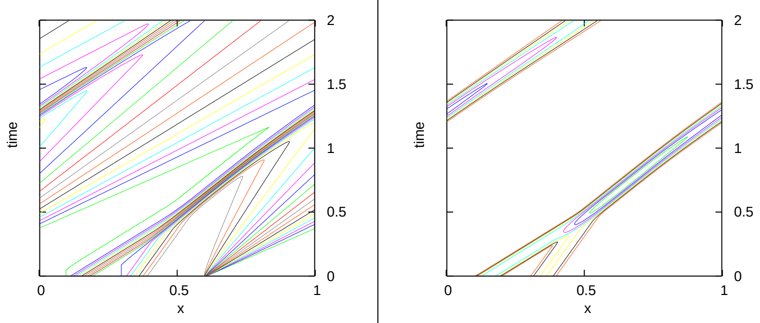
# Periodic Burgers equation

| $x = 0.35, t = 0.2\ s$ | $x = 0.45, t = 0.4\ s$ | $x = 0.55, t = 0.6\ s$ |
|---|---|---|
|  |  |  |

Stochastic solution $U(x, t, (\xi_1, \xi_2))$ as a function of $(\xi_1, \xi_2)$ for different spatial positions and times. $\mathrm{No} = \mathrm{Nr} = 3$.

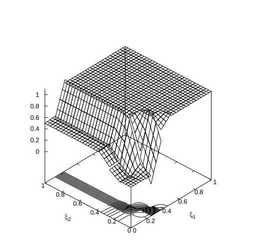
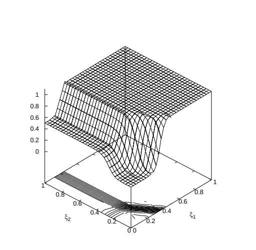
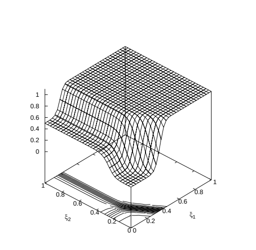
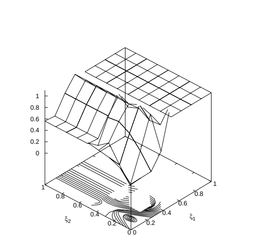
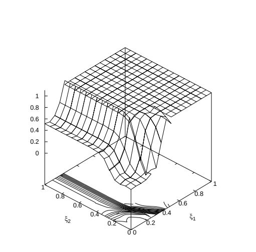
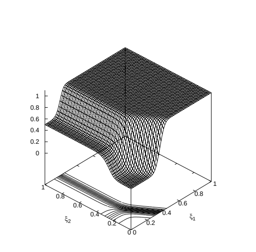
# Periodic Burgers equation

| Expectation | Standard Deviation |
|---|---|



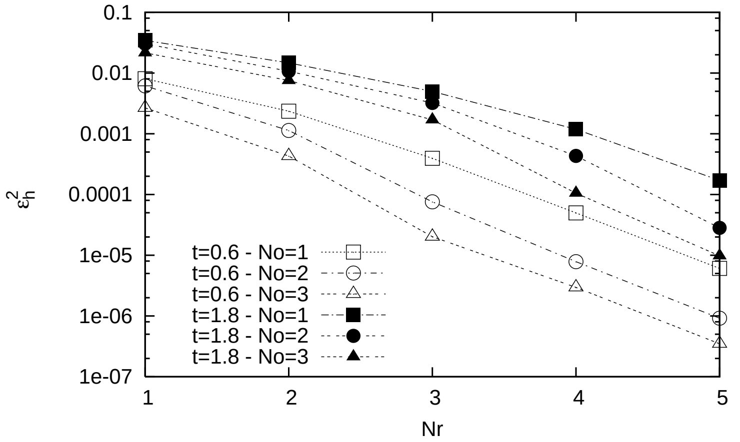

Space-time isolines of the expectation $\langle U(x,t,\cdot)\rangle$ and standard deviation $\sigma(U(x,t,\cdot))$ of the stochastic solution. $\mathrm{No} = \mathrm{Nr} = 3$.

# Periodic Burgers equation

| $\mathrm{Nr} = 3, \mathrm{No} = 1$ | $\mathrm{Nr} = 3, \mathrm{No} = 2$ | $\mathrm{Nr} = 3, \mathrm{No} = 3$ |
|---|---|---|
|  |  |  |
| $\mathrm{Nr} = 1, \mathrm{No} = 3$ | $\mathrm{Nr} = 2, \mathrm{No} = 3$ | $\mathrm{Nr} = 4, \mathrm{No} = 3$ |
|  |  |  |

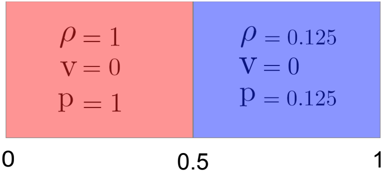
Stochastic solution of the Burgers equation as a function of $(\xi_1, \xi_2)$ at $x = 0.5$ and $t = 0.5$ for different $\mathrm{Nr}$ and $\mathrm{No}$.

# Burgers equation




$$\epsilon_h^2(t) := \frac{1}{M}\sum_{i=1}^{M}\int_\Omega \left(U_h^{\mathrm{No,Nr}}(x,t,\xi^{(i)}) - U_h^{MC}(x,t,\xi^{(i)})\right)^2.$$

# Euler equations (Sod Shock Tube)




$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,$$

$$U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E+p))^T,$$

$$v = \frac{q}{\rho}, \quad p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right).$$

$$\boxed{\gamma(\xi) = 1.4 + 0.2\,\xi, \quad \xi \sim U[0,1].}$$

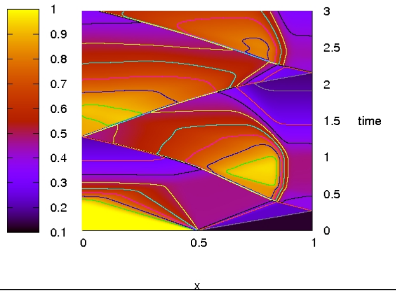
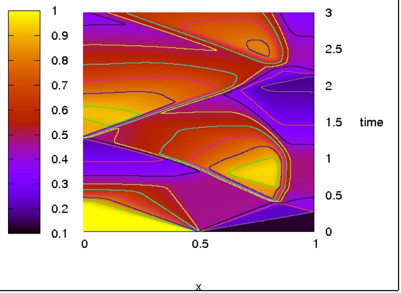
## Euler equations

Computation of the Galerkin flux and Jacobian matrix : using a pseudo-spectral approximation (Debusschere et al. 04)

- $a \times b \approx a * b = \sum_{\alpha=0}^{P} (a * b)_\alpha \Psi_\alpha$,
  $(a * b)_\alpha = \sum_{\beta,\delta=0}^{P} a_\beta b_\gamma \mathcal{M}_{\alpha\beta\gamma}, \quad \mathcal{M}_{\alpha\beta\gamma} = \langle \Psi_\alpha \Psi_\beta \Psi_\gamma \rangle$
- $1/a \approx a^{-*}$ obtained by solving $a * a^{-*} = 1$
- $\sqrt{a} \approx a^{*/2}$ obtained by solving $(a^{*/2}) * (a^{*/2}) = a$

$$\rightarrow p \approx p^* = (\gamma - 1) * (E - ((q * q) * \rho^{-*})/2)$$

# Euler equations

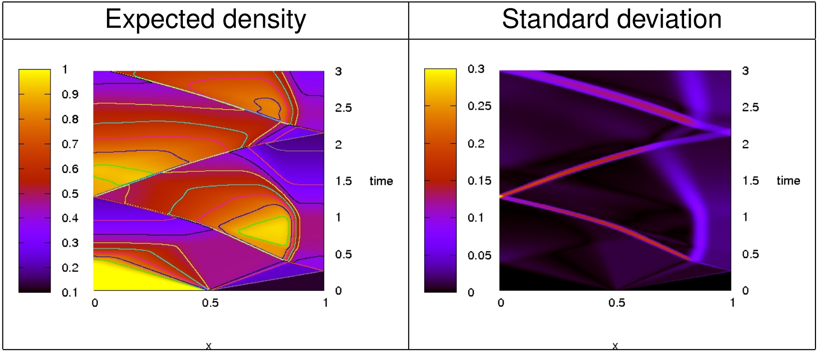
| Deterministic problem ($\gamma = 1.5$) | Expected density |
|---|---|
|  |  |





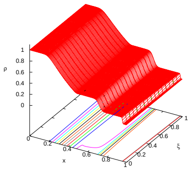
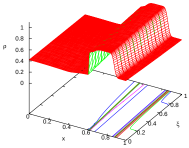
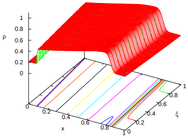
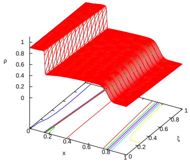
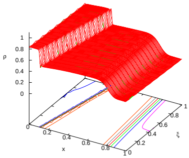
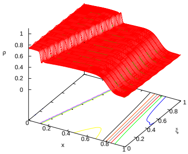
Computations with $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

# Euler equations



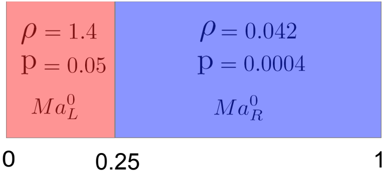

Computations with $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

# Euler equations

| $t = 0.25\ s$ | $t = 0.75\ s$ | $t = 1.25\ s$ |
|---|---|---|
|  |  |  |
| $t = 1.50\ s$ | $t = 3.25\ s$ | $t = 5\ s$ |
|  |  |  |

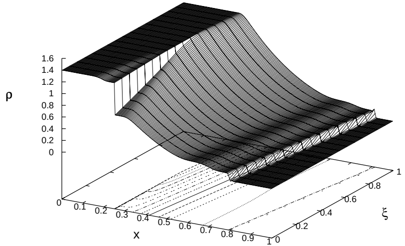
Stochastic density as a function of $(x, \xi)$. $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

# Entropy corrector
# Euler equations with sonic points




$$Ma^0(x,\xi) = \begin{cases} Ma_L^0(\xi), x \in [0, 1/4], \\ Ma_R^0(\xi), x \in ]1/4, 1], \end{cases} \qquad \xi \sim U[0,1].$$

s.t. $\exists$ sonic points for $\xi \in [0, 0.6]$.



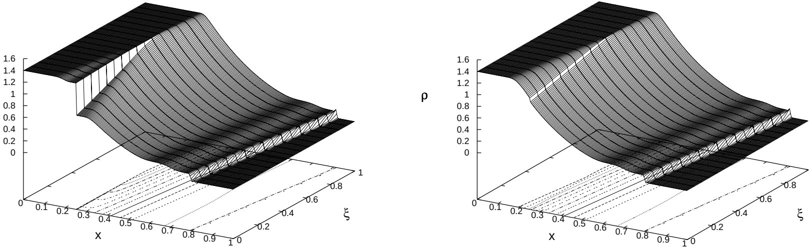
## Entropy corrector

Non-parametrized entropy corrector proposed by Dubois and Mehlmann (96) for Roe solver in the deterministic case

- Nonlinear modification of the numerical flux in the vicinity of sonic points
- Detection of sonic expansion waves based on reconstruction of intermediate states for each couple of left and right states and test on sign of eigenvalues of the Roe linearized matrix

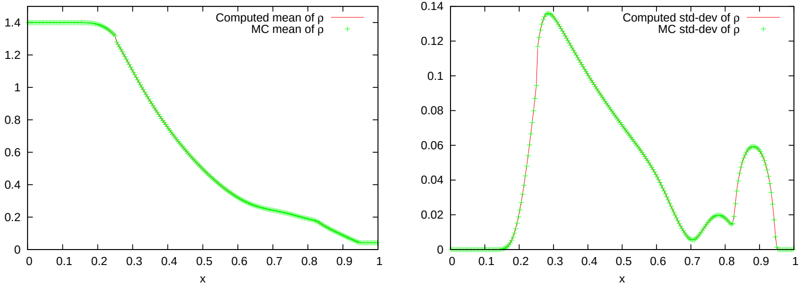
$\rightarrow$ Adaptation to the present context using the approximate eigenvalues and eigenvectors of $a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)$.

# Entropy corrector
# Euler equations with sonic points




Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without (left) and with (right) the entropy corrector using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

# Entropy corrector
# Euler equations with sonic points




Comparison of the mean and standard deviation of the numerical density at $t = 1$, computed with a Galerkin method (using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$) and a MC method.

# Entropy corrector
# Euler equations with sonic points

## CPU improvements

- Only the eigenvalue $v - c$ can change its sign.
- Mean value averaged criterium → for each interface $LR$, only test the entropy correction for the SE s.t.

$$E^{SE}[(v_L - c_L)] - ctol < 0 < E^{SE}[(v_R - c_R)] + ctol.$$

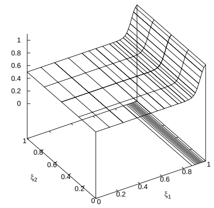
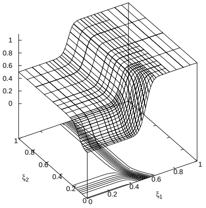
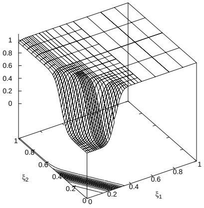
| | $\mathrm{No}=1, \mathrm{Nr}=3$ | | $\mathrm{No}=2, \mathrm{Nr}=3$ | | $\mathrm{No}=3, \mathrm{Nr}=3$ | |
|---|---|---|---|---|---|---|
| $\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$ | 16 | | 24 | | 32 | |
| | $T_{CPU}$ | $R_{cor}$ | $T_{CPU}$ | $R_{cor}$ | $T_{CPU}$ | $R_{cor}$ |
| $ctol = +\infty$ | 11.7 | 49% | 16.1 | 42% | 21.6 | 38% |
| $ctol = 1e^{-1}$ | 8.2 | 19% | 11.8 | 16% | 16.4 | 14% |
| $ctol = 1e^{-2}$ | 6.5 | 4% | 9.8 | 3% | 13.9 | 2% |
| $ctol = 1e^{-3}$ | 6.1 | <1% | 9.3 | <1% | 13.5 | <1% |
| $\epsilon_h$ | 1.32e-3 | | 7.17e-4 | | 2.88e-4 | |

# Conclusion

- intrusive stochastic Galerkin method
- Roe-type solver with upwind matrices efficiently computed by an original and fast method
- entropy correction in the presence of sonic points only requiring marginal costs
- accurate, stable and robust method
- computational costs scale as $dim\ S^{\mathrm{No},\mathrm{Nr}}$ (at least for moderate $\mathrm{No}$)

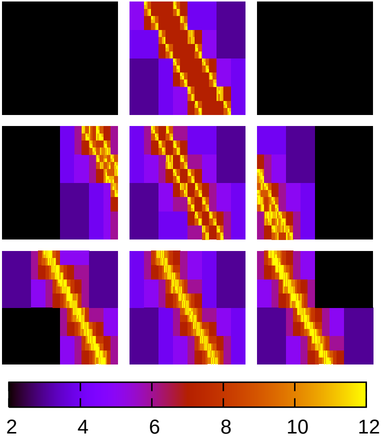
Under investigation :
savings in computational costs for problems with higher stochastic dimensions
$\rightarrow$ adaptive stochastic mesh refinement

# Adaptive stochastic mesh refinement
# Periodic Burgers equation

| $x = 0.35, t = 0.2\ s$ | $x = 0.45, t = 0.4\ s$ | $x = 0.55, t = 0.6\ s$ |
|---|---|---|
|  |  |  |

Stochastic solution $U(x, t, (\xi_1, \xi_2))$ as a function of $(\xi_1, \xi_2)$ for different spatial positions and times.

# Adaptive stochastic mesh refinement Burgers equation




Partition of the parameter space $\Xi$ at times 0.2, 0.4 and 1.5 (from top to bottom) and spatial locations $x = 0.48$, 0.5 and 0.52 (from left to right). The color scale gives the depth of the leafs.

## References :

Intrusive Projection Methods with Upwinding for Uncertain Nonlinear Hyperbolic Systems,
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern,
*J. Comput. Phys.*, Under Review

Roe solver with Entropy Corrector for Uncertain Hyperbolic Systems,
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern,
*J. Comput. Appl. Math.*, Under Review

# Euler equations

| | $\mathrm{Nr} = 2$ | | $\mathrm{Nr} = 3$ | | $\mathrm{Nr} = 4$ | |
|---|---|---|---|---|---|---|
| | $T_{CPU}$ | $\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$ | $T_{CPU}$ | $\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$ | $T_{CPU}$ | $\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$ |
| $\mathrm{No} = 0$ | 4.0 | (4) | 8.1 | (8) | 16.1 | (16) |
| $\mathrm{No} = 1$ | 6.9 | (8) | 13.9 | (16) | 27.8 | (32) |
| $\mathrm{No} = 2$ | 11.8 | (12) | 23.2 | (24) | 46.5 | (48) |
| $\mathrm{No} = 3$ | 17.1 | (16) | 34.1 | (32) | 68.1 | (64) |
| $\mathrm{No} = 4$ | 24.8 | (20) | 49.3 | (40) | 98.0 | (80) |

Normalized computational times $T_{CPU}$ for different stochastic discretization parameters $\mathrm{Nr}$ and $\mathrm{No}$. $\mathrm{Nc} = 250$.

$\rightarrow$ computational costs scale as *dim* $S^{\mathrm{No},\mathrm{Nr}}$ at least for moderate $\mathrm{No}$.

## Some details

- Parallelisation of the procedure on each stochastic element $\alpha_\sigma$, $1 \leq \alpha_\sigma \leq \mathrm{P}_\sigma$
- Compute the $m\mathrm{P}_\pi$ characteristic variables $\{\beta'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$
$$u_L - u_R \approx \sum_{\gamma=1}^{m\mathrm{P}_\pi} \beta'_\gamma r'_\gamma(u_{LR}^{\mathrm{Roe}}).$$
- Reconstruct the $m\mathrm{P}_\pi$ intermediate states at each physical interface
$$u'_\gamma = u'_{\gamma-1} + \beta'_\gamma r'_\gamma(u_{LR}^{\mathrm{Roe}}).$$
- Determine the set of sonic indices :
$$S' = \{\gamma, \lambda'_\gamma(u'_{\gamma-1}) < 0 < \lambda'_\gamma(u'_\gamma)\}.$$
- The indexing of $\{\lambda'_\gamma\}_\gamma$ and $\{r'_\gamma\}_\gamma$, $\gamma = 1, \dots, m\mathrm{P}_\pi$, provides a correspondence between approximate eigenvalues and eigenvectors and is central to determine $S'$.

# Case of random variables with non-uniform distribution functions

## Stochastic parametrization

$\xi = (\xi_1, \ldots, \xi_N)$ vector of random variables with known independent distribution functions.

## Change of variables

$x(\xi) = (x_1(\xi_1), \ldots, x_N(\xi_N)) = (p_1(\xi_1), \ldots, p_N(\xi_N))$ with $(p_d(\xi_d))_{d=1,\ldots,N}$ cumulative density functions

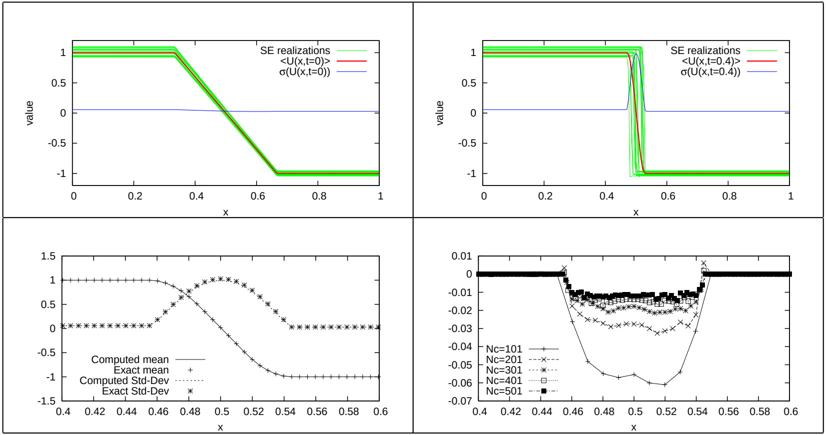
$$\rightarrow \boxed{x(\xi) \sim \mathcal{U}([0,1]^N).}$$

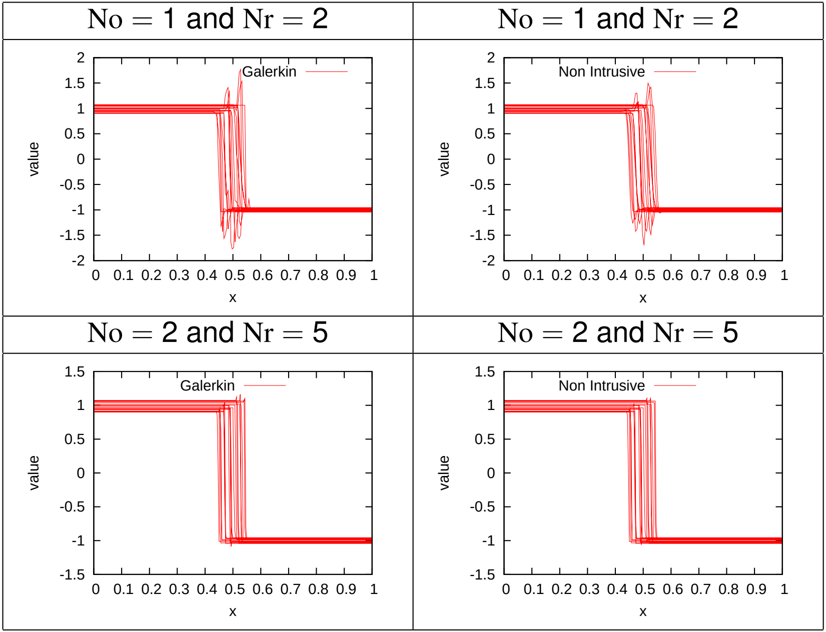
## Expansion of a process

$$\boxed{H(\xi) = \tilde{H}(x(\xi)) = \sum_{\alpha=1}^{P} \tilde{H}_\alpha \Psi_\alpha(x(\xi)).}$$

# Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},$$
$$U^+(\xi_1) = 1 + 0.1(2\xi_1 - 1), \qquad \xi_1 \sim \mathcal{U}[0,1],$$
$$U^-(\xi_2) = -1 + 0.05(2\xi_2 - 1), \quad \xi_2 \sim \mathcal{U}[0,1].$$




No = 1 and Nr = 2
Galerkin
No = 1 and Nr = 2
Non Intrusive
No = 2 and Nr = 5
Galerkin
No = 2 and Nr = 5
Non Intrusive
value
x