

Convergence Rates of Sparse sGFEM for elliptic sPDEs

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Elliptic BVP with stochastic data

Given:

- probability space (Ω, Σ, P) on data space $X(D) \subseteq L^\infty(D)$, $V \subseteq H^1(D)$,
- random diffusion coefficient $a(x, \omega) \in L^\infty(\Omega, dP; X(D))$,
- deterministic source term $f \in H^{-1}(D) = (H_0^1(D))' =: V'$,

(sBVP) Find $u(x, \omega) \in L^2(\Omega, dP; V) = L^2(\Omega, dP; H_0^1(D))$ such that

$$\mathbb{E} \left[\int_D a(x, \cdot) \nabla_x u(x, \cdot) \cdot \nabla_x v(x, \cdot) dx \right] = \mathbb{E} \left[\int_D f(x) v(x, \cdot) dx \right]$$

for all $v \in L^2(\Omega, dP; V)$

$a \in L^\infty(\Omega, dP; X(D))$ and $a_{max} \geq a(\cdot, \cdot) \geq a_{min} > 0 \Rightarrow \exists! u \in L^2(\Omega, dP; H_0^1(D))$.

Karhunen-Loève expansion

- separation of deterministic and stochastic variables -

Proposition 1 (Karhunen-Loève)

If $a \in L^2(\Omega, dP; L^\infty(D))$ then in $L^2(\Omega, dP; L^2(D))$,

$$a(x, \omega) = \mathbb{E}[a](x) + \sum_{m \geq 1} \psi_m(x) Y_m(\omega) = \mathbb{E}[a](x) + \sum_{m \geq 1} \sqrt{\lambda_m} \varphi_m(x) Y_m(\omega),$$

- $(\lambda_m, \varphi_m)_{m \geq 1}$ eigensequence of (cpt., s.a.) **covariance operator**

$$C[a] : L^2(D) \rightarrow L^2(D) \quad (C[a]v)(x) := \int_D C_a(x, x') v(x') dx' \quad \forall v \in L^2(D),$$

•

$$C_a = \mathbb{E} [(a - \mathbb{E}[a]) \otimes (a - \mathbb{E}[a])]$$

•

$$Y_m(\omega) := \frac{1}{\sqrt{\lambda_m}} \int_D (a(x, \omega) - \mathbb{E}[a](x)) \varphi_m(x) dx : \Omega \rightarrow \Gamma_m \subseteq \mathbb{R} \quad m = 1, 2, \dots$$

Parametric Deterministic BVP

$$a(x, \omega) = \mathbb{E}[a](x) + \sum_{j \geq 1} \psi_j(x) Y_j(\omega) \simeq a(x, y) := \mathbb{E}[a](x) + \sum_{j \geq 1} \psi_j(x) y_j,$$

$$y = (y_1, y_2, \dots) \in U = (-1, 1)^\infty, \quad dP(\omega) \sim \rho(dy) \sim \rho(y) dy \sim \bigotimes_{j \geq 1} \rho_j(y_j) dy_j$$

parametric deterministic BVP on U (variational stochastic formulation):

$$\text{Find } u \in L^2(U, V, d\rho) \quad \text{s.t.} \quad B(u, v) = F(v) \quad \forall v \in L^2(U, V, d\rho),$$

$$B(u, v) = \int_U \left(\int_D a(x, y) \nabla u(x, y) \cdot \nabla v(x, y) dx \right) \rho(dy), \quad F(v) = \int_U \left(\int_D f(x) v(x, y) dx \right) \rho(dy)$$

sGFEM

\mathcal{F} : set of all sequences $\nu = (\nu_j)_{j \geq 1}$ of nonnegative integers such that only finitely many ν_j are non-zero.

Notation:

$$|\nu| := \sum_{j \geq 1} \nu_j = \|\nu\|_{\ell^1}, \quad \nu! = \prod_{j \geq 1} \nu_j!, \quad \nu \in \mathcal{F}.$$

Tensorized ($L^2(-1, 1; 1/2dx)$ -normalized) Legendre polynomials:

$$L_\nu(y) = \prod_{j \geq 1} L_{\nu_j}(y_j).$$

Proposition 2:

Assume $\rho(dy_j) = 1/2dy_j$. Then

- $(L_\nu)_{\nu \in \mathcal{F}}$ is a complete orthonormal system in $L^2(U, d\mu)$.
- each $v \in L^2(U, V, d\mu)$ has a representation

$$v = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu, \quad \text{where} \quad v_\nu = \int_U v(\cdot, y) L_\nu(y) d\mu(y) \in V$$

and there holds Parseval's equation

$$\|v\|_{L^2(U,V,d\mu)} = \|(\|v_\nu\|_V)\|_{\ell^2(\mathcal{F})}.$$

sGFEM

For any finite subset $\Lambda \subset \mathcal{F}$, define

$$X_\Lambda := \{v_\Lambda(x, y) = \sum_{\nu \in \Lambda} v_\nu(x) L_\nu(y) ; v_\nu \in V\} \subset L^\infty(U, V) \subset L^2(U, V, d\mu)$$

where $\{L_\nu\}_{\nu \in \mathcal{F}}$ is the basis of Legendre polynomials.

Galerkin approximation: $u_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu \in X_\Lambda$ s.t.

$$u_\Lambda \in X_\Lambda \quad \text{such that} \quad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \quad \forall v_\Lambda \in X_\Lambda.$$

Cea's lemma:

$$\|u - u_\Lambda\|_{L^2(U, V, d\mu)} \leq C_1 \inf_{v_\Lambda \in X_\Lambda} \|u - v_\Lambda\|_{L^2(U, V, d\mu)},$$

where $C_1 := \sqrt{\frac{a_{\max}}{a_{\min}}}$.

sGFEM

Legendre expansion of $u(x, y)$:

$$u(x, y) = \sum_{\nu \in \mathcal{F}} v_\nu(x) L_\nu(y) \quad \text{where} \quad v_\nu := \int_U u(\cdot, y) L_\nu(y) d\mu(y) \in V.$$

$$\|u - u_\Lambda\|_{L^2(U, V, d\mu)} \leq C_1 \left(\sum_{\nu \notin \Lambda} \|v_\nu\|_V^2 \right)^{\frac{1}{2}}.$$

Choice of Λ ? Summability of sequence $\alpha_\nu := \|v_\nu\|_V$, $\nu \in \mathcal{F}$.

Lemma 3 (Stechkin):

Let $0 < p \leq q \leq \infty$ and assume $\alpha = (\alpha_\nu)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$.

If \mathcal{F}_N is the set of indices corresponding to N largest $|\alpha_\nu|$,

$$\left(\sum_{\nu \notin \mathcal{F}_N} |\alpha_\nu|^q \right)^{\frac{1}{q}} \leq \|\alpha\|_{\ell^p(\mathcal{F})} N^{-r},$$

where $r := \frac{1}{p} - \frac{1}{q} \geq 0$.

Regularity

$$b = (b_j)_{j=1}^{\infty}, \quad b_j := \frac{\|\psi_j\|_{L^\infty(D)}}{a_{min}}, \quad j = 1, 2, \dots$$

Theorem 4

Assume

$$\|b\|_{\ell_1(\mathbb{N})} = \sum_{j \geq 1} \frac{\|\psi_j\|_{L^\infty(D)}}{a_{min}} \leq \kappa < 1.$$

Then

$$\|v_\nu\|_V \leq \left(\frac{|\nu|!}{\nu!} b^\nu \right) \frac{\|f\|_{V^*}}{a_{min}}, \quad b^\nu = b_1^{\nu_1} b_2^{\nu_2} \dots, \quad \nu \in \mathcal{F}.$$

Regularity

Theorem 5 (Sequence Summability)

For $0 < p \leq 1$,

$$\left(\frac{|\nu|!}{\nu!}b^\nu\right)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F}) \text{ iff } \|b\|_{\ell^1(\mathbb{N})} \leq \bar{\kappa} < 1 \wedge b \in \ell^p(\mathbb{N}).$$

Moreover,

$$\left\|\left(\frac{|\nu|!}{\nu!}b^\nu\right)\right\|_{\ell^p(\mathcal{F})} \leq \frac{2}{\eta} \exp\left(\frac{2(1-p)(J(\eta) + \|b\|_{\ell^p(\mathbb{N})}^p)}{p^2\eta}\right),$$

where $\eta := (1 - \|b\|_{\ell^1(\mathbb{N})} a_{min})/2$ and $J(\eta)$ is the smallest positive integer such that $\sum_{j>J} |b_j|^p \leq \frac{\eta}{2}$.

Convergence rates

- semi-discretization in stochastic variable -

Focus on approximation in L^2 (mean square, maximum likelihood) and L^∞ (worst case).

Tool: Classical Legendre basis $(P_n)_{n \geq 0}$:

$$\|P_n\|_{L^\infty([-1,1])} = P_n(1) = 1. \quad (1)$$

L^2 normalized Legendre Sequence $L_n(t) = \sqrt{2n+1}P_n(t)$,

$$\int_{-1}^1 |L_n(t)|^2 \frac{dt}{2} = 1.$$

Note

$$L_0 = P_0 = 1.$$

For $\nu \in \mathcal{F}$, consider tensorized polynomials

$$P_\nu(y) := \prod_{j \geq 1} P_{\nu_j}(y_j) \quad \text{and} \quad L_\nu(y) := \prod_{j \geq 1} L_{\nu_j}(y_j). \quad (2)$$

Note:

- $(L_\nu)_{\nu \in \mathcal{F}}$ orthonormal basis of $L^2(U, d\mu)$, $\mu = \bigotimes_{j=1}^{\infty} \frac{1}{2} \lambda_j$
- every $u \in L^\infty(U, V, d\mu) \subset L^2(U, V, d\mu)$ admits expansions

$$u(y) = \sum_{\nu \in \mathcal{F}} u_\nu P_\nu(y) = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu(y), \quad (3)$$

in $L^2(U, V, d\mu)$ with $u_\nu, v_\nu \in V$ defined by

$$v_\nu := \int_U u(y) L_\nu(y) d\mu(y) \text{ and } u_\nu := \left(\prod_{j \geq 1} (1 + 2\nu_j) \right)^{1/2} v_\nu. \quad (4)$$

Theorem 6:

Assume

- $a(x, z)$ satisfies **UEAC**(r, R) for some $0 < r \leq R < \infty$
- $(\|\psi_j\|_{L^\infty(D)})_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p < 1$,

Then

$$(\|u_\nu\|_V)_{\nu \in \mathcal{F}}, \quad (\|v_\nu\|_V)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$$

for the same value of p .

The Legendre expansions (3) converge in $L^\infty(U, V)$:

i.e. $S_{\Lambda_N} u(y) := \sum_{\nu \in \Lambda_N} u_\nu(x) P_\nu(y) = \sum_{\nu \in \Lambda_N} v_\nu(x) L_\nu(y)$ converge

$$\lim_{N \rightarrow +\infty} \sup_{y \in U} \|u(y) - S_{\Lambda_N} u(y)\|_V = 0 \quad (5)$$

for $(\Lambda_N)_{N \geq 1} \subset \mathcal{F}$ any sequence of finite sets which exhausts \mathcal{F} .

Moreover,...

1. L^∞ (“worst case”) convergence:

If Λ_N is the set of $\nu \in \mathcal{F}$ corresponding to indices of the largest $\|u_\nu\|_V$, then holds the convergence estimate

$$\sup_{y \in U} \|u(y) - S_{\Lambda_N} u(y)\|_V \leq \|(\|u_\nu\|_V)\|_{\ell^p(\mathcal{F})} N^{-r}, \quad r = \frac{1}{p} - 1 \geq 0. \quad (6)$$

2. L^2 (“maximum likelihood”) convergence:

If Λ_N is the set of $\nu \in \mathcal{F}$ corresponding to indices of the largest $\|v_\nu\|_V$, then

$$\|u - S_{\Lambda_N} u\|_{L^2(U, V, d\mu)} \leq \|(\|v_\nu\|_V)\|_{\ell^p(\mathcal{F})} N^{-r}, \quad r = \frac{1}{p} - \frac{1}{2} \geq \frac{1}{2}. \quad (7)$$

Key estimate:

Lemma 7:

Assume $\text{UEAC}(r, R)$ for some $0 < r \leq R < \infty$. Let $\rho = (\rho_j)_{j \geq 1}$ be δ -admissible, i.e.

$$\exists \delta > 0 \quad \forall x \in D : \quad \sum_{j \geq 1} \rho_j |\psi_j(x)| \leq \bar{a}(x) - \delta. \quad (8)$$

such that $\rho_j > 1$ for all j with $\nu_j \neq 0$.

Then for any $\nu \in \mathcal{F}$ we have the estimate

$$\|u_\nu\|_V \leq \frac{\|f\|_{V^*}}{\delta} \prod_{j \geq 1, \nu_j \neq 0} \phi(\rho_j) (2\nu_j + 1) \rho_j^{-\nu_j}, \quad (9)$$

where $\phi(t) := \frac{\pi t}{2(t-1)}$ for $t > 1$.

Choice of ρ_j : fix $1 < \kappa \leq 2$ such that

$$(\kappa - 1) \sum_{j \geq 1} \|\psi_j\|_{L^\infty(D)} \leq \frac{r}{8}. \quad (10)$$

Choose J_0 as smallest integer such that

$$\sum_{j > J_0} \|\psi_j\|_{L^\infty(D)} \leq \frac{r(\kappa - 1)}{18\pi\kappa}, \quad (11)$$

Then define

$$\rho_j := \begin{cases} \kappa & \text{if } 1 \leq j \leq J_0, \\ 2 + \frac{r\nu_j}{4|\nu_F|\|\psi_j\|_{L^\infty(D)}}, & \forall j \geq J_0 \text{ such that } \nu_j \neq 0, \\ 1 & \forall j \geq J_0 \text{ such that } \nu_j = 0. \end{cases} \quad (12)$$

Then ρ is $\frac{r}{2}$ admissible.

Number Theory (finding Λ_ℓ) - I

Problem: Theorem does *not* give a constructive way of finding sets

$$\Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda_\ell \subset \dots \mathcal{F}$$

Two strategies: 1. Adaptive sGFEM, 2. A-priori selection of the Λ_ℓ .

Consider 2:

given $\mu = (\mu_1, \mu_2, \dots) > 0$ s.t. $1 > \mu_1 \geq \mu_2 \geq \dots \geq \mu_m \rightarrow 0$ and $\varepsilon > 0$, define

$$\Lambda_\varepsilon(\mu) := \{\nu \in \mathcal{F} \mid \mu^\nu \geq \varepsilon\} = \{\nu \in \mathcal{F} \mid \mu^{-\nu} \leq 1/\varepsilon\}$$

If our bounds on coefficients $\|v_\nu\|_V$ are sharp, $\Lambda_\varepsilon(b)$ will contain essentially the $\#\Lambda_\varepsilon(b)$ 'largest coefficients' $\|v_\nu\|_V$.

Monotonicity: for any $\mu, \bar{\mu} \in \Lambda$ as above and any $\bar{\varepsilon}, \varepsilon > 0$

1. $M_\varepsilon(\mu) := \max_{m \in \mathbb{N}} \{\mu_m \geq \varepsilon\} < \infty$,
2. $\forall \alpha \in \Lambda_\varepsilon(\mu) : \text{supp}(\alpha) \subset \{1, 2, \dots, M_\varepsilon(\mu)\}$,
3. $\bar{\varepsilon} \geq \varepsilon$ implies $\Lambda_{\bar{\varepsilon}}(\mu) \subseteq \Lambda_\varepsilon(\mu)$,
4. $\mu \preceq \bar{\mu}$ implies $\Lambda_\varepsilon(\mu) \subseteq \Lambda_\varepsilon(\bar{\mu})$.

Number Theory (finding Λ_ℓ) - II

Consider *comparison sequences* $\nu_\sigma \in \ell^\infty(\mathbb{N})$ such that

$$b \preceq \nu_\sigma := \{(m+1)^{-\sigma} : m = 1, 2, \dots\}, \quad \sigma > 0.$$

Note:

$$b \preceq \nu_\sigma \Rightarrow \forall \nu \in \Lambda : b^\nu \leq \nu_\sigma^\nu$$

Scaling: given $\sigma > 0$, find

$$\Lambda_\varepsilon(\nu_\sigma) = \Lambda_{\varepsilon^{1/\sigma}}(\nu_1) = \left\{ \nu \in \mathbb{N}_0^{\mathbb{N}} : \prod_{m \in \mathbb{N}} (m+1)^{\nu_m} \leq \varepsilon^{-1/\sigma} \right\}, \quad \varepsilon \rightarrow 0.$$

$\nu \in \Lambda_\varepsilon(\nu_\sigma)$ iff ν is a multiplicative partition of some integer $x < \varepsilon^{-1/\sigma}$.

Complexity (finding Λ_ℓ)

Proposition 8 (R. Andreev '08)

1. $\Lambda_\varepsilon(\nu_\sigma)$ can be localized in work and memory growing log-linearly in $\#\Lambda_\varepsilon(\nu_\sigma)$.
2. As $\varepsilon \rightarrow 0$,

$$\#\Lambda_\varepsilon(\nu_\sigma) \sim x \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}} \quad \text{with} \quad x = \varepsilon^{-1/\sigma},$$

Discretization in D

Smoothness in D :

$$W = \{v \in V : \Delta v \in L^2(D)\} \subset V.$$

Norms:

$$\|v\|_W := \|v\|_V + |v|_W, \quad |v|_W = \|\Delta v\|_{L^2(D)}.$$

Note: D convex $\implies W = H^2(D) \cap V$.

Theorem 9

Assume $f \in L^2(D)$ and $a(x, z)$ satisfies **UEAC**(r, R) for $0 < r \leq R < \infty$.

If for some $0 < p < 1$

$$(\|\psi_j\|_{L^\infty(D)})_{j \geq 1} \in \ell^p(\mathbb{N}) \quad \text{and} \quad (\|\nabla \psi_j\|_{L^\infty(D)})_{j \geq 1} \in \ell^p(\mathbb{N})$$

then $(\|u_\nu\|_W)_{\nu \in \mathcal{F}}, (\|v_\nu\|_W)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$.

Discretization in D

Finite Element Approximation in D :

D Convex \implies

$$\inf_{v_h \in V_h} \|w - v_h\|_V \leq CM^{-\frac{1}{d}} |w|_W.$$

Non-convex polyhedra D : $W \not\subset H^2(D)$.

Convergence as $M = \dim(V_h) \rightarrow \infty$ reduced to

$$\inf_{v_h \in V_h} \|w - v_h\|_V \leq C_t M^{-t} |w|_W \quad \text{some } 0 < t < \frac{1}{d}.$$

Discretization in D

Finite Element Approximation in D :

Theorem 10:

Assume

- FE spaces $(V_h)_{h>0}$ in $D \subset \mathbb{R}^d$ have approximation order $0 < t \leq 1/d$,
- a satisfies **UEAC**(r, R) and $(\|\psi_j\|_{W^{1,\infty}(D)})_{j=1}^\infty \in \ell^p(\mathbb{N})$ some $0 < p \leq 1$.

Then

- (i) With Λ_N the set of indices corresponding to N largest $\|t_\nu\|_V$, there exist finite element spaces V_ν of dimensions M_ν , $\nu \in \Lambda_N$, such that

$$\sup_{y \in U} \|u(y) - \sum_{\nu \in \Lambda_N} \tilde{t}_\nu y^\nu\|_V \leq C N_{dof}^{-\min\{r,t\}}, \quad r := \frac{1}{p} - 1,$$

where $N_{dof} = \sum_{\nu \in \Lambda_N} M_\nu$ and $C = (\bar{C}_t + \|(\|t_\nu\|_V)\|_{\ell^p(\mathcal{F})}) \|(\|t_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

(ii) With Λ_N the set of indices corresponding to N largest $\|u_\nu\|_V$, there exist finite element spaces V_ν of dimensions M_ν , $\nu \in \Lambda_N$, such that

$$\sup_{y \in U} \|u(y) - \sum_{\nu \in \Lambda_N} \tilde{u}_\nu P_\nu(y)\|_V \leq C N_{dof}^{-\min\{r,t\}}, \quad r := \frac{1}{p} - 1,$$

where $N_{dof} = \sum_{\nu \in \Lambda_N} M_\nu$ and $C = (\bar{C}_t + \|(\|u_\nu\|_V)\|_{\ell^p(\mathcal{F})}) \|(\|u_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

(iii) With Λ_N the set of indices corresponding to N largest $\|v_\nu\|_V$, there exist finite element spaces V_ν of dimension M_ν , $\nu \in \Lambda_N$, such that

$$\|u - \sum_{\nu \in \Lambda_N} \tilde{v}_\nu L_\nu\|_{L^2(U,V,d\mu)} \leq C N_{dof}^{-\min\{r,t\}}, \quad r := \frac{1}{p} - \frac{1}{2},$$

where $N_{dof} = \sum_{\nu \in \Lambda_N} M_\nu$ and $C = (\bar{C}_t^2 + \|(\|v_\nu\|_V)\|_{\ell^p(\mathcal{F})}^2)^{\frac{1}{2}} \|(\|v_\nu\|_W)\|_{\ell^p(\mathcal{F})}$.

Here, \tilde{t}_ν , \tilde{u}_ν and \tilde{v}_ν are V -projections of t_ν , u_ν and v_ν , respectively, onto V_ν .

Example (R. Andreev)

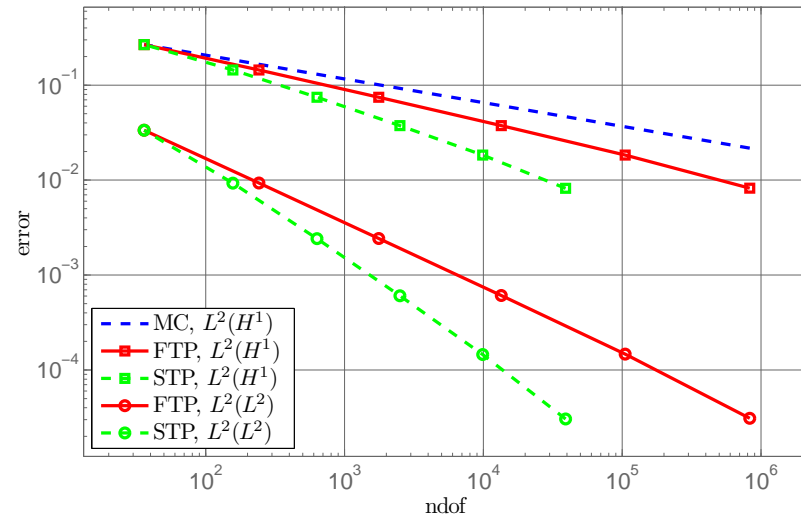
$$D = (-1, 1), \mathbb{E}_a(x) = 5 + x, C_a(x, x') = \frac{\min\{x, x'\} + 1}{2} \in H^{1,1}(D \times D).$$

KL Eigenpairs:

$$\lambda_m = \frac{8}{\pi^2(2m-1)^2}, \quad \psi_m(x) = \sin((x+1)/\sqrt{2\lambda_m})$$

Algebraic KL decay with rate 2.

Error vs. N_{total}



Conclusion

- gpc: Transform SPDE into parametric, deterministic PDE on $U = (-1, 1)^\infty$

- Sparse tensor approximations of

$$B_1(\ell_\infty) \ni y \rightarrow u(y, \cdot) \in V$$

- Stochastic Regularity = p -summability of gpc coefficients
- Sparse tensorization of D and Ω Galerkin discretizations: Dimension independent convergence rates in terms of N_{total} .
- requires *mixed regularity* in *both*, D and U : if

$$\{\|\psi_j\|_{W^{1,\infty}(D)}\}_{j=1}^\infty \in \ell^p(\mathbb{N}) \quad \text{then} \quad \{\|u_\nu\|_W : \nu \in \mathcal{F}\} \in \ell^p(\mathcal{F})$$

- L^2 (mean square) and L^∞ (worst case) convergence rates.
- Adaptivity essential.

- Diffusion problems in physical dimension $d = 2$, with $\lambda_m \sim m^{-2.2}$ and $\text{supp}\Lambda \subset \{1, \dots, 1500\}$ on PC (R. Andreev, M. Bieri and CS (SISC 2010)) (superior to Smolyak construction).
- Sparse **collocation** on input adapted 'lattices' of points in $U = (-1, 1)^\infty$ (M. Bieri Dissertation (2009) and SISC (2010))
- Lognormal Case (C.J. Gittelsohn)
- Other Differential Equations
- Fully Adaptive, Convergent Algorithm (Cohen, DeVore)

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