Workshop on:
Numerical Solution of Stochastic Partial Differential Equations
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INTEGRATION OF FICTITIOUS DOMAIN AND CHAOS COLLOCATION METHODS FOR THE ANALYSIS OF GEOMETRIC UNCERTAINTIES IN FLUID DYNAMICS

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OUTLINE

- INTRODUCTION
- THEORY

UNCERTAINTY QUANTIFICATION METHODS: TeCC METHOD
FICTITIOUS DOMAIN APPROACH WITH LEAST-SQUARES SPECTRAL ELEMENT APPROXIMATION
FORMULATION OF POLYNOMIAL CHAOS METHODOLOGIES COUPLED TO FICTITIOUS DOMAIN

- APPLICATIONS

WANNIER FLOW WITH UNCERTAINTIES ON GEOMETRY AND BOUNDARY CONDITIONS

BACKWARD-FACING STEP WITH GEOMETRIC UNCERTAINTIES

- CONCLUSIONS

INTRODUCTION

Deterministic mathematical models are rough simplifications of reality.

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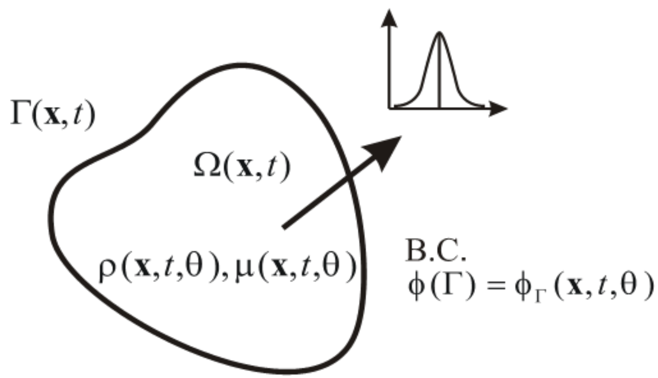
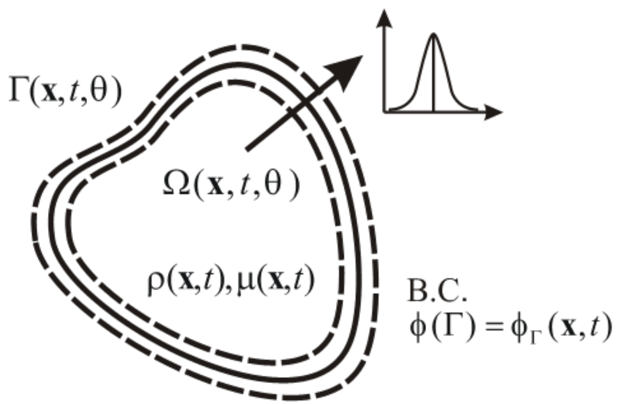
Uncertainty quantification is necessary.



Differential problem with stochastic material properties and stochastic boundary conditions, where θ is uncertainty.



Differential problem with stochastic definition domain, where θ is uncertainty.

INTRODUCTION

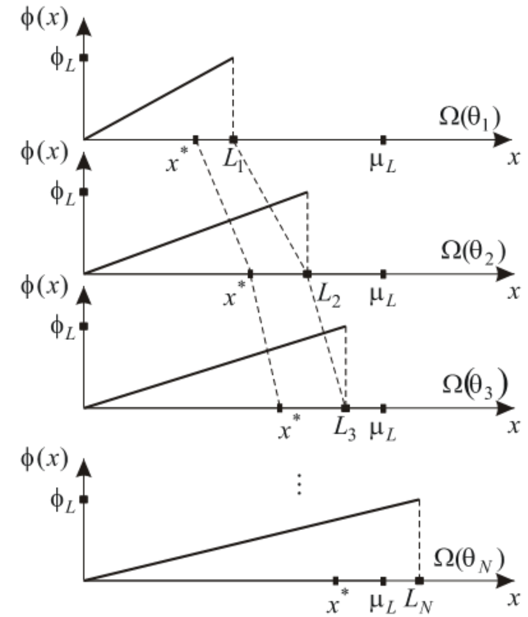
GEOMETRIC UNCERTAINTIES: CONCEPT AND ANALYSIS

An approach to geometric uncertainties has been investigated in *Xiu and Tartakovsky* (2006), where a mapping methodology is introduced.

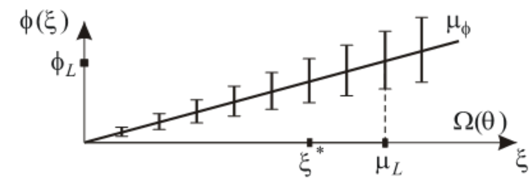
Let us consider this one-dimensional differential problem defined on stochastic domain, as example:

$$\frac{d^2\phi}{dx^2} = 0 \quad \text{in } [0, L]$$

with $\phi(0) = \phi_0$, $\phi(L) = \phi_L$
and $L = N(L_{Mean}, \sigma_L)$.



Representation of different deterministic solutions ϕ(x) corresponding to different domains $\Omega(\theta_i)$ of problem under study.



Representation of stochastic solution ϕ(ξ,θ) of problem under study: it refers to relative coordinates.

INTRODUCTION

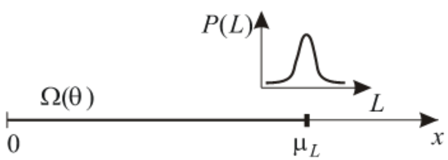
GEOMETRIC UNCERTAINTIES: CONCEPT AND ANALYSIS

Let us consider this one-dimensional differential problem defined in a stochastic domain, as example:

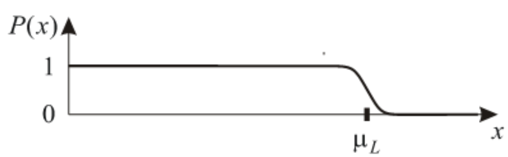
$$\frac{d^2\phi}{dx^2} = 0 \quad \text{in } [0, L]$$

with

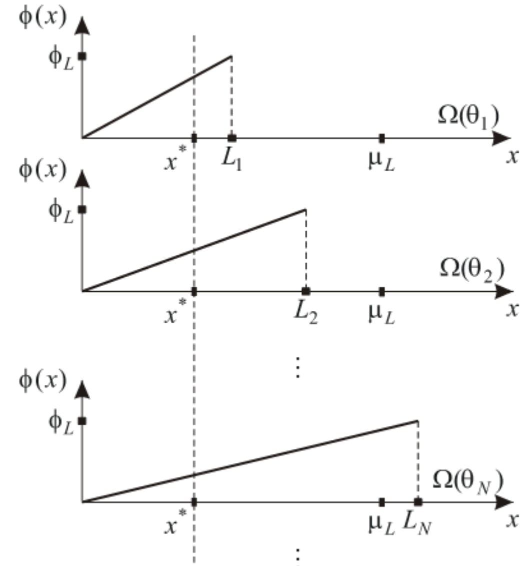
$$\phi(0) = \phi_0 \,, \quad \phi(L) = \phi_L \quad \text{and} \quad L = N(L_{Mean}, \sigma_L).$$



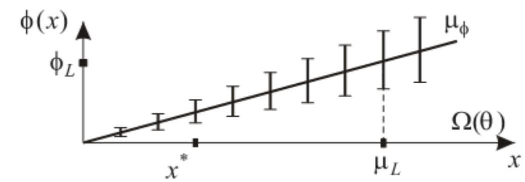
Representation of the stochastic domain $\Omega(\theta)$.



Probability P(x) of a point of belonging to domain $\Omega(\theta)$.



Representation of different deterministic solutions $\phi(x)$ corresponding to different domains $\Omega(\theta_i)$ of problem under study.



Representation of stochastic solution $\phi(x,\theta)$ of problem under study: the mean value μ_ϕ and the uncertainty bars $\mu_\phi \pm \sigma_\phi$ are shown, referring to absolute coordinates.

Example: S. Hosder, R.W. Walters, and R. Perez, A Non-Intrusive Polynomial Chaos Method for Uncertainty Propagation in CFD Simulations, 44th AIAA Aerospace Sciences Meeting and Exhibit, Paper No. AIAA 2006-891. Reno, NV, Jan. 2006.

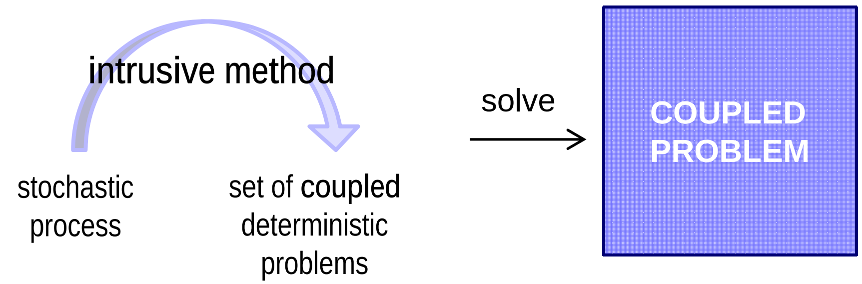
INTRODUCTION

To face problems with uncertainty:

INTRUSIVE APPROACH

- coupled system of deterministic equations
- needs to modify the solver obtaining an efficient tool but limited to solve just a set of problems

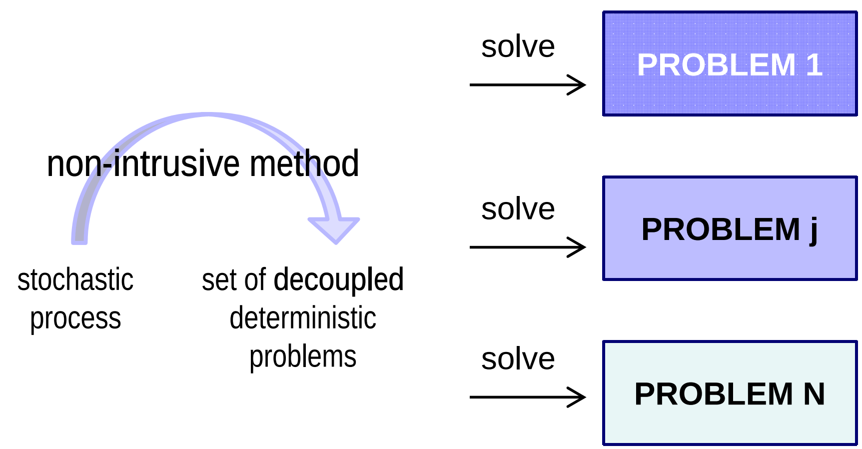
Chaos Polynomial method



NON-INTRUSIVE APPROACH

- decoupled system of deterministic equations
- allows the use of existing deterministic solvers

Monte Carlo method
Stochastic Collocation method
Chaos Collocation method
Tensorial-expanded Chaos Collocation method



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UNCERTAINTY QUANTIFICATION METHODS

Let us consider the stochastic differential equation: $L(\mathbf{x},t,\theta;\phi) = f(\mathbf{x},t,\theta)$

with $\mathbf{x} \in \Re^d$	space	t	time	θ	random parameters
L	differential operator	f	source term	ϕ	solution

We refer to the solution in terms of spectral representation $\phi(\mathbf{x},t,\theta) = \sum_{i=0}^{\infty} \phi_i(\mathbf{x},t)\Psi_i(\xi(\theta))$

For practical cases, the series has to be truncated to a finite number of terms, here denoted with N. $\phi(\mathbf{x},t,\theta) = \sum_{i=0}^{N} \phi_i(\mathbf{x},t)\Psi_i(\xi(\theta))$

$$L\left(\mathbf{x},t,\theta;\sum_{i=0}^{N} \phi_i(\mathbf{x},t)\Psi_i(\xi(\theta))\right) \cong f(\mathbf{x},t,\theta)$$

to solve this differential equation

Method of Weighted Residuals

UNCERTAINTY QUANTIFICATION METHODS

By Galerkin projection :

$$\left\langle L\left(\mathbf{x},t,\theta;\sum_{i=0}^{N}\phi_i(\mathbf{x},t)\Psi_i(\xi(\theta))\right),\Psi_j\right\rangle = \left\langle f(\mathbf{x},t,\theta),\Psi_j\right\rangle \quad j=0,\ldots,N$$

Chaos Polynomial
intrusive form

By Collocation projection :

$$L(\mathbf{x},t,\theta_j;\phi_j) = f(\mathbf{x},t,\theta_j) \quad j=0,\ldots,N$$

Chaos Collocation
non-intrusive form

EXPECTED VALUE

$$E_{PC}(\phi) = \mu_\phi = \phi_0(\mathbf{x},t)$$

VARIANCE

$$Var_{PC}(\phi) = \sigma_\phi^2 = \sum_{i=1}^{N}\phi_i^2(\mathbf{x},t)\left\langle \Psi_i^2\right\rangle$$

UNCERTAINTY QUANTIFICATION METHODS

Tensorial-expanded Chaos Collocation (TeCC)

non-intrusive form

Tensorial-expanded Chaos Collocation method and Chaos Collocation method differ for

- the choice of expansion polynomials

$$\phi(\mathbf{x},t,\theta)=\sum_{ijk=0}^{N}\phi_{ijk}(\mathbf{x},t)\Psi_i(\xi_1)\Psi_j(\xi_2)\ldots\Psi_k(\xi_n)$$

- the selection of collocation points

If n is the number of uncertain variables and p is the expansion polynomial order the collocation points are the full-factorial of roots of the one dimensional orthogonal polynomial of order p+1 with n factors.

$N+1=(p+1)^n$ number of terms of spectral representation

UNCERTAINTY QUANTIFICATION METHODS

In this work we examine problems with multi uncertain **geometric** parameters with Gaussian distribution applying the **Tensorial-expanded Chaos Collocation** theory.

If we have to compute the solution of a differential equation defined on a stochastic domain, we have to remesh the computational domain for each new simulation.

↓

Fictitious Domain methodology

- the stochastic domain does not coincide with the computational domain, which is the same for all simulations
- for every new geometry the trace of Lagrange multipliers, which enforce the boundary conditions immersed in the computational domain, has just to be modified

References:

Canuto C, Kozubek T. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. Numerische Mathematik 2007;107(2):257–93.

Parussini L, Pediroda V. Fictitious domain with least-squares spectral element method to explore geometric uncertainties by non-intrusive polynomial chaos method. CMES Comput Model Eng Sci 2007;22(1):41–64.

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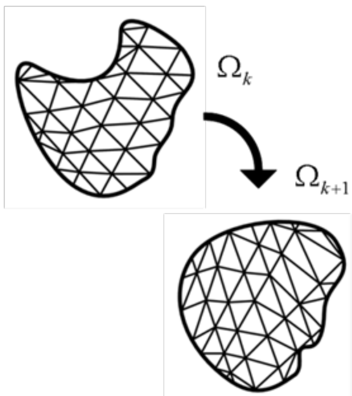
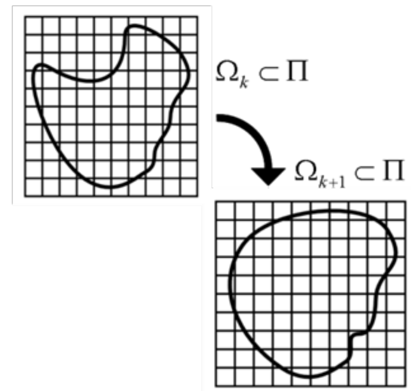
FICTITIOUS DOMAIN APPROACH



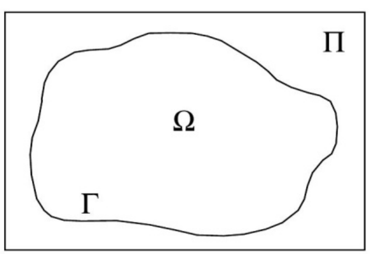
Classical approach based on the boundary variation technique to solve differential problems defined on domain changing in time and space.



Fictitious Domain approach to solve differential problems defined on domain changing in time and space.

BASIC IDEA:

to extend the operator and domain Ω of original differential problem to a simple shaped domain Π



Example of a fictitious rectangular domain Π containing the original domain Ω. Π represents the computational domain, Ω is the definition domain of the state problem.

- Algebraic approach
- Functional analytic approach

FICTITIOUS DOMAIN APPROACH

DEFINITION

A variational principle specifies a scalar quantity, the functional J, which is defined by an integral form

$$J = \int_{\Omega} F\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots\right) d\Omega + \int_{\Gamma} E\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots\right) d\Gamma$$

The solution to the continuum problem is a function ϕ which make J stationary with respect to small changes $\delta\phi$

$$\delta J = 0$$

J' is the constrained fictitious domain functional:

$$J' = \int_{\Pi} F\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots\right) d\Pi + \int_{\Gamma} \lambda(\mathbf{x}) E\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots\right) d\Gamma.$$

Here $\lambda(\mathbf{x})$ is an undetermined Lagrangian multiplier which is in general a function of position, because the local condition must be satisfied at every point of Γ, rather than being a global restriction.

The solution to the continuum problem are the functions ϕ and λ which make J' stationary with respect to small changes $\delta\phi$ and $\delta\lambda$.

FICTITIOUS DOMAIN APPROACH

Let us consider the vorticity based Navier-Stokes equations:

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+\frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\omega}=\mathbf{f} \quad in\ \Omega$$

$$\boldsymbol{\omega}-\nabla\times\mathbf{u}=\mathbf{0} \quad in\ \Omega$$

$$\nabla\cdot\mathbf{u}=\mathbf{0} \quad in\ \Omega$$

$$\nabla\cdot\boldsymbol{\omega}=\mathbf{0} \quad in\ \Omega$$

$$\mathbf{u}=\mathbf{u}^s \quad on\ \Gamma_u$$

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}=\mathbf{f}^s \quad on\ \Gamma_f$$

<u>Fictitious Domain Least-Squares functional</u>

$$J(\mathbf{u},p,\boldsymbol{\omega},\lambda;\mathbf{f})=\frac{1}{2}\left\|(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+\frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\omega}-\mathbf{f}\right\|^2_{0,\Pi}+$$
$$\frac{1}{2}\|\boldsymbol{\omega}-\nabla\times\mathbf{u}\|^2_{0,\Pi}+\frac{1}{2}\|\nabla\cdot\mathbf{u}\|^2_{0,\Pi}+\frac{1}{2}\|\nabla\cdot\boldsymbol{\omega}\|^2_{0,\Pi}+$$
$$\left\langle\lambda,(\mathbf{u}-\mathbf{u}^s)\right\rangle_{0,\Gamma}+\left\langle\lambda,(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}-\mathbf{f}^s)\right\rangle_{0,\Gamma}$$

with $\mathbf{X}=\{(\mathbf{u},p,\boldsymbol{\omega})\in\mathbf{H}^1(\Pi)\times H^1(\Pi)\times\mathbf{H}^1(\Pi)\}$ and

$$M=\{\lambda\in H^{-1/2}(\Gamma)\}$$

This yields:

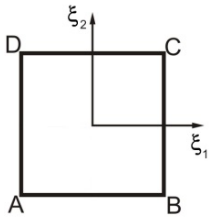
$$\begin{cases}\text{Find } (\mathbf{u},p,\boldsymbol{\omega},\lambda)\in\mathbf{X}\times M \text{ such that}\\ a((\mathbf{u},p,\boldsymbol{\omega}),(\mathbf{v},q,\boldsymbol{\psi}))+b((\mathbf{v},q,\boldsymbol{\psi}),\lambda)=l((\mathbf{v},q,\boldsymbol{\psi})) \quad \forall(\mathbf{v},q,\boldsymbol{\psi})\in\mathbf{X}\\ b((\mathbf{u},p,\boldsymbol{\omega}),\mu)=g(\mu) \quad \forall\mu\in M\end{cases}$$

SPECTRAL ELEMENT APPROXIMATION

$$\mathbf{u}(\xi,\eta)=\sum_{i=1}^{Q}\tilde{\mathbf{u}}_i\varphi_i(\xi,\eta) \quad in\ \Pi_{st}$$

$$p(\xi,\eta)=\sum_{i=1}^{Q}\tilde{p}_i\varphi_i(\xi,\eta) \quad in\ \Pi_{st}$$

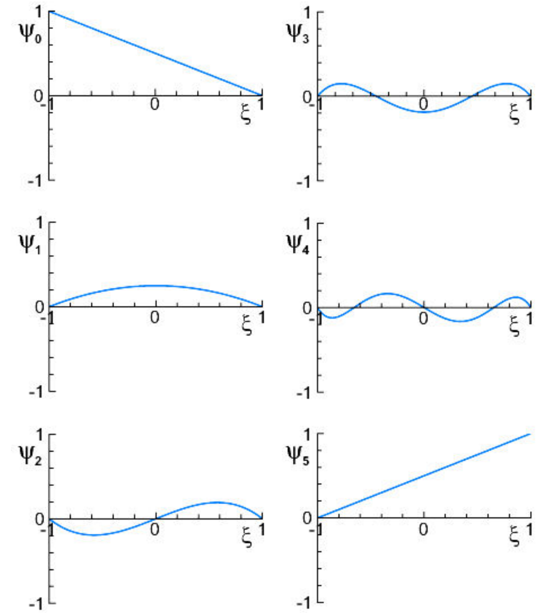
$$\boldsymbol{\omega}(\xi,\eta)=\sum_{i=1}^{Q}\tilde{\boldsymbol{\omega}}_i\varphi_i(\xi,\eta) \quad in\ \Pi_{st}$$



In similar manner:

$$\lambda(\xi)=\sum_{p=1}^{P}\tilde{\lambda}_p\psi_p(\xi) \quad in\ \Gamma_{st}$$

$$\psi_p=\begin{cases}\dfrac{1-\xi}{2} & \text{for} \quad p=0\\ \dfrac{1-\xi}{2}\dfrac{1+\xi}{2}P_{p-1}^{\alpha,\beta} & \text{for} \quad 0<p<P, P\geq 1\\ \dfrac{1+\xi}{2} & \text{for} \quad p=P\end{cases}$$



Shape of modal expansion modes for a polynomial order of P=5.

References:
Karniadakis GE, Sherwin SJ. Spectral/hp element methods for CFD. Oxford: Oxford University Press; 1999.

FICTITIOUS DOMAIN APPROACH AND SPECTRAL ELEMENT APPROXIMATION

REMARK

Thanks to Least-Squares formulation the choice of primal variables $\mathbf{u}$, p and $\boldsymbol{\omega}$ discrete spaces has not to satisfy compatibility conditions, so we can choose the same finite element subspace for each one, but the choice of Lagrange multipliers discrete space is not independent by the discrete spaces of the other variables.

Ladyzhenskaja-Babuska-Brezzi (LBB)-condition

To ensure the convergence of the solution of discretized model to that one of the continuous problem the LBB-condition has to be satisfied:

$$\sup_{\psi \in H^1(\Pi)} \frac{\int_\Gamma \mu \psi ds}{\|\psi\|_{H^1(\Pi)}} \geq \overline{\beta} \|\mu\|_{H^{1/2}(\Gamma)}, \forall \mu \in H^{1/2}(\Gamma)$$

for some $\overline{\beta} > 0$ indipendent of discretization.

FICTITIOUS DOMAIN APPROACH AND SPECTRAL ELEMENT APPROXIMATION

The FD-LSqSEM produces symmetric indefinite matrices:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{\Phi} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}$$

where $\mathbf{A}$ is symmetric positive definite and $\mathbf{B}$ is the matrix coupling the primal variables $\mathbf{u}$, p, $\boldsymbol{\omega}$ and the Lagrange multipliers λ.

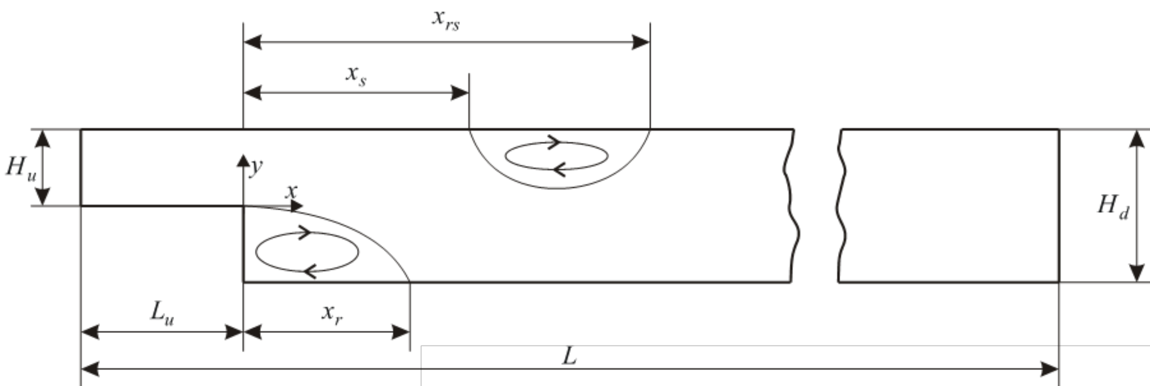
The information on the geometry is just in $\mathbf{B}$ and $\mathbf{g}$.

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{u} \\ p \\ \boldsymbol{\omega} \end{pmatrix} = \mathbf{A}^{-1}\left(-\mathbf{B}^T\boldsymbol{\lambda} + \mathbf{f}\right) \quad \longrightarrow \quad \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{B}\mathbf{A}^{-1}\mathbf{f} - \mathbf{g}$$

For problems with moving boundaries or if the same problem has to be solved for several different boundary conditions, the matrix $\mathbf{A}$ remains unchanged.
A modification of Γ only affects $\mathbf{B}$ and $\mathbf{g}$, but not $\mathbf{A}$, so the inversion of $\mathbf{A}$ has to be performed just once and the remeshing of Π is not necessary.

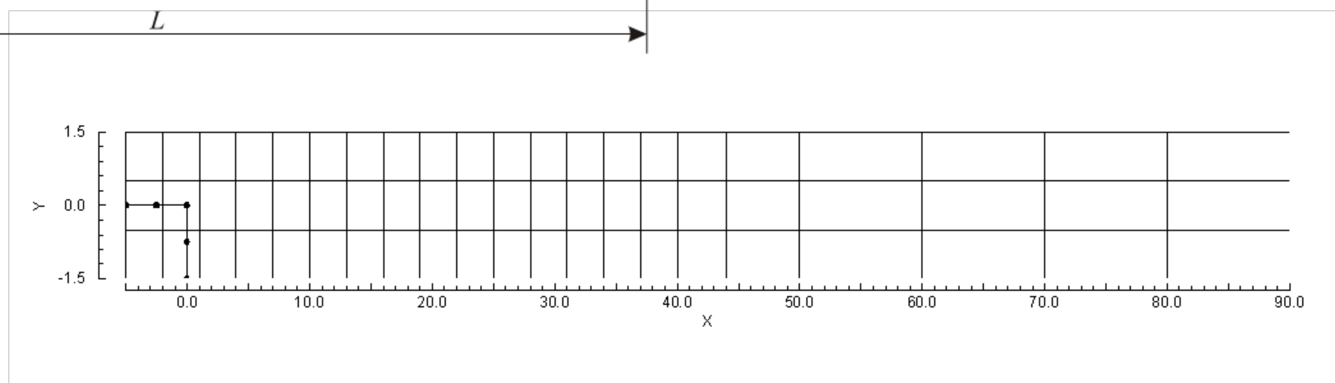
VALIDATION OF FD-LSqSEM SOLVER

Backward-facing step: two-dimensional steady flow over a backward-facing step at Re = 800

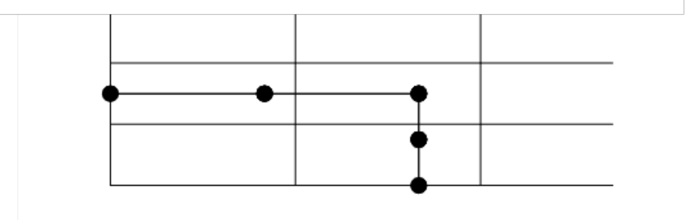


Schematic illustration of flow over a backward-facing step: geometry of flow field.

Computational domain and mesh for flow over a backward-facing step: connected model. The reference system is such that x : y = 1 : 4.

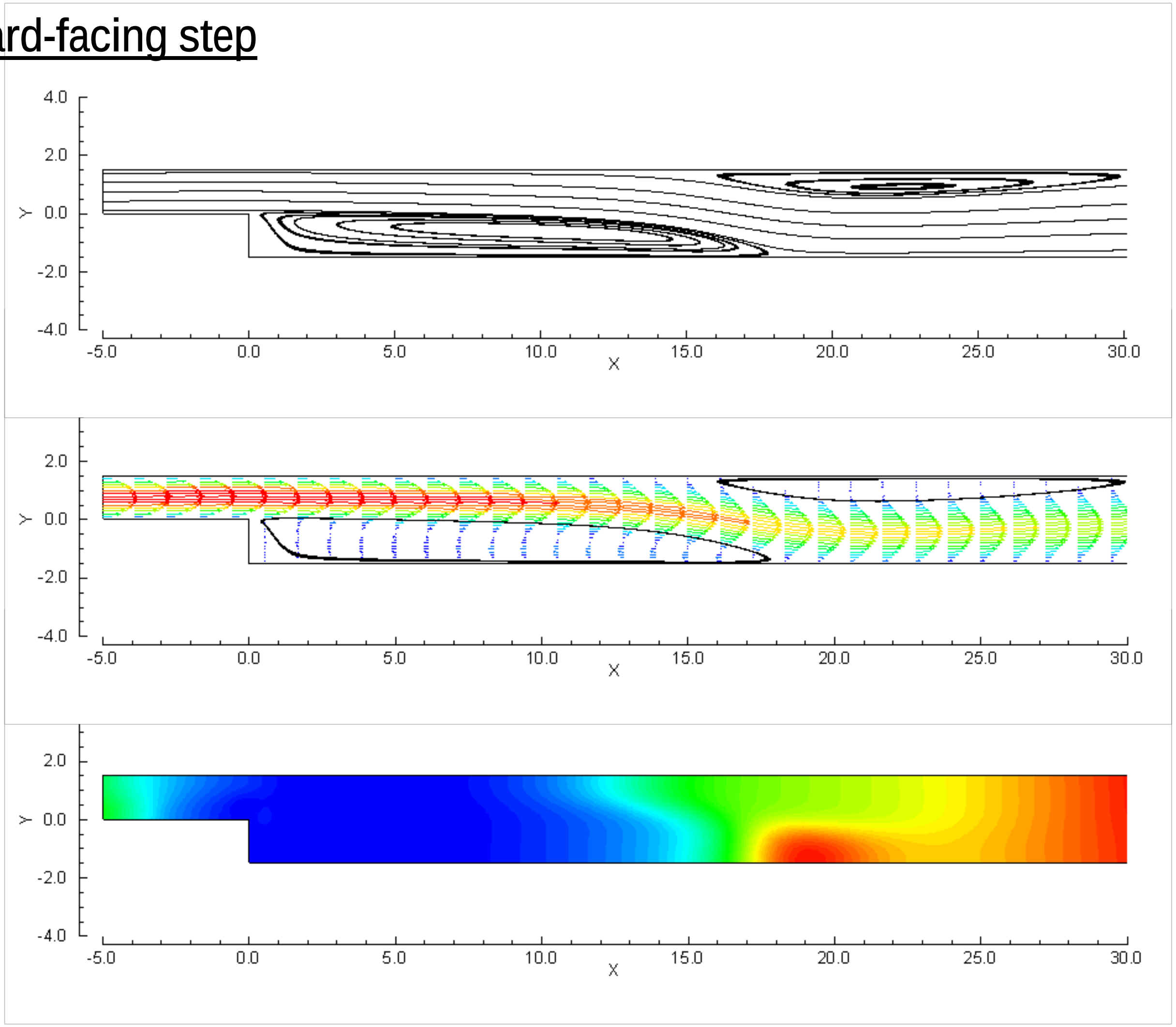


Computational domain and mesh for flow over a backward-facing step: close-up view of the immersed boundary geometric discretization.



VALIDATION OF FD-LSqSEM SOLVER

Backward-facing step



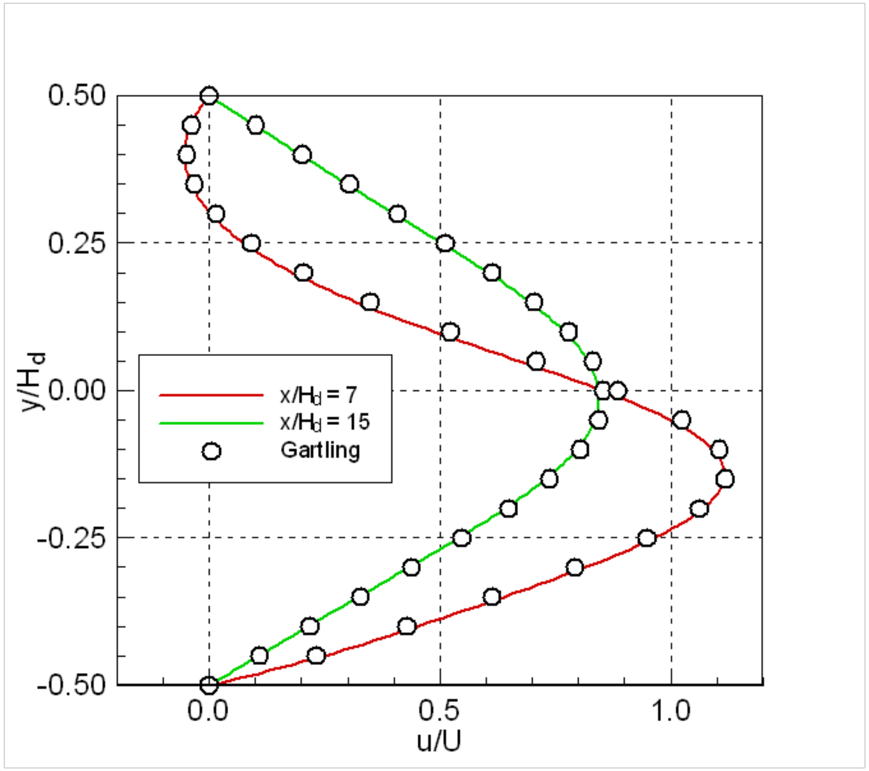
Streamlines.

Vector velocity field.

Pressure contours.

VALIDATION OF FD-LSqSEM SOLVER

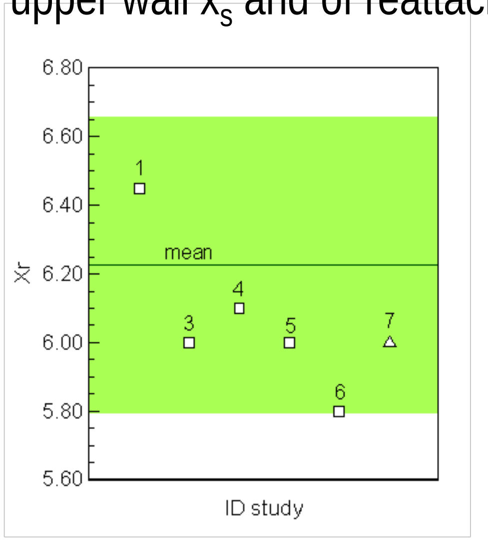
Backward-facing step

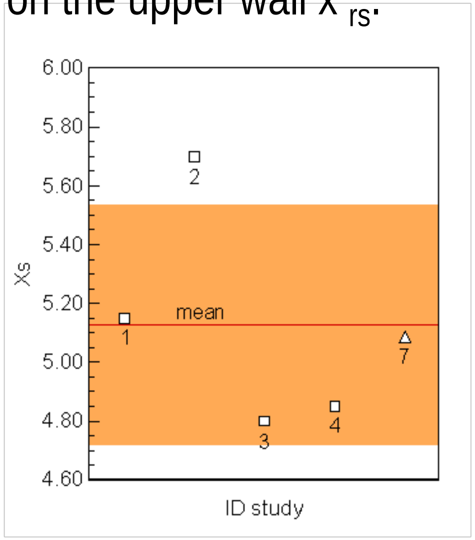


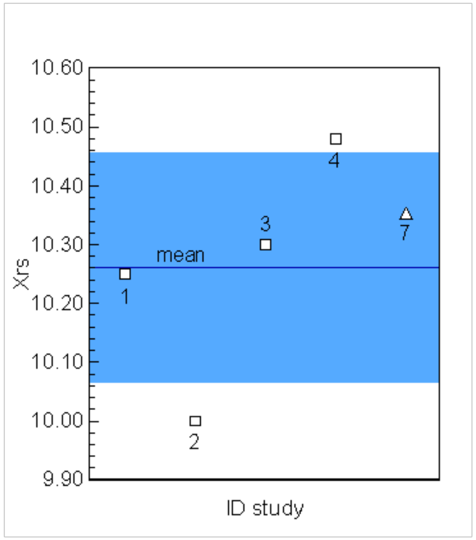
Flow over a backward-facing step at Re = 800: horizontal velocity profiles along the height of the channel at x/H_d=7 and x/H_d=15. Comparison with the benchmark solution of *Gartling* (1990).

VALIDATION OF FD-LSqSEM SOLVER

<u>Backward-facing step</u>: dimensionless length of reattachment on the lower wall x_r, of separation on the upper wall x_s and of reattachment on the upper wall x_{rs}.







ID study	References	xr	xs	xrs
1	Lee and Mateescu (1998) MHFS data	6.45	5.15	10.25
2	Armaly (1983)	7.0	5.70	10.00
3	Lee and Mateescu (1998)	6.0	4.80	10.30
4	Gartling (1990)	6.1	4.85	10.48
5	Kim and Moin (1985)	6.0	-	-
6	Sohn (1988)	5.8	-	-
7	Current	6.00	5.08	10.35

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TeCC METHOD COUPLED TO FICTITIOUS DOMAIN APPROACH

Let us consider the vorticity based Navier-Stokes equations defined in the stochastic domain $\Omega(\theta)$:

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\omega} = \mathbf{f} && \text{in } \Omega(\theta)\\
&\boldsymbol{\omega} - \nabla\times\mathbf{u} = \mathbf{0} && \text{in } \Omega(\theta)\\
&\nabla\cdot\mathbf{u} = \mathbf{0} && \text{in } \Omega(\theta)\\
&\nabla\cdot\boldsymbol{\omega} = \mathbf{0} && \text{in } \Omega(\theta)\\
&\mathbf{u} = \mathbf{u}^s && \text{on } \Gamma_u\\
&\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \mathbf{f}^s && \text{on } \Gamma_f
\end{aligned}
$$

We define the Lagrangian $L : \mathbf{X}\times M \to \Re$

$$
L(\mathbf{u}, p, \boldsymbol{\omega}, \lambda; \mathbf{f}) = \frac{1}{2}\left\|(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\omega} - \mathbf{f}\right\|^2_{0,\Pi} + \frac{1}{2}\|\boldsymbol{\omega} - \nabla\times\mathbf{u}\|^2_{0,\Pi} + \frac{1}{2}\|\nabla\cdot\mathbf{u}\|^2_{0,\Pi} + \frac{1}{2}\|\nabla\cdot\boldsymbol{\omega}\|^2_{0,\Pi} + \left\langle\lambda, (\mathbf{u} - \mathbf{u}^s)\right\rangle_{0,\Gamma_u(\theta)} + \left\langle\lambda, (\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} - \mathbf{f}^s)\right\rangle_{0,\Gamma_f(\theta)}
$$

with $\mathbf{X} = \{(\mathbf{u}, p, \boldsymbol{\omega}) \in \mathbf{H}^1(\Pi)\times H^1(\Pi)\times\mathbf{H}^1(\Pi)\}$ and $M(\theta) = \{\lambda \in H^{-1/2}(\Gamma(\theta))\}$

TeCC METHOD COUPLED TO FICTITIOUS DOMAIN APPROACH

This yields:

$$\begin{cases} \text{Find } (\mathbf{u}, p, \boldsymbol{\omega}, \lambda) \in \mathbf{X} \times M(\theta) \text{ such that} \\ a((\mathbf{u}, p, \boldsymbol{\omega}), (\mathbf{v}, q, \boldsymbol{\psi})) + b((\mathbf{v}, q, \boldsymbol{\psi}), \lambda) = l((\mathbf{v}, q, \boldsymbol{\psi})) \quad \forall (\mathbf{v}, q, \boldsymbol{\psi}) \in \mathbf{X} \\ b((\mathbf{u}, p, \boldsymbol{\omega}), \mu) = g(\mu) \quad \forall \mu \in M(\theta) \end{cases}$$

Substituting the polynomial Chaos series:

$$\mathbf{u}^*(\mathbf{x}, \theta) = \sum_{i=0}^{N} \mathbf{u}_i(\mathbf{x}) H_i(\boldsymbol{\xi}) \qquad p^*(\mathbf{x}, \theta) = \sum_{i=0}^{N} p_i(\mathbf{x}) H_i(\boldsymbol{\xi})$$

$$\boldsymbol{\omega}^*(\mathbf{x}, \theta) = \sum_{i=0}^{N} \boldsymbol{\omega}_i(\mathbf{x}) H_i(\boldsymbol{\xi}) \qquad \lambda^*(\mathbf{x}, \theta) = \sum_{i=0}^{N} \lambda_i(\mathbf{x}) H_i(\boldsymbol{\xi})$$

$$\begin{cases} \text{Find } (\mathbf{u}^*, p^*, \boldsymbol{\omega}^*, \lambda^*) \in L^2_\rho(\mathbf{I}; \mathbf{X}) \times L^2_\rho(\mathbf{I}; M^*) \text{ such that} \\ a((\mathbf{u}^*, p^*, \boldsymbol{\omega}^*), (\mathbf{v}^*, q^*, \boldsymbol{\psi}^*)) + b((\mathbf{v}^*, q^*, \boldsymbol{\psi}^*), \lambda^*) = l((\mathbf{v}^*, q^*, \boldsymbol{\psi}^*)) \quad \forall (\mathbf{v}^*, q^*, \boldsymbol{\psi}^*) \in L^2_\rho(\mathbf{I}; \mathbf{X}) \\ b((\mathbf{u}^*, p^*, \boldsymbol{\omega}^*), \mu^*) = g(\mu^*) \quad \forall \mu^* \in L^2_\rho(\mathbf{I}; M^*) \end{cases}$$

with $M^* = \{\lambda^* \in H^{-1/2}(\Gamma^*)\}$

TeCC METHOD COUPLED TO FICTITIOUS DOMAIN APPROACH

Performing Collocation projection:

$$\begin{cases} \text{Find } (\mathbf{u}, p_i, \boldsymbol{\omega}_i, \lambda_i) \in \mathbf{X} \times M_i \text{ such that} \\ a_i(\cdot,\cdot) + b_i(\cdot,\cdot) = l_i(\cdot) \quad \forall (\mathbf{v}_i, q_i, \boldsymbol{\psi}_i,) \in \mathbf{X} \quad \text{with } i = 1,\ldots,N \quad \text{where } M_i = \left\{ \lambda_i \in H^{-1/2}(\Gamma_i) \right\} \\ b_i(\cdot,\cdot) = g_i(\cdot) \quad \forall \mu_i \in M_i \end{cases}$$

To reconstruct the stochastic solution

$$E_{PC}(\mathbf{u}) = \mu_{\mathbf{u}} = \mathbf{u}_0(\mathbf{x}) \qquad Var_{PC}(\mathbf{u}) = \sigma_{\mathbf{u}}^2 = \sum_{i=1}^{N} \mathbf{u}_i^2(\mathbf{x}) \langle H_i^2 \rangle$$

$$E_{PC}(p) = \mu_p = p_0(\mathbf{x}) \qquad Var_{PC}(p) = \sigma_p^2 = \sum_{i=1}^{N} p_i^2(\mathbf{x}) \langle H_i^2 \rangle$$

$$E_{PC}(\boldsymbol{\omega}) = \mu_{\boldsymbol{\omega}} = \boldsymbol{\omega}_0(\mathbf{x}) \qquad Var_{PC}(\boldsymbol{\omega}) = \sigma_{\boldsymbol{\omega}}^2 = \sum_{i=1}^{N} \boldsymbol{\omega}_i^2(\mathbf{x}) \langle H_i^2 \rangle$$

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APPLICATION

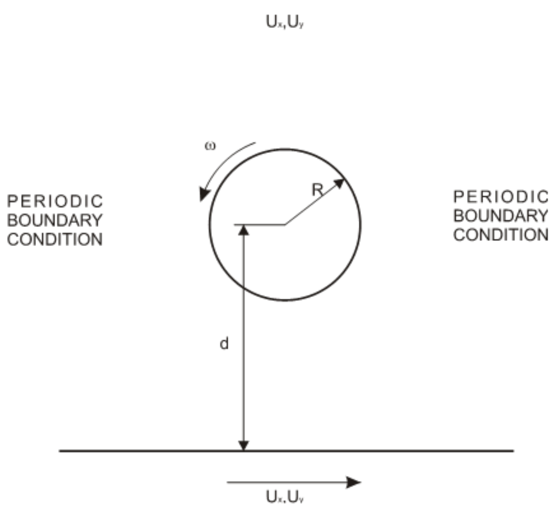
Wannier flow with geometric tolerances and uncertain boundary conditions

Flow past a 2 D spinning cylinder near a moving wall

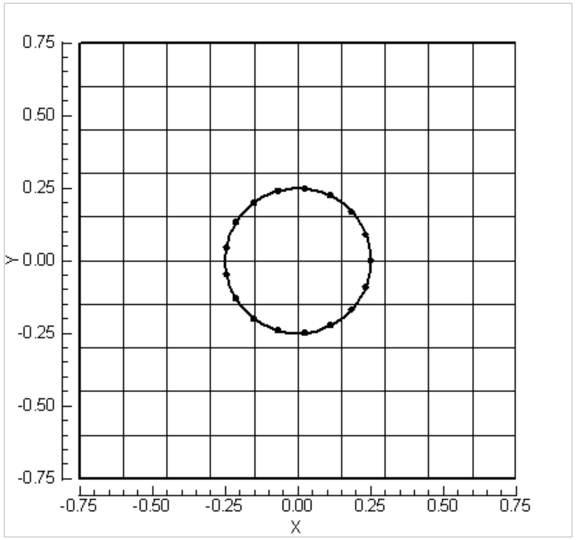
steady incompressible Stokes equations

$$\nabla p - \frac{1}{\mathrm{Re}} \nabla \cdot \left[(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \right] = \mathbf{f}$$

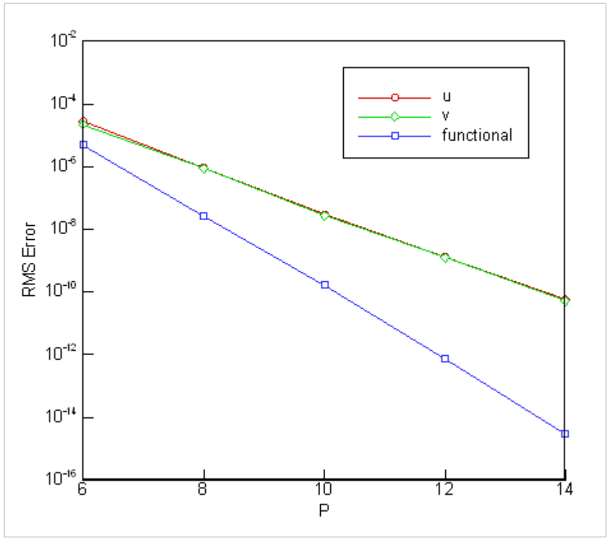
$$\nabla \cdot \mathbf{u} = 0$$



Boundary conditions.



Numerical grid used for calculations.



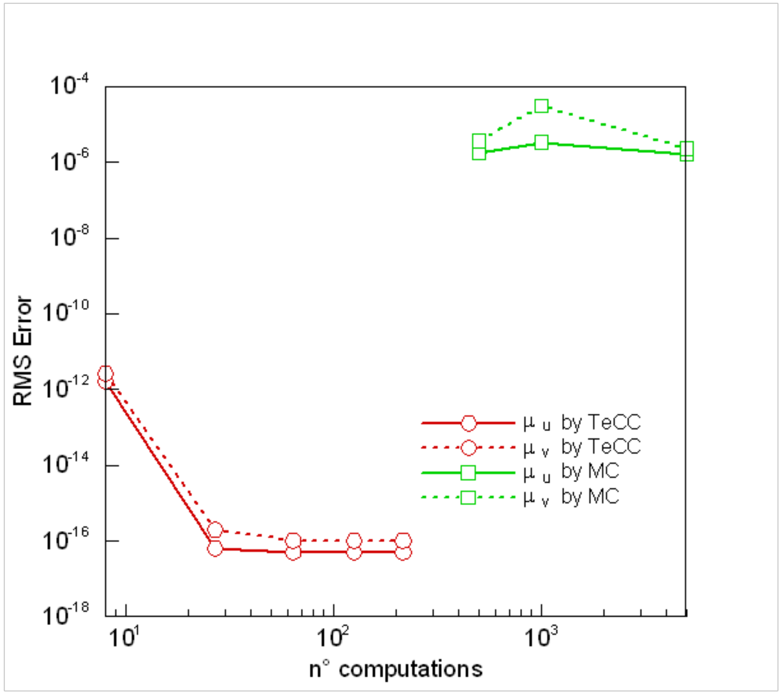
Accuracy analysis of deterministic model.

APPLICATION

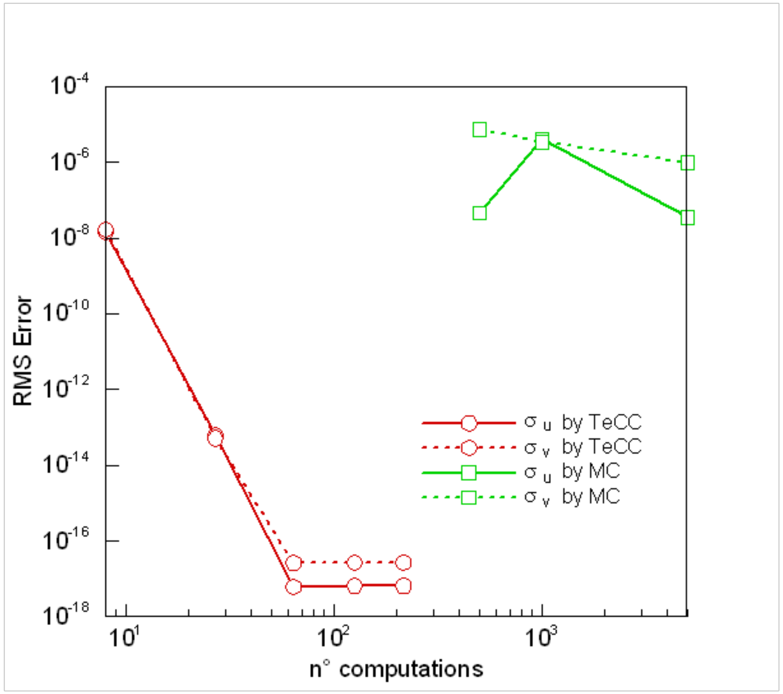
Wannier flow with geometric tolerances and uncertain boundary conditions

Flow past a 2 D spinning cylinder near a moving wall

Uncertain parameters: $R = N(0.250, 0.005)$, $U = N(1.0, 0.03)$, $d = N(0.75, 0.02)$



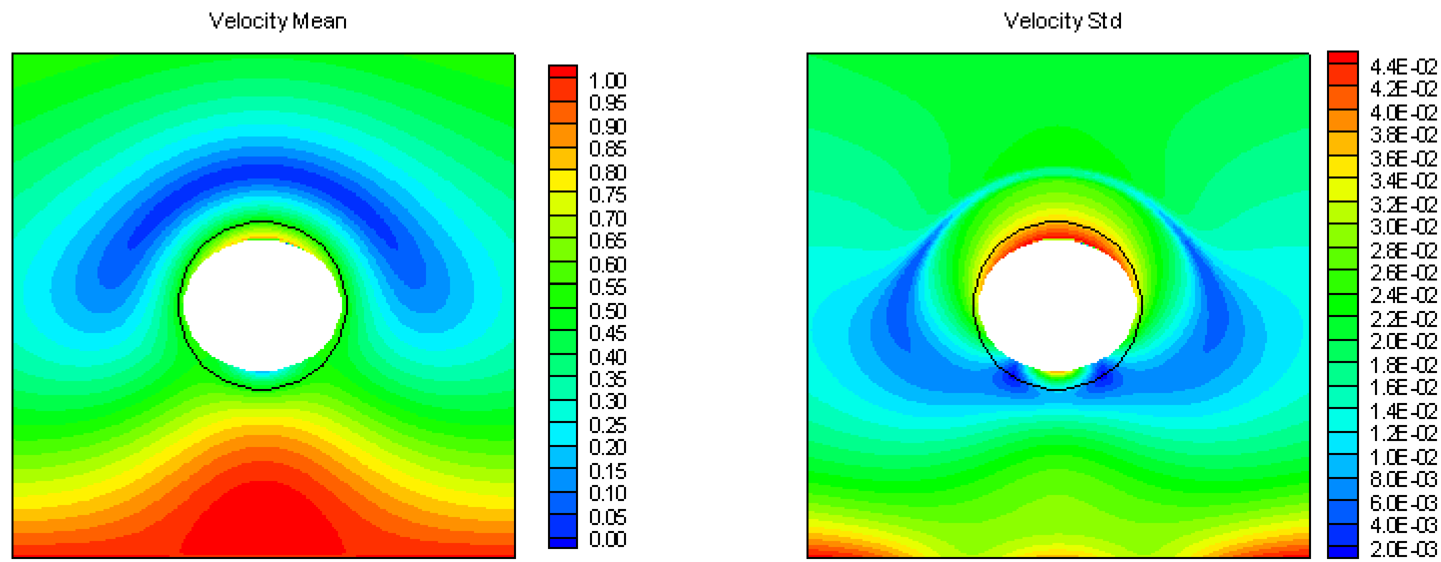
Error analysis of mean value of stochastic model.



Error analysis of standard deviation of stochastic model.

APPLICATION

Wannier flow with geometric tolerances and uncertain boundary conditions



Contours of the velocity mean and contours of the velocity standard deviation. The fields are obtained by TeCC method with expansion polynomial order 3 coupled to the FD-LSqSEM solver.

	μ_u	σ_u	μ_v	σ_v
RMSE	$3.03\cdot10^{-13}$	$1.33\cdot10^{-13}$	$3.40\cdot10^{-14}$	$2.64\cdot10^{-14}$

Stochastic Wannier flow: RMS Error of the mean and standard deviation of u and v velocity components obtained by TeCC method with expansion polynomial order 3 coupled to the FD-LSqSEM solver, respect to the analytical solutions.

APPLICATION

<u>Wannier flow with geometric tolerances and uncertain boundary conditions</u>

SENSITIVITY ANALYSIS

The first-order approximation of velocity module V using Taylor series is:

$$V(\mathbf{x},R,U,d) = V(\mathbf{x},R^0,U^0,d^0) +$$

$$\left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial R}\right]_{(R^0,U^0,d^0)}\left(R-R^0\right)+\left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial U}\right]_{(R^0,U^0,d^0)}\left(U-U^0\right)+\left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial d}\right]_{(R^0,U^0,d^0)}\left(d-d^0\right)$$

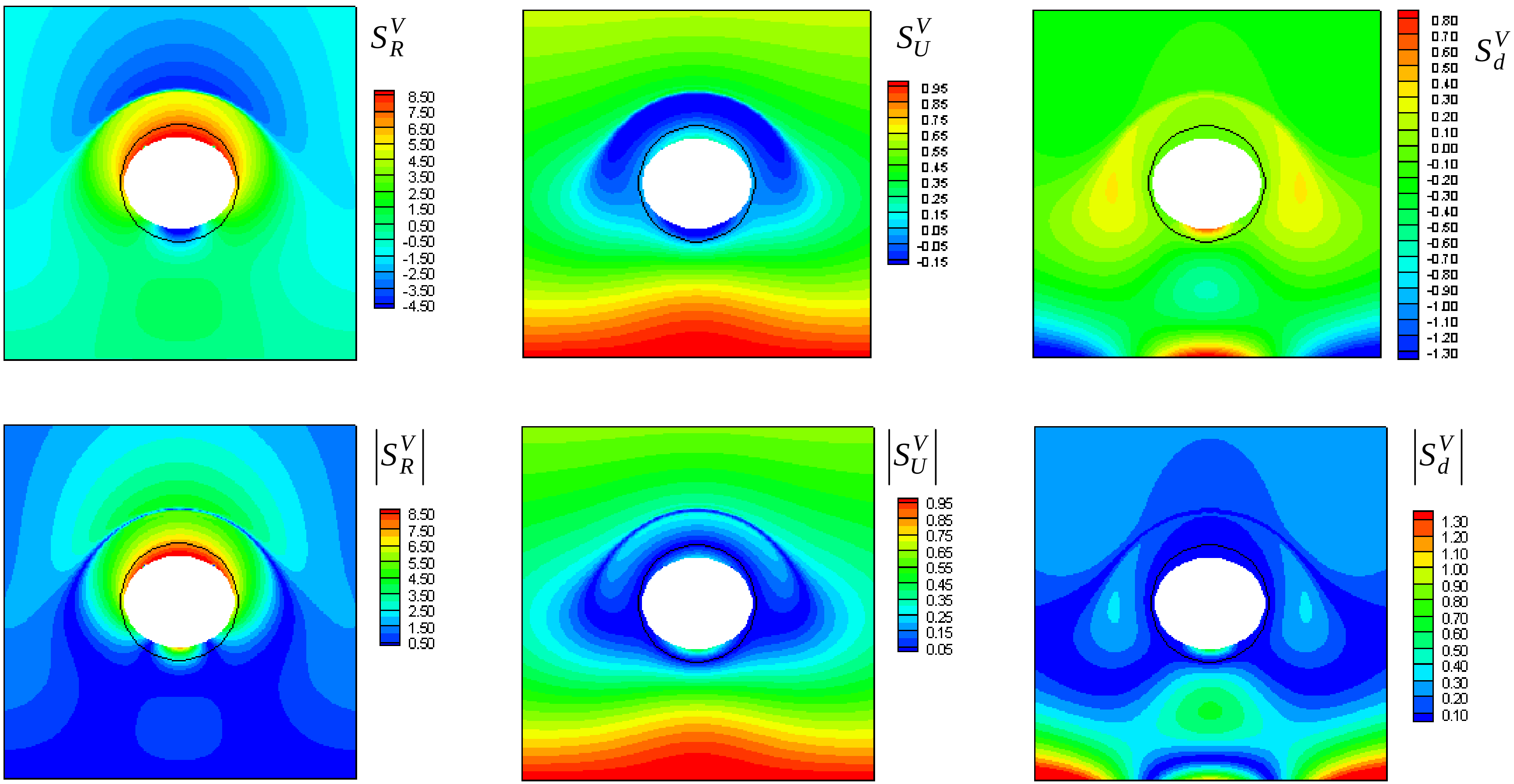
$$S_R^V = \left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial R}\right]_{(R^0,U^0,d)^0} \qquad S_U^V = \left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial U}\right]_{(R^0,U^0,d^0)} \qquad S_d^V = \left[\frac{\partial V(\mathbf{x},R,U,d)}{\partial d}\right]_{(R^0,U^0,d^0)}$$

are the **first-order sensitivity coefficients** of V to parameters variation

with (R^0, U^0, d^0) the nominal parameters value for which the sensitivity analysis is performed

APPLICATION

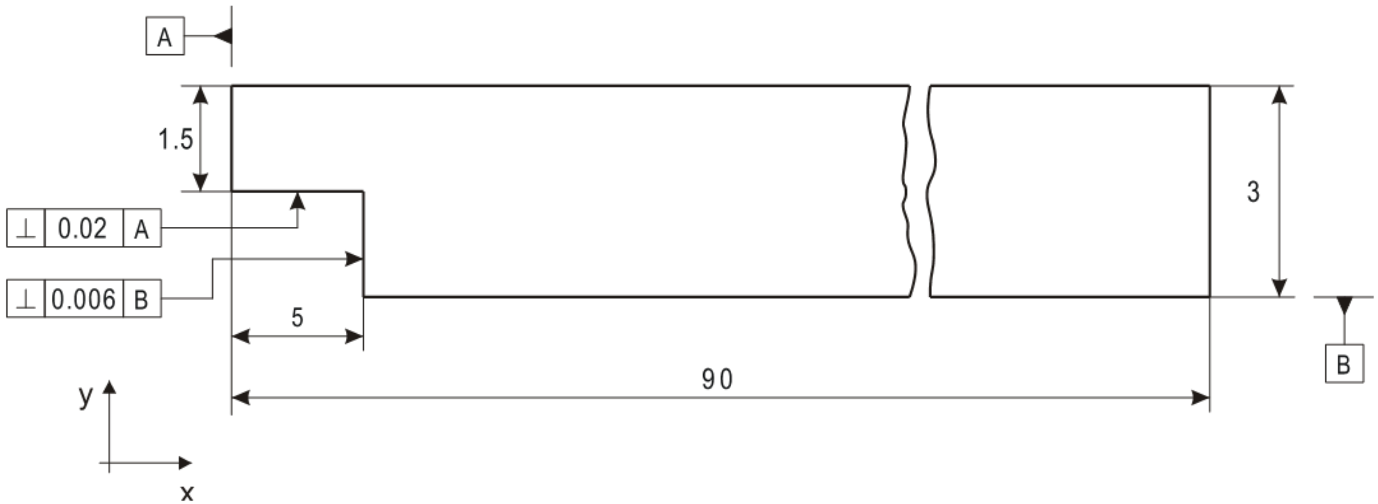
<u>Wannier flow with geometric tolerances and uncertain boundary conditions</u>



Stochastic Wannier flow: value and absolute value of the first-order sensitivity analysis coefficient of velocity respect to R, U and d. The coefficients are computed in the mean values of the uncertain parameters.

APPLICATION

Backward-facing step with geometric tolerances of perpendicularity on the step

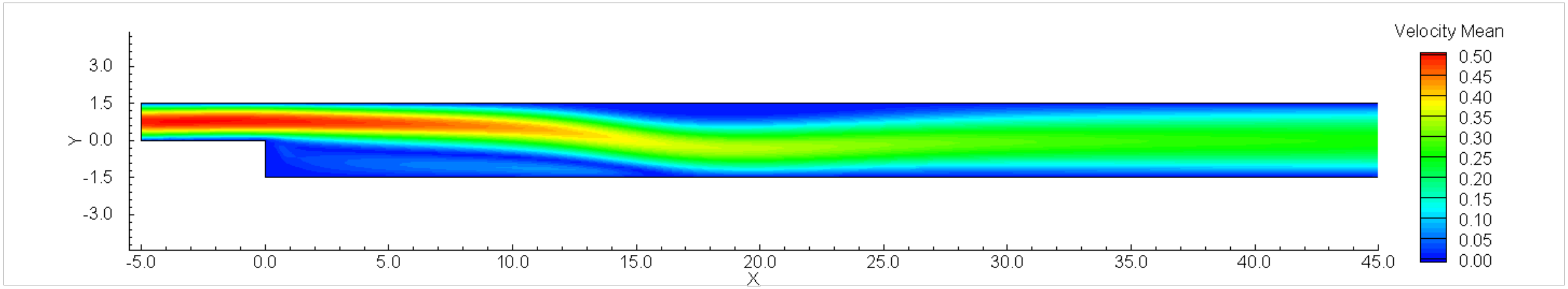


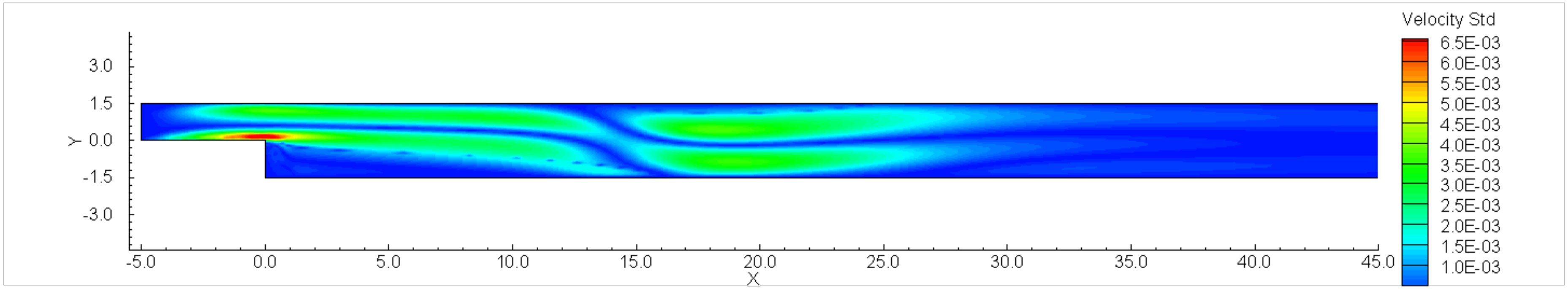
Stochastic flow over a backward-facing step at Re = 600: geometric uncertainties.

APPLICATION

<u>Backward-facing step with geometric tolerances of perpendicularity on the step</u>

the mean field and the standard deviation field of the vector velocity

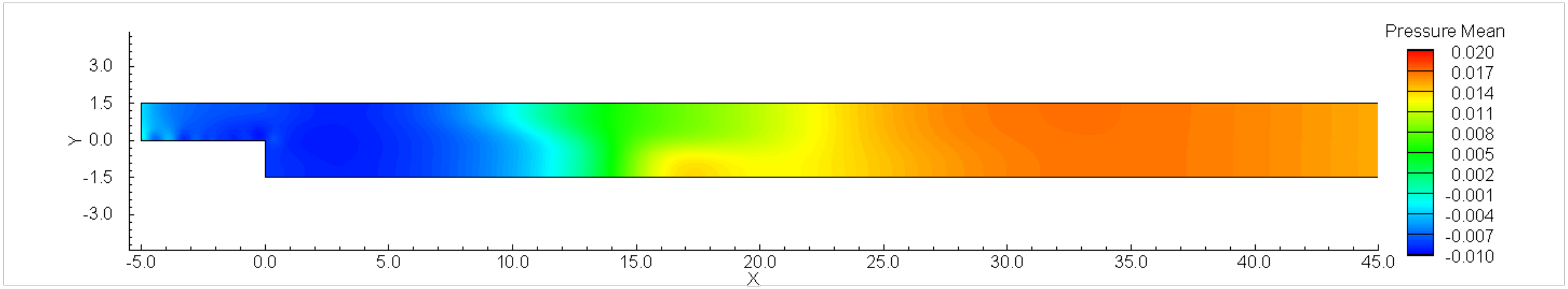


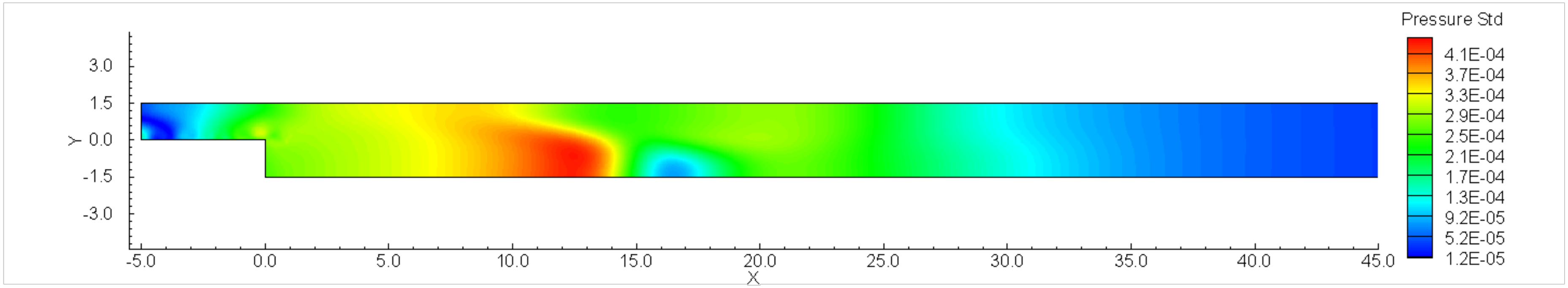


APPLICATION

<u>Backward-facing step with geometric tolerances of perpendicularity on the step</u>

the mean field and the standard deviation field of the pressure

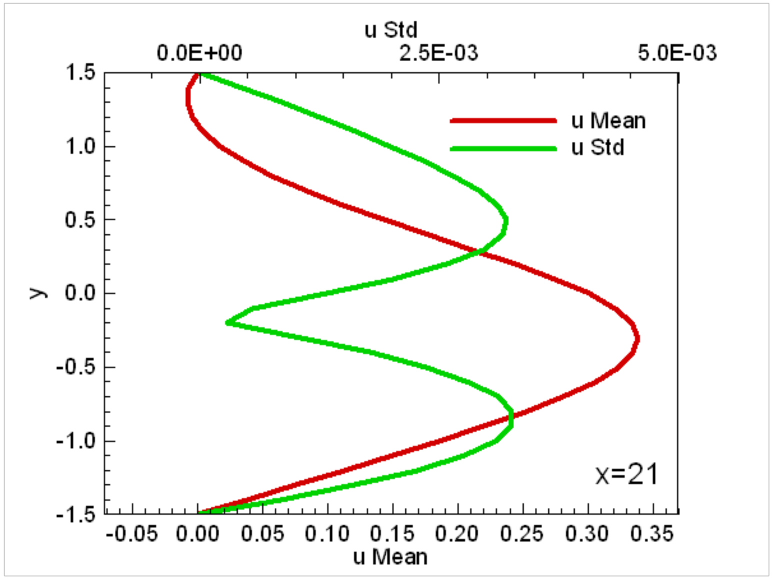


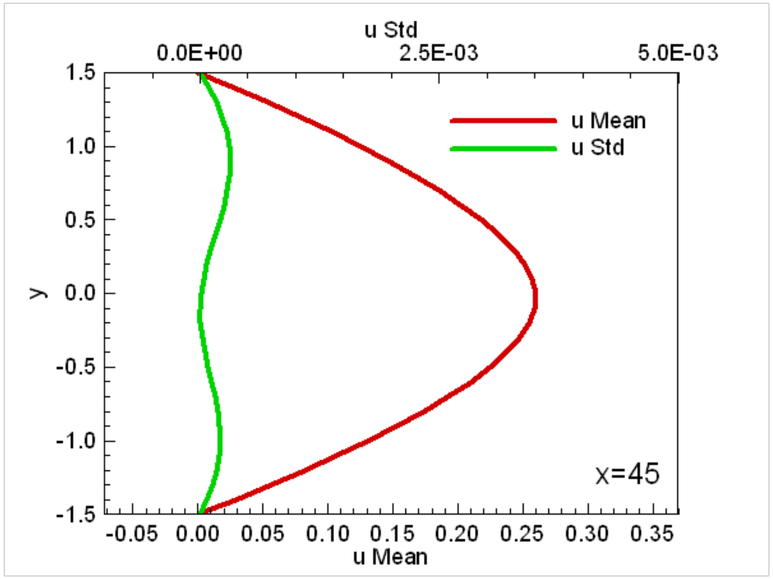


APPLICATION

<u>Backward-facing step with geometric tolerances of perpendicularity on the step</u>

horizontal velocity profiles along the height of the channel at x = 21 and x = 45





APPLICATION

Backward-facing step with geometric tolerances of perpendicularity on the step

SENSITIVITY ANALYSIS

The first-order approximation of variables *u* and *p* using Taylor series is:

$$u(\mathbf{x},\theta_1,\theta_2) = u(\mathbf{x},\theta_1^0,\theta_2^0) + \left[\frac{\partial u(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_1}\right]_{(\theta_1^0,\theta_2^0)} \left(\theta_1 - \theta_1^0\right) + \left[\frac{\partial u(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_2}\right]_{(\theta_1^0,\theta_2^0)} \left(\theta_2 - \theta_2^0\right)$$

$$p(\mathbf{x},\theta_1,\theta_2) = p(\mathbf{x},\theta_1^0,\theta_2^0) + \left[\frac{\partial p(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_1}\right]_{(\theta_1^0,\theta_2^0)} \left(\theta_1 - \theta_1^0\right) + \left[\frac{\partial p(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_2}\right]_{(\theta_1^0,\theta_2^0)} \left(\theta_2 - \theta_2^0\right)$$

$$S_1^u = \left[\frac{\partial u(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_1}\right]_{(\theta_1^0,\theta_2^0)} \quad S_2^u = \left[\frac{\partial u(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_2}\right]_{(\theta_1^0,\theta_2^0)} \quad S_1^p = \left[\frac{\partial p(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_1}\right]_{(\theta_1^0,\theta_2^0)} \quad S_2^p = \left[\frac{\partial p(\mathbf{x},\theta_1,\theta_2)}{\partial \theta_2}\right]_{(\theta_1^0,\theta_2^0)}$$

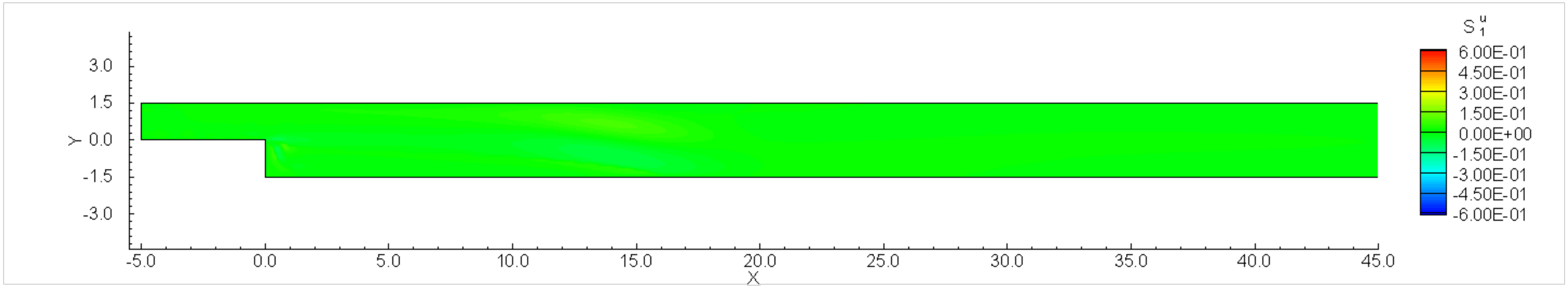
are the **first-order sensitivity coefficients** of *u* and *p* to parameters variation

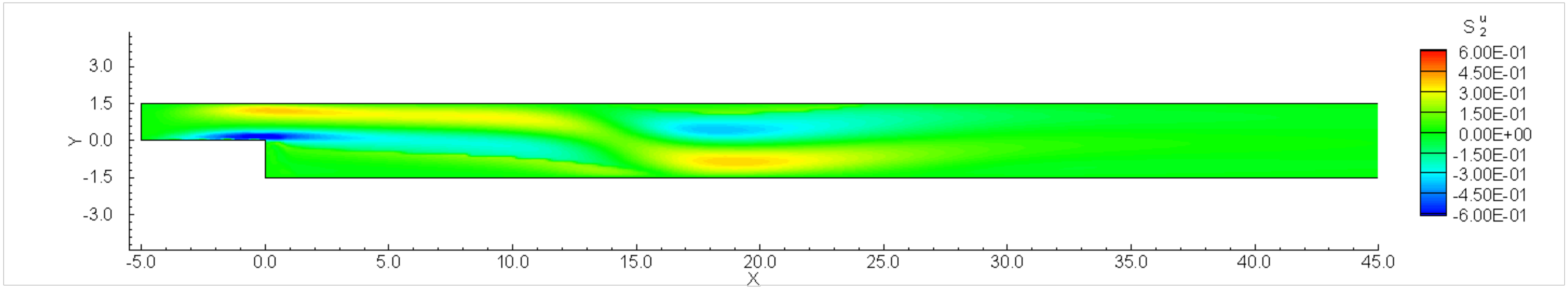
with (θ_1^0, θ_2^0) the nominal parameters value for which the sensitivity analysis is performed

APPLICATION

<u>Backward-facing step with geometric tolerances of perpendicularity on the step</u>

first-order sensitivity coefficients of *u*

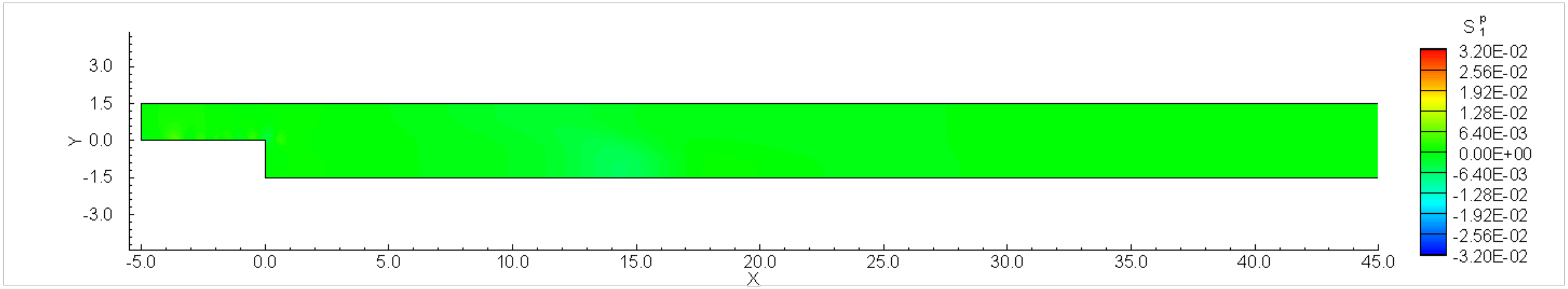


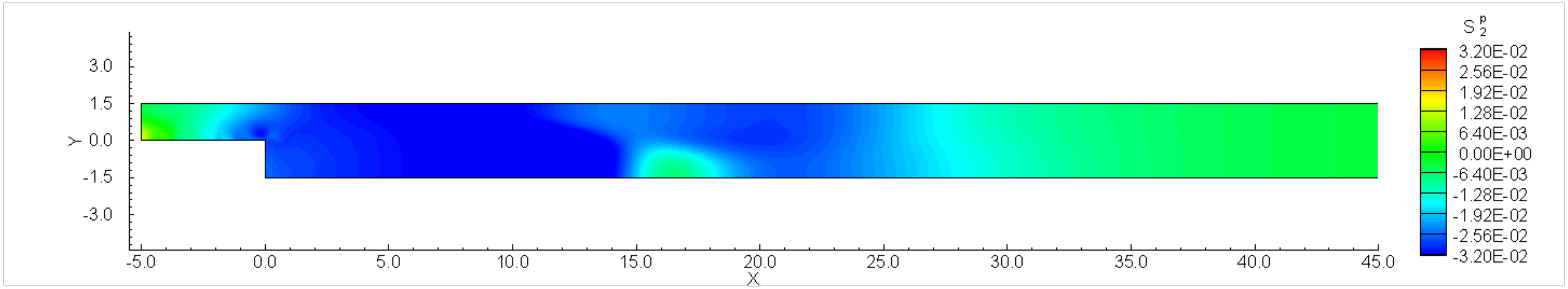


APPLICATION

<u>Backward-facing step with geometric tolerances of perpendicularity on the step</u>

first-order sensitivity coefficients of *p*





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REMARKS AND CONCLUSIONS

The **Tensorial-expanded Chaos Collocation** methodology coupled to a **Fictitious Domain** solver has been presented, in order to solve fluid dynamic problems with multi geometric uncertainties.

This formulation is of particular interest to study problems defined on stochastic domain, since the Fictitious Domain approach allows avoiding the remeshing of computational domain in the presence of geometric uncertainties.

The approach, which is used for the solution of problems with geometric tolerances, is a novelty in Fluid Dynamics and it promises to have interesting applications in the future.

ACKNOWLEDGMENTS

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