Trace Functionals for a Class of Pseudo-differential Operators in $\mathbb{R}^n$

Fabio Nicola
Dipartimento di Matematica
via Carlo Alberto 10,
10123 Torino, Italy.
E-mail nicola@dm.unito.it, telephone number 0116702827, fax number 0116702878

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Abstract. In this paper we define trace functionals on the algebra of pseudo-differential operators with cone-shaped exits to infinity. Furthermore, we improve the Weyl formula on the asymptotic distribution of eigenvalues and make use of it in order to establish inclusion relations between the interpolation normed ideals of compact operators in $L^2(\mathbb{R}^n)$ and the above operator classes.

Keywords: Dixmier trace, noncommutative residue, pseudo-differential operators, trace functionals, Weyl formula

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1. Introduction

In this paper we study pseudo-differential operators in $\mathbb{R}^n$ with symbols satisfying estimates of product type. The basic ideas of this calculus go back to Shubin [29], Parenti [22], Feygin [10], Grushin [13], Cordes [5],[6], Schrohe [23], and its properties follow from the general Weyl calculus of Hörmander, see [15], Chapter XVIII. In fact, the corresponding symbol classes $S^{\mu,\rho}$ are just the classes $S(m,g)$ with weight function $m(x,\xi) = \langle x\rangle^{\rho}\langle\xi\rangle^{\mu}$ and slowly varying metric

$$g_{x,\xi} = \frac{|dx|^2}{1+|x|^2} + \frac{|d\xi|^2}{1+|\xi|^2}. \tag{1.1}$$

These operator classes play an important role in Scattering Theory, as the resolvent of the Laplacian can be viewed as a holomorphic family taking values in the space $L^{-2,0}$ ($L^{\mu,\rho} = \mathrm{Op}(S^{\mu,\rho})$). Actually, here as in other applications it is mostly the subalgebra $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ of operators which are classical both in x and ξ (see Definition 2.1, below) which arises, cf. the recent book of Schulze [27].

We observe that the corresponding calculus can be easily transferred to non-compact manifolds with cone-shaped exits to infinity, i.e. defined in terms of changes of variables which are classical symbols in x; we keep however the $\mathbb{R}^n$ frame in the following, for simplicity.

The first problem under investigation in the present paper is the existence of trace functionals (i.e. functionals which vanish on commutators) on the algebra $\bigcup_{\mu,\rho\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ of all operators of integer order. This problem was studied in Wodzicki [31],[32] for classical operators on compact manifolds and extended to several operator algebras by Guillemin [12], Fedosov, Golse, Leichtnam and Schrohe, [8],[9], Melrose [19], Melrose and Nistor [20], Schrohe [24],[25],[26], Boggiatto and Nicola [2].

Following ideas of Melrose and Nistor [20] we start from the usual trace functional

$$\mathrm{Tr}(a^{\mathrm{w}}) = \iint a(x,\xi)\, dx\, d\xi, \tag{1.2}$$

defined on trace class operators in $L^2(\mathbb{R}^n)$ and extend it using holomorphic families; here a^{w} denotes the pseudo-differential operator with Weyl symbol a. Precisely, if in (1.2) we replace a by any holomorphic family $a(\tau,z)\in S^{\mu+z,\rho+\tau}_{\mathrm{cl}(\xi,x)}$ such that $a(0,0)=a$, we see that $\mathrm{Tr}(a(\tau,z)^{\mathrm{w}})$ is defined and holomorphic for $\Re z<-\mu-n, \Re\tau<-\rho-n$, and extends to a meromorphic function of τ,z with at most simple poles on the surfaces $z=-\mu-n+j,\ \tau=-\rho-n+k,\ j,k\in\mathbb{N}$. In a neighborhood of $(0,0)\in\mathbb{C}\times\mathbb{C}$ we shall have

$$\mathrm{Tr}(a(\tau,z)^{\mathrm{w}}) = \frac{1}{\tau z}\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) - \frac{1}{z}\widehat{\mathrm{Tr}}_{\psi}(a^{\mathrm{w}}) - \frac{1}{\tau}\widehat{\mathrm{Tr}}_{e}(a^{\mathrm{w}}) + \sum_{h+i\geq 2} c_{h,i}\tau^{h-1}z^{i-1},$$

defining in this way the functionals $\mathrm{Tr}_{\psi,e}, \widehat{\mathrm{Tr}}_{\psi}, \widehat{\mathrm{Tr}}_{e}$. We shall determine an explicit expression for each of these functionals and shall see that $\mathrm{Tr}_{\psi,e}$ is a trace on the algebra $\bigcup_{\mu,\rho\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, whereas the restrictions Tr_{ψ} and Tr_{e} of $\widehat{\mathrm{Tr}}_{\psi}$ and $\widehat{\mathrm{Tr}}_{e}$ to $\bigcup_{\mu\in\mathbb{Z}}\bigcap_{\rho\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ and $\bigcup_{\rho\in\mathbb{Z}}\bigcap_{\mu\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ respectively, define traces on these algebras. The uniqueness of the traces involved is investigated for quotient algebras.

In Section 4 we study inclusion relations between the classes $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, where $\mu<0,\rho<0$, and the interpolation normed ideals $\mathcal{L}^{(p,\infty)}(L^2(\mathbb{R}^n))$, $1\leq p<\infty$, (cf. Gohberg and Krein [11], Connes [4]). In fact the spaces $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, under our assumption on μ and ρ, are all contained in the ideal $\mathcal{K}(L^2(\mathbb{R}^n))$ of compact operators in $L^2(\mathbb{R}^n)$, obtaining, in particular, trace class operators when $\mu<-n,\rho<-n$. On the other hand, the spaces $\mathcal{L}^{(p,\infty)}(L^2(\mathbb{R}^n))$, defined as the sets of compact operators with an eigenvalue sequence $\mu_k(|T|)=O\left(k^{-1/p}\right)$, also are contained in $\mathcal{K}(L^2(\mathbb{R}^n))$ (more precisely they define a filtration of $\mathcal{K}(L^2(\mathbb{R}^n))$) and in turn contain the ideal $\mathcal{B}_1(L^2(\mathbb{R}^n))$ of trace class operators.

In order to establish such inclusion relations we investigate the asymptotic bahaviour of the spectrum of elliptic self-adjoint operators of positive order. We make use of the results of Hörmander [14] on the spectrum of operators with positive Weyl symbol $p \in S(p,g)$, to improve the Weyl formula of Maniccia and Panarese [18], obtaining better estimates of the remainder. We point out that in the case $\mu = \rho$ in the Weyl formula for the counting function $N(\lambda)$ a factor $\log \lambda$ appears (cf. Nilsson [21], and Boggiatto, Buzano and Rodino [1], Section 2.7 for the case of ordinary differential operators). This phenomenon leads us to introduce new normed ideals, denoted by $\mathcal{L}^{(p,\infty)}_{\log}(L^2(\mathbb{R}^n)), 1 \le p < \infty$, see Definition 4.2.

The limit cases of the normed ideals $\mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^n))$ and $\mathcal{L}^{(1,\infty)}_{\log}(L^2(\mathbb{R}^n))$ are particularly important, as they correspond to domains of two Dixmier traces (cf. [7]).

Dixmier traces, in a broad sense, are defined taking a class of compact operators for which the usual trace diverges at a given (suitable) rate. Then, with any such operator it is associated, via a normalizing sequence, a bounded sequence, and 2-dilatation invariant states in $l^\infty(\mathbb{N})$ provide (non normal) traces. Informally, if $\alpha = (\alpha_N)$ is a convenient sequence, we define

$$\mathrm{Tr}_{\alpha,\omega} T = \lim_{\omega} \frac{1}{\alpha_N} \sum_{k=0}^{N} \mu_k(T) = \text{Dixmier trace for } T \ge 0,\ \mu_k(T) \searrow,$$

see Section 4 for the precise definition.

We emphasize that, in a strict sense, by Dixmier trace is generally meant a trace which "sums" logarithmic divergences, i.e. whose domain is the ideal $\mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^n))$; however we shall consider the above more general construction, since for instance, we shall also need to consider the trace associated with the sequence $\alpha_N = (\log N)^2$ and therefore with domain $\mathcal{L}^{(1,\infty)}_{\log}(L^2(\mathbb{R}^n))$. Indeed, we shall prove the following theorem.

Theorem 1.1. *We have*

$$\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) = 2n^2 \mathrm{Tr}_{\alpha,\omega}(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,-n}_{\mathrm{cl}(\xi,x)}, \tag{1.3}$$

$$\mathrm{Tr}_{\psi}(a^{\mathrm{w}}) = n \mathrm{Tr}_{\alpha,\omega}(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,\rho}_{\mathrm{cl}(\xi,x)} \text{ with } \rho \in \mathbb{Z}, \rho < -n, \tag{1.4}$$

$$\mathrm{Tr}_{e}(a^{\mathrm{w}}) = n \mathrm{Tr}_{\alpha,\omega}(a^{\mathrm{w}}) \quad \text{for } a \in S^{\mu,-n}_{\mathrm{cl}(\xi,x)} \text{ with } \mu \in \mathbb{Z}, \mu < -n, \tag{1.5}$$

independently of ω, *where* $\alpha_N = (\log N)^2$ *in* (1.3), *whereas* $\alpha_N = \log N$ *in* (1.4) *and* (1.5).

It is well known that the analogous result for classical operators on compact manifolds is due to Connes [3].

After completing the present paper, we were acquainted of the recent contribution of Lauter and Moroianu [17], which obtain results corresponding to those put forward in Section 3 in the frame of the double-edge pseudo-differential calculus on manifolds with fibered boundaries.

2. Basic calculus and holomorphic families

We recall in short the definitions of some symbol classes and the basic properties of the corresponding operators; we follow Schulze [27] in notations and terminology.

The symbol classes $S^{\mu,\rho} := S^{\mu,\rho}(\mathbb{R}^n \times \mathbb{R}^n)$, $\mu, \rho \in \mathbb{R}$ are defined by the following inequalities:

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{\mu-|\beta|}(1+|x|)^{\rho-|\alpha|}$$

for all $x, \xi \in \mathbb{R}$. We denote by $L^{\mu,\rho}$ the space of the corresponding pseudo-differential operators. If a is a symbol in $S^{\mu,\rho}$ we shall choose the notation a^{w} for the pseudo-differential operator with Weyl symbol a.

In order to consider classical symbols, we need to introduce some function spaces.

We denote by $S_\xi^{(\mu)}$ ($S_\xi^{[\mu]}$) the space of all functions $a \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ (resp. in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$) homogeneous of degree μ with respect to ξ (resp. homogeneous for $|\xi| \geq 1$); at the same way we define the spaces $S_x^{(\rho)}$ and $S_x^{[\rho]}$. Then we set $S^{\mu,[\rho]} := S^{\mu,\rho} \cap S_x^{[\rho]}$ and $S^{[\mu],\rho} := S^{\mu,\rho} \cap S_\xi^{[\mu]}$, and we define $S_{\mathrm{cl}(\xi)}^{\mu,[\rho]} \subset S^{\mu,[\rho]}$ as the space of all $a(x,\xi) \in S^{\mu,[\rho]}$ such that there are elements $a_j \in S_\xi^{[\mu-j]} \cap S_x^{[\rho]}$, $j \in \mathbb{N}$, with

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-(N+1),\rho}$$

for every $N \in \mathbb{N}$. Analogously we obtain the spaces $S_{\mathrm{cl}(x)}^{[\mu],\rho}$.

Definition 2.1. *A symbol $a \in S^{\mu,\rho}(\mathbb{R}^n \times \mathbb{R}^n)$ is called classical in x and ξ if it has the following properties:*

(i) *there are symbols $a_j \in S_{\mathrm{cl}(x)}^{[\mu-j],\rho}$, $j \in \mathbb{N}$, such that*

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-(N+1),\rho}$$

for every $N \in \mathbb{N}$.

(ii) *there are symbols* $b_k \in S^{\mu,[\rho-k]}_{\mathrm{cl}(\xi)}$, $k \in \mathbb{N}$, *such that*

$$a(x,\xi) - \sum_{k=0}^{N} b_k(x,\xi) \in S^{\mu,\rho-(N+1)}$$

for every $N \in \mathbb{N}$.

We shall denote by $S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ *the space of classical symbols in* x, ξ *and by* $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ *the space of the corresponding pseudo-differential operators.*

For instance, the symbol $a(x,\xi) = (1+|x|^2)^{\rho/2}(1+|\xi|^2)^{\mu/2}$ is in $S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$.

As in the standard calculus one sees that every classical symbol a determines its asymptotic expansions in homogeneous terms in a unique way, i.e. there are unique mappings

$$\sigma^{\mu-j}_{\psi} : S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to S^{(\mu-j)}_{\xi} \tag{2.1}$$

$$\sigma^{\rho-k}_{e} : S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to S^{(\rho-k)}_{x} \tag{2.2}$$

for all $j, k \in \mathbb{N}$. Furthermore, one can also consider the homogeneous component of degree $\rho - k$ (with respect to x) of $\sigma^{\mu-j}_{\psi}(a)$ (which in turn coincides with the homogeneous component of degree $\mu - j$ of $\sigma^{\rho-k}_{e}(a)$); in this way one obtains unique mappings

$$\sigma^{\mu-j,\rho-k}_{\psi,e} : S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to S^{(\mu-j)}_{\xi} \cap S^{(\rho-k)}_{x} \tag{2.3}$$

for all $k, j \in \mathbb{N}$.

Definition 2.2. *A symbol* $a(x,\xi) \in S^{\mu,\rho}(\mathbb{R}^n \times \mathbb{R}^n)$ *is called elliptic if for suitable constants* $C, R > 0$ *it satisfies the following inequality:*

$$|a(x,\xi)| \geq C(1+|x|)^{\rho}(1+|\xi|)^{\mu} \tag{2.4}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, *with* $|x| + |\xi| \geq R$.

For a classical symbol $a \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ the ellipticity condition (2.4) is equivalent to require that the principal symbols $\sigma^{\mu}_{\psi}(a), \sigma^{\rho}_{e}(a), \sigma^{\mu,\rho}_{\psi,e}(a)$ do not vanish on their definition domains.

It is also useful to define a scale of weighted Sobolev spaces as follows.

Definition 2.3. *Let* $s, r \in \mathbb{R}$; *we set* $H^{s,\delta}(\mathbb{R}^n) = \{\langle x\rangle^{-\delta}u; u \in H^s(\mathbb{R}^n)\}$.

Proposition 2.4. *Let* $A \in L^{\mu,\rho}(\mathbb{R}^n)$. *Then* A *induces a continuous map* $H^{s,\delta}(\mathbb{R}^n) \to H^{s-\mu,\delta-\rho}(\mathbb{R}^n)$.

For the proof of Proposition 2.4 see for example Schulze [27].

In the following we shall be interested in the following operator algebras, constructed beginning from $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$.

Definition 2.5. *Let $\mathcal{I} := \mathrm{Op}(\mathcal{S}(\mathbb{R}^{2n}))$ and $\mathcal{A} := \cup_{\rho\in\mathbb{Z}} \cup_{\mu\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}/\mathcal{I}$. We define the two-sided ideals of $\mathcal{A}$*

$$\mathcal{I}_\psi = \bigcup_{\mu\in\mathbb{Z}} \bigcap_{\rho\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}/\mathcal{I}, \qquad \mathcal{I}_e = \bigcup_{\rho\in\mathbb{Z}} \bigcap_{\mu\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}/\mathcal{I}$$

and the quotient algebras

$$\mathcal{A}_\psi = \mathcal{A}/\mathcal{I}_e, \qquad \mathcal{A}_e = \mathcal{A}/\mathcal{I}_\psi, \qquad \mathcal{A}_{\psi,e} = \mathcal{A}/(\mathcal{I}_\psi + \mathcal{I}_e).$$

In the next section we use holomorphic families of classical operators, according to the following definition, cf. Melrose and Nistor [20].

Definition 2.6. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open subsets of the complex plane and let $h_1 : \Omega_1 \to \mathbb{C}, h_2 : \Omega_2 \to \mathbb{C}$ be holomorphic functions. We call holomorphic (symbol) families of order $(h_1(z), h_2(\tau))$ functions of the particular product type*

$$\Omega_1 \times \Omega_2 \ni (\tau, z) \mapsto a(\tau, z) = [x]^{h_2(\tau)}[\xi]^{h_1(z)} b(\tau, z, x, \xi) \in S^{\Re h_1(z),\Re h_2(\tau)}_{\mathrm{cl}(\xi,x)}$$

where $[\cdot]$ denotes an arbitrary strictly positive C^∞ function on $\mathbb{R}^n$ with $[y] = |y|$ for $|y| \geq 1$ and $b(\tau, z, x, \xi) \in S^{0,0}_{\mathrm{cl}(\xi,x)}$ is holomorphic as function of (τ, z) for every fixed x, ξ.

We denote by $FS^{h_1(z),h_2(\tau)}_{\mathrm{cl}(\xi,x)}$ the space of these holomorphic families.

Remark 2.7. Given $\tilde{a}(x, \xi) \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, there exist always a holomorphic family $a(\tau, z) \in S^{\mu+z,\rho+\tau}_{\mathrm{cl}(\xi,x)}$ with $a(0, 0) = \tilde{a}$; it suffices to choose in the previous definition $a(\tau, z, x, \xi) = [x]^\tau[\xi]^z \tilde{a}(x, \xi)$.

3. Trace functionals

The main result of this section is the explicit construction of trace functionals for each of the algebras in Definition 2.5. These traces come from residues of the trace of holomorphic operator families, according to ideas of Wodzicki [31].

Firstly we recall that on the ideal of regularizing operators every trace is a multiple of the functional

$$\mathrm{Tr}(a^{\mathrm{w}}) = (2\pi)^{-n} \iint a(x, \xi)\, dx\, d\xi, \tag{3.1}$$

i.e. the usual operator trace. The integral (3.1) extends by continuity to $a \in S^{\mu,\rho}$ provided $\mu < -n, \rho < -n$. In order to extend it further we need to regularize the resultant divergent integral; we do this using holomorphic families (cf. Definition 2.6).

Lemma 3.1. *If* $a(\tau, z) \in FS^{\mu+z,\rho+\tau}_{\mathrm{cl}(\xi,x)}$ *is a holomorphic symbol family, then the function* $t(\tau, z) := \mathrm{Tr}(a(\tau, z)^{\mathrm{w}})$ *is defined and holomorphic for* $\Re z < -\mu - n, \Re\tau < -\rho - n$, *and extends to a meromorphic function of* τ, z *with at most simple poles on the surfaces* $z = -\mu - n + j,\ \tau = -\rho - n + k,\ j, k \in \mathbb{N}$

Proof. We can write $a(\tau, z, x, \xi) = [x]^\tau [\xi]^z a'(\tau, z, x, \xi)$, where $a'(\tau, z, x, \xi) \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ is holomorphic with respect to τ, z. So we have

$$t(\tau, z) = (2\pi)^{-n} \iint [x]^\tau [\xi]^z a'(\tau, z, x, \xi)\, dx\, d\xi.$$

Now we write $t(\tau, z) = t_1(\tau, z) + t_2(\tau, z) + t_3(\tau, z) + t_4(\tau, z)$ where t_1, t_2, t_3, t_4 are the integrals respectively on $A_1 = \{|x| \le \epsilon, |\xi| \le 1\},\ A_2 = \{|x| \le \epsilon, |\xi| \ge 1\},\ A_3 = \{|x| \ge \epsilon, |\xi| \le 1\},\ A_4 = \{|x| \ge \epsilon, |\xi| \ge 1\}$. To prove Lemma 3.1 it would suffice to set $\epsilon = 1$, but in view of future developments it is useful to work with an arbitrary $\epsilon \ge 1$.

Clearly $t_1(\tau, z)$ is an entire function.

As t_2 is concerned, we note that for $|\xi| \ge 1$ and every $p \in \mathbb{N}, p \ge 1$, we have

$$a(x, \xi) = \sum_{j=0}^{p-1} \sigma_\psi^{\mu-j}(a(\tau, z))(x, \xi/|\xi|)|\xi|^{z+\mu-j} + r_p(\tau, z, x, \xi),$$

with a remainder $r_p \in S^{z+\mu-p,\tau+\rho}_{\mathrm{cl}(\xi,x)}$. Substituting this expression for $a(x, \xi)$ in the integral

$$t_2(\tau, z) = \int_{|x|\le\epsilon} \int_{|\xi|\ge 1} a(x, \xi)\, dx\, d\xi$$

and introducing polar coordinates for the integration in the variables ξ, we obtain

$$t_2(\tau, z) = -(2\pi)^{-n} \sum_{j=0}^{p-1} \frac{1}{z + \mu + n - j} \int_{|x|\le\epsilon} \int_{\mathbb{S}^{n-1}} \sigma_\psi^{\mu-j}(a(\tau, z))\, d\theta\, dx + R_{p,\epsilon}(\tau, z), \quad (3.2)$$

where $R_{p,\epsilon}(\tau, z)$ is holomorphic for $\Re z < -\mu - n + p$ and all $\tau \in \mathbb{C}$.

Interchanging the roles of the variables x, ξ we obtain

$$t_3(\tau,z) = -(2\pi)^{-n} \sum_{k=0}^{q-1} \frac{\epsilon^{\tau+\rho+n-k}}{\tau+\rho+n-k} \int_{|\xi|\leq 1} \int_{\mathbb{S}^{n-1}} \sigma_e^{\rho-k}(a(\tau,z))\, d\theta\, d\xi + R'_{q,\epsilon}(\tau,z), \quad (3.3)$$

where $R'_{q,\epsilon}(\tau,z)$ is holomorphic for $\Re\tau < -\rho-n+q$ and all $z \in \mathbb{C}$.

Finally, repeating the same argument twice, we get

$$t_4(\tau,z) = (2\pi)^{-n} \sum_{j=0}^{p-1}\sum_{k=0}^{q-1} \frac{1}{z+\mu+n-j} \frac{\epsilon^{\tau+\rho+n-k}}{\tau+\rho+n-k} \times \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{\mu-j,\rho-k}(a(\tau,z))\, d\theta\, d\theta' + \sum_{j=0}^{p-1} \frac{1}{z+\mu+n-j} R''_{q,j,\epsilon}(\tau,z) + R'''_{p,\epsilon}(\tau,z), \quad (3.4)$$

where $R''_{q,j,\epsilon}(\tau,z)$ is holomorphic for $\Re\tau < -\rho-n+q$ and all $z \in \mathbb{C}$, and $R'''_{p,\epsilon}(\tau,z)$ is holomorphic for $\Re z < -\mu+n+p$ and all $\tau \in \mathbb{C}$.

So, we have verified that $t(\tau,z)$ extends to a meromorphic function on $\Re\tau < -\rho-n+q, \Re z < -\mu+n+p$. As p and q are arbitrary, this concludes the proof. □

Remark 3.2. Note that if $a(\tau,z) \in FS^{\mu+z,\rho+\tau}_{\mathrm{cl}(\xi,x)}$ is a holomorphic family with $a(0,0)=0$, then $\mathrm{Tr}(a(\tau,z)^{\mathrm{w}})$ is holomorphic near $(0,0)$.

Now, let $a \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}, \mu,\rho \in \mathbb{Z}$ be a classical symbol and let $a(\tau,z)$ be an holomorphic family with $a(0,0)=a$ (cf. Remark 2.7). Consider the functionals defined by

$$\tau z \mathrm{Tr}(a(\tau,z)^{\mathrm{w}}) = \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) - \tau \widehat{\mathrm{Tr}}_{\psi}(a^{\mathrm{w}}) - z\widehat{\mathrm{Tr}}_e(a^{\mathrm{w}}) + \tau^2 V + \tau z V' + z^2 V'' \quad (3.5)$$

where V, V', V'' are holomorphic near $(0,0)$. In view of Remark 3.2 they do not depend on the choice of the holomorphic family $a(\tau,z)$.

Proposition 3.3. *The functionals* $\mathrm{Tr}_{\psi,e}, \widehat{\mathrm{Tr}}_{\psi}, \widehat{\mathrm{Tr}}_e$ *defined in* (3.5) *have the following explicit expressions:*

$$\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{-n,-n}(a)\, d\theta\, d\theta', \quad (3.6)$$

$$\widehat{\mathrm{Tr}}_{\psi}(a^{\mathrm{w}}) = (2\pi)^{-n} \lim_{\epsilon\to+\infty} \Bigg(\int_{|x|\leq\epsilon} \int_{\mathbb{S}^{n-1}} \sigma_{\psi}^{-n}(a)\, d\theta\, dx - (2\pi)^n \log\epsilon \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) - \sum_{i=1}^{\rho+n} \frac{\epsilon^i}{i} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{-n,i-n}(a)\, d\theta\, d\theta' \Bigg), \quad (3.7)$$

$$\widehat{\mathrm{Tr}}_e(a^{\mathrm{w}}) = (2\pi)^{-n} \lim_{\epsilon\to+\infty} \left(\int_{|\xi|\le\epsilon} \int_{\mathbb{S}^{n-1}} \sigma_e^{-n}(a)\, d\theta\, d\xi - (2\pi)^n \log\epsilon \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) \right.$$
$$\left. - \sum_{i=1}^{\mu+n} \frac{\epsilon^i}{i} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{i-n,-n}(a)\, d\theta\, d\theta' \right). \quad (3.8)$$

Proof. We refer to the proof of Lemma 3.1, where now we take $p > \mu + n, q > \rho + n$.

(3.6) follows at once as limit $\lim_{(\tau,z)\to(0,0)} \tau z \mathrm{Tr}(a(\tau,z)^{\mathrm{w}})$ using the expressions (3.2), (3.3), (3.4).

To prove (3.7), we observe that we can obtain $\widehat{\mathrm{Tr}}_\psi$ as

$$\widehat{\mathrm{Tr}}_\psi(a^{\mathrm{w}}) = -\lim_{\tau\to 0} \tau^{-1} \lim_{z\to 0} (\tau z \mathrm{Tr}(a(\tau,z)^{\mathrm{w}}) - \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}})). \quad (3.9)$$

When we perform the most internal limit, the expressions $\tau z t_1(\tau,z)$ and $\tau z t_3(\tau,z)$ vanish, as well as $\tau z R'''_{p,\epsilon}(\tau,z)$ and the terms of the sums in (3.2) and (3.4) for $j \ne \mu - n$. What remains obviously is independent of ϵ but, on the other hand, as $\epsilon \to +\infty$ the expression $\tau R''_{q,\mu+n,\epsilon}(\tau,0)$ and the terms of the sum in (3.4) with $k > \rho + n$ tend to zero uniformly for small τ. Then we have

$$\lim_{z\to 0}(\tau z \mathrm{Tr}(a(\tau,z)^{\mathrm{w}}) - \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}})) =$$
$$(2\pi)^{-n}\tau \lim_{\epsilon\to 0} \left(-\int_{|x|\le\epsilon} \int_{\mathbb{S}^{n-1}} \sigma_\psi^{\mu-j}(a(\tau,0))\, d\theta\, dx \right.$$
$$+ \frac{\epsilon^\tau}{\tau} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{-n,-n}(a(\tau,0))\, d\theta\, d\theta' - \frac{(2\pi)^n}{\tau} \mathrm{Tr}_{\psi,e}(a^{\mathrm{w}})$$
$$\left. + \sum_{k=0}^{\rho+n-1} \frac{\epsilon^{\tau+\rho+n-k}}{\tau+\rho+n-k} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_{\psi,e}^{-n,\rho-k}(a(\tau,0))\, d\theta\, d\theta' \right),$$

from which, by (3.9), (3.7) follows. At the same way one proves (3.8). □

Remark 3.4. Let us note that the restrictions Tr_ψ and Tr_e of $\widehat{\mathrm{Tr}}_\psi$ and $\widehat{\mathrm{Tr}}_e$ to $\bigcup_{\mu\in\mathbb{Z}} L^{\mu,-n-1}_{\mathrm{cl}(\xi,x)}$ and $\bigcup_{\rho\in\mathbb{Z}} L^{-n-1,\rho}_{\mathrm{cl}(\xi,x)}$ are given by

$$\mathrm{Tr}_\psi(a^{\mathrm{w}}) = (2\pi)^{-n} \int_{\mathbb{R}^n_x} \int_{\mathbb{S}^{n-1}} \sigma_\psi^{-n}(a)\, d\theta\, dx, \quad a \in \bigcup_{\mu\in\mathbb{Z}} S^{\mu,-n-1}_{\mathrm{cl}(\xi,x)}, \quad (3.10)$$

$$\mathrm{Tr}_e(a^{\mathrm{w}}) = (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{S}^{n-1}} \sigma_e^{-n}(a)\, d\theta\, d\xi, \quad a \in \bigcup_{\rho\in\mathbb{Z}} S^{-n-1,\rho}_{\mathrm{cl}(\xi,x)}, \quad (3.11)$$

and $\widehat{\mathrm{Tr}}_\psi$ and $\widehat{\mathrm{Tr}}_e$ turn out just the finite parts of the integrals in (3.10) and (3.11) when $a \in \bigcup_{\mu\in\mathbb{Z},\rho\in\mathbb{Z}} S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$. Furthermore, the functional $\widehat{\mathrm{Tr}}_\psi$ and $\widehat{\mathrm{Tr}}_e$ vanish on $\mathcal{I}_e$ and $\mathcal{I}_\psi$ respectively, so that they are well defined on $\mathcal{A}_\psi$ and $\mathcal{A}_e$ as continuous extensions of Tr_ψ and Tr_e.

Theorem 3.5. *The functional* $\mathrm{Tr}_{\psi,e}$ *defines a trace on the algebra* $\mathcal{A}$ *which vanishes on* $\mathcal{I}_\psi$ *and* $\mathcal{I}_e$ *and therefore it induces traces on* $\mathcal{A}_\psi, \mathcal{A}_e$ *and* $\mathcal{A}_{\psi,e}$*. On* $\mathcal{I}_\psi$ *and* $\mathcal{I}_e$ *trace functionals are given respectively by* Tr_ψ *and* Tr_e *defined in* (3.10) *and* (3.11)*.*

For all these algebras, the above functionals are the unique traces up to multiplication by a constant.

Proof. In all cases the statement easily follows by the same arguments of the proof of Theorem 1.4 of Fedosov, Golse, Leichtnam and Schrohe [9]. See also Boggiatto and Nicola [2] for a version in $\mathbb{R}^n$, and the other papers on Wodzicki's residue listed in the references. To avoid an overweight of the paper, we prefer then to omit any detail. □

Remark 3.6. Theorem 3.5 says us that for each of the algebras in Definition 2.5 vanishing of the corresponding trace characterizes commutators.

4. Dixmier traces

We begin by reviewing the construction of non-normal Dixmier traces. We consider traces whose natural domain is contained in the ideal $\mathcal{K}(H)$ of compact operators on the Hilbert space H.

For $T \in \mathcal{K}(H)$, let $\mu_n(T)$, $n \in \mathbb{N}$, be the sequence of the eigenvalues of $|T|$, counted with their multiplicity and labelled in decreasing order and let $\sigma_N(T) = \sum_{n=0}^{N} \mu_n(T)$, $N \in \mathbb{N}$. For a fixed sequence α of positive numbers α_N such that

(i) $\alpha_N \to +\infty$;

(ii) $\alpha_0 > \alpha_1 - \alpha_0$ and $\alpha_{N+1} - \alpha_N \geq \alpha_{N+2} - \alpha_{N+1}$ for $N \in \mathbb{N}$;

(iii) $\alpha_N^{-1}\alpha_{2N} \to 1$,

we define the ideal $I_\alpha(H) := \{T \in \mathcal{K}(H) : \alpha_N^{-1}\sigma_N(T) \in l^\infty(\mathbb{N})\}$.

Then, consider a linear form ω on $C_b(1,\infty)$, the space of the continuous bounded function on $[1,\infty]$, with $\omega \geq 0$, $\omega(1) = 1$ and $\omega(f) = 0$ if $\lim_{x\to+\infty} f(x) = 0$. Given a bounded sequence $a = (a_n)_{n\geq 1}$, we construct the function $f_a = \sum_{n\geq 1} a_n \chi_{[n-1,n)} \in L^\infty(\mathbb{R}_+)$ and define the ω-limit $\lim_\omega a_n = \omega(Mf_a)$ where, for $g \in L^\infty(\mathbb{R}_+)$, $Mg(t) := \frac{1}{\log t}\int_1^t \frac{g(s)}{s}\,ds$ is the Cesàro mean of g. In the case of convergent sequences the ω-limit coincides with the usual limit.

Definition 4.1. *Let $\alpha = (\alpha_N)$ be a sequence as above and $T \in I_\alpha(H)$, $T \geq 0$. We define the Dixmier trace of T as*

$$\mathrm{Tr}_{\alpha,\omega}(T) = \lim_\omega \alpha_N^{-1}\sigma_N(T).$$

Dixmier's trace extends to a linear map on $I_\alpha(H)$.

The case of the sequence $\alpha_N = \log N$ will be of particular relevance in the following; we shall use the notation Tr_ω for the Dixmier trace associated with that sequence (cf. Connes [4]) whereas we shall denote by $\mathcal{L}^{(1,\infty)}(H)$ its domain, cf. the following more general definition.

Definition 4.2. *For $1 < p < \infty$ we define the subspace $\mathcal{L}^{(p,\infty)}(H) \subset \mathcal{K}(H)$ as the set of all compact operators T with $\sigma_N(T) = O(N^{1-1/p})$. Similarly we define $\mathcal{L}^{(1,\infty)}(H) \subset \mathcal{K}(H)$ by the condition $\sigma_N(T) = O(\log N)$.*

For $1 < p < \infty$, we define the subspace $\mathcal{L}^{(p,\infty)}_{\log}(H) \subset \mathcal{K}(H)$ as the set of all compact operators T with $\sigma_N(T) = O(N^{1-1/p}(\log N)^{-1/p})$; $\mathcal{L}^{(1,\infty)}_{\log}(H) \subset \mathcal{K}(H)$ will be defined by the condition $\sigma_N(T) = O((\log N)^2)$.

All these spaces are normed ideals contained in $\mathcal{K}(H)$ and containing the ideal $\mathcal{B}_1(H)$ of trace class operators.

Remark 4.3. Let us observe that for $p = 1$ the ideal $\mathcal{L}^{(1,\infty)}_{\log}(H)$ is the natural domain of the Dixmier trace associated with the sequence $\alpha_N = (\log N)^2$. In short we shall denote it by Tr'_ω.

All spaces $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ with $\mu < 0, \rho < 0$ are contained in $\mathcal{K}(L^2(\mathbb{R}^n))$. In order to establish relations between these spaces and the ideals in Definition 2.5, we have to study the asymptotic bahaviour of the spectrum of such operators. We begin by observing that Theorem 3.4 of Hörmander [14], when applied to our symbol classes, gives the following result for operators of positive order (in our case, with the metric $g_{x,\xi}$ given in (1.1), we have $g^\sigma_{x,\xi} = \langle\xi\rangle^2|dx|^2 + \langle x\rangle^2|d\xi|^2$ and $h^2(x,\xi) := \sup g_{x,\xi}/g^\sigma_{x,\xi} = \langle x\rangle^{-2}\langle\xi\rangle^{-2}$).

Proposition 4.4. *Let $a \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}, \mu > 0, \rho > 0$, be a positive elliptic symbol. Then the corresponding operator a^{w} is self-adjoint in $L^2(\mathbb{R}^n)$; it is bounded from below and has discrete spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$ diverging to $+\infty$.*

Under the hypotheses of Proposition 4.4, it makes sense to consider the function $N(\lambda) := \sum_{j:\lambda_j \le \lambda} 1$ which "counts" the number of eigenvalues not greater than λ. We are going to give an asymptotic estimation for this function.

In the proof of the following theorem we shall use the notation $f(y) \prec g(y)$ for functions $f, g : Y \to \mathbb{R}$ when there exist a constant $C > 0$ such that $f(y) \le Cg(y)$ for all $y \in Y$; the constant C may depend on parameters, indices etc. possibly appearing in the expression of f and g, but not on $y \in Y$.

Theorem 4.5. *(Weyl Formula) Let $a \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, $\mu > 0, \rho > 0$, be a strictly positive elliptic symbol and denote by $N(\lambda)$ the counting function associated with the operator a^{w}. Then for every $0 < \delta_1 < \frac{2}{3\rho}, 0 < \delta_2 < \frac{2}{3\mu}$, we have*

$$N(\lambda) = \begin{cases} C_\mu \lambda^{\frac{n}{\mu}} \log\lambda + O\left(\lambda^{\frac{n}{\mu}}\right) & \text{for } \mu = \rho, \\ C'_\mu \lambda^{\frac{n}{\mu}} + O\left(\lambda^{\frac{n}{\mu}-\delta_1}\right) & \text{for } \mu < \rho, \\ C''_\rho \lambda^{\frac{n}{\rho}} + O\left(\lambda^{\frac{n}{\rho}-\delta_2}\right) & \text{for } \mu > \rho, \end{cases} \tag{4.1}$$

where

$$C_\mu = \frac{(2\pi)^{-n}}{n\mu} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma^{\mu,\mu}_{\psi,e}(a)^{-\frac{n}{\mu}}\, d\theta\, d\theta', \tag{4.2}$$

$$C'_\mu = \frac{(2\pi)^{-n}}{n} \int_{\mathbb{R}^n_x} \int_{\mathbb{S}^{n-1}} \sigma^{\mu}_{\psi}(a)^{-\frac{n}{\mu}}\, d\theta\, dx, \tag{4.3}$$

$$C''_\rho = \frac{(2\pi)^{-n}}{n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{S}^{n-1}} \sigma^{\rho}_{e}(a)^{-\frac{n}{\rho}}\, d\theta\, d\xi. \tag{4.4}$$

Proof. It follows from Theorem 4.1 of Hörmander [14] that

$$|N(\lambda) - W(\lambda)| \prec W\left(\lambda + \lambda^{1-\delta}\right) - W\left(\lambda - \lambda^{1-\delta}\right), \tag{4.5}$$

for $0 < \delta < 2/(3\max\{\mu,\rho\})$ and

$$W(\lambda) = (2\pi)^{-n} \int\limits_{a(x,\xi)\le\lambda} dx\, d\xi. \tag{4.6}$$

Hence we have to estimate the Weyl term $W(\lambda)$. We first consider the case $\mu = \rho$.

It is easy to convince ourselves, by the ellipticity of a, that for every fixed $x \in \mathbb{R}^n$ and large λ the subset $\{a(x,\xi) \le \lambda\} \subset \mathbb{R}^n_\xi$ is star-shape with respect to the origin (in fact for fixed $x, u \in \mathbb{R}^n$, $|u| = 1$, $a(x,tu)$ is increasing as a function of t for large t), so that, if we introduce polar coordinates (r,θ) in the integral with respect to the variable ξ in (4.6), we shall have to integrate on a set of the type

$$\{(x,r,\theta)\in\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}^{n-1}: r\le f(\lambda,x,\theta)\},$$

for a suitable non-negative function $f(\lambda,x,\theta)$. Now again by (2.4), which we suppose satisfied for every $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$ in view of the strict positivity of a, we have

$$\max\{C'\lambda^{\frac{1}{\mu}}(1+|x|)^{-1}-1,0\}\le f(\lambda,x,\theta)\le\max\{C\lambda^{\frac{1}{\mu}}(1+|x|)^{-1}-1,0\}, \tag{4.7}$$

for suitable constants $C,C'>0$, so that, in particular, $f(\lambda,x,\theta)$ vanishes for $|x|\ge C\lambda^{\frac{1}{\mu}}-1$. Then we have

$$(2\pi)^n W(\lambda)=\frac{1}{n}\int\limits_{|x|\le C\lambda^{\frac{1}{\mu}}-1}\int\limits_{\mathbb{S}^{n-1}} f(\lambda,x,\theta)^n\,d\theta\,dx. \tag{4.8}$$

Now we write $a=\sigma^\mu_\psi(a)+a'$, where $a'\in C^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ satisfies the estimate

$$|a'(x,\xi)|\prec(1+|x|)^\mu(1+|\xi|)^{\mu-1}. \tag{4.9}$$

Let $a_0(x,\theta)=\sigma^\mu_\psi(a)(x,\xi(1,\theta))$. By definition of $f(\lambda,x,\theta)$, by (4.9) and the homogeneity of $\sigma^\mu_\psi(a)$ we deduce

$$|\lambda^{-1}a_0(x,\theta)f(\lambda,x,\theta)^\mu-1|\prec\lambda^{-1}(1+|x|)^\mu(1+f(\lambda,x,\theta))^{\mu-1}. \tag{4.10}$$

Then in (4.8) we write f as $(f^\mu)^{\frac{1}{\mu}}$ and replace f^μ by $f^\mu=\lambda a_0^{-1}(1+(\lambda^{-1}a_0f^\mu-1))$; we obtain

$$(2\pi)^n W(\lambda)=\frac{1}{n}\lambda^{\frac{n}{\mu}}\int\limits_{|x|\le C\lambda^{\frac{1}{\mu}}-1}\int\limits_{\mathbb{S}^{n-1}} a_0(x,\theta)^{-\frac{n}{\mu}}(1+R(\lambda,x,\theta))^{\frac{n}{\mu}}\,d\theta\,dx, \tag{4.11}$$

where, in view of (4.10) and (4.7), on the integration domain the function $R=\lambda^{-1}a_0f^\mu-1$ satisfies the estimate

$$|R(\lambda,x,\theta)|\prec\lambda^{-1}(1+|x|)^\mu(1+f(\lambda,x,\theta))^{\mu-1}\prec\lambda^{-\frac{1}{\mu}}(1+|x|). \tag{4.12}$$

From (4.11) it follows that

$$(2\pi)^n W(\lambda)=\frac{1}{n}\lambda^{\frac{n}{\mu}}\int\limits_{|x|\le C\lambda^{\frac{1}{\mu}}-1}\int\limits_{\mathbb{S}^{n-1}} a_0(x,\theta)^{-\frac{n}{\mu}}\,d\theta\,dx+g(\lambda), \tag{4.13}$$

with

$$|g(\lambda)| \prec \lambda^{\frac{n}{\mu}-1} \int\limits_{|x|\le C\lambda^{\frac{1}{\mu}}-1} (1+|x|)^{-n+1}\,dx, \tag{4.14}$$

as one sees from the fact that $(1+t)^\alpha \prec t$ for $0 \le t \le T < +\infty$ and by using (4.12). Introducing polar coordinates in (4.14) we get

$$g(\lambda) = O\left(\lambda^{\frac{n}{\mu}}\right),$$

and therefore it remains only to estimate the integral in (4.13). To do this, we write

$$a_0(x,\theta) = \sigma^{\mu,\mu}_{\psi,e}(a)(x,\xi(1,\theta)) + a''(x,\theta);$$

arguing as above we easily deduce

$$(2\pi)^n W(\lambda) = \frac{1}{n}\lambda^{\frac{n}{\mu}} \int\limits_{|x|\le C\lambda^{\frac{1}{\mu}}-1} \int\limits_{\mathbb{S}^{n-1}} \sigma^{\mu,\mu}_{\psi,e}(a)^{-\frac{n}{\mu}}\,d\theta\,dx + g(\lambda) + g'(\lambda), \tag{4.15}$$

with $g'(\lambda) = O\left(\lambda^{\frac{n}{\mu}}\right)$. Switching to polar coordinates in the integral with respect to the variables x in (4.15) and using the homogeneity of $\sigma^{\mu,\mu}_{\psi,e}(a)$ we obtain

$$W(\lambda) = C_\mu \lambda^{\frac{n}{\mu}} \log\lambda + O\left(\lambda^{\frac{n}{\mu}}\right), \tag{4.16}$$

where C_μ is given in (4.2). Replacing λ with $\lambda \pm \lambda^{1-\delta}$ in (4.16) we get

$$W(\lambda \pm \lambda^{1-\delta}) = C_\mu \lambda^{\frac{n}{\mu}} \log\lambda + O\left(\lambda^{\frac{n}{\mu}}\right),$$

from which, by (4.5), the first formula in (4.1) follows.

In the same way, a simpler version of the above argument proves the other two formulas in (4.1). In the case $\mu < \rho$ for instance, one obtains

$$W(\lambda) = C'_\mu \lambda^{\frac{n}{\mu}} + O\left(\lambda^{\frac{n-1}{\mu}}\right),$$

where C'_μ is given in (4.3), and therefore

$$W\left(\lambda \pm \lambda^{1-\delta}\right) = C'_\mu \lambda^{\frac{n}{\mu}} + O\left(\lambda^{\frac{n}{\mu}-\delta}\right),$$

where $0 < \delta < \frac{2}{3\rho}$.

This concludes the proof. □

Of course, (4.1) could be rewritten in the form $N(\lambda) = W(\lambda) + R(\lambda)$ where $W(\lambda)$ is given by (4.6) and $R(\lambda) = O(\lambda^{n/\mu})$ for $\mu = \rho$, $R(\lambda) = O(\lambda^{n/\mu-\delta_1})$ for $\mu < \rho$, $R(\lambda) = O(\lambda^{n/\rho-\delta_2})$ for $\mu > \rho$. However the computation of the volume $W(\lambda)$ and therefore the more explicit formula (4.1) will be essential in the following.

We shall need the following simple lemma.

Lemma 4.6. *For $1 \le p < \infty$, let g_p be the inverse function of $f_p : (1,\infty) \to \mathbb{R}_+$, $f_p(x) = x^p \log x$. Then*

(a) *if (a_n) and (b_n) are positive sequences with $a_n \sim b_n$ we have $g_p(a_n) \sim g_p(b_n)$;*

(b) *for every positive sequence (k_n) diverging to $+\infty$ we have $g_p(k_n) \sim (pk_n/\log k_n)^{1/p}$.*

Proof. (a) The statement follows at once observing that, for $0 < x < x'$, we have

$$0 < \frac{\log g_p(x) - \log g_p(x')}{x - x'} \le \frac{1}{px},$$

as one verifies by Lagrange's formula.
(b) Note that $f_p\left((pk_n/\log k_n)^{1/p}\right) \sim k_n$ and then use (a). □

Theorem 4.7. *Let $\mu < 0, \rho < 0$, with $\mu \ge -n$ or $\rho \ge -n$, so that $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \subset \mathcal{K}(L^2(\mathbb{R}^n))$ but $L^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \not\subset \mathcal{B}_1(L^2(\mathbb{R}^n))$. Then the following inclusions hold:*

$$L^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \subset \begin{cases} \mathcal{L}^{(-n/\mu,\infty)}_{\log}(L^2(\mathbb{R}^n)) & \text{if } \mu = \rho, \\ \mathcal{L}^{(-n/\mu,\infty)}(L^2(\mathbb{R}^n)) & \text{if } \mu > \rho, \\ \mathcal{L}^{(-n/\rho,\infty)}(L^2(\mathbb{R}^n)) & \text{if } \mu < \rho. \end{cases} \tag{4.17}$$

Furthermore we have

$$\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) = 2n^2 \mathrm{Tr}'_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,-n}_{\mathrm{cl}(\xi,x)}, \tag{4.18}$$

$$\mathrm{Tr}_\psi(a^{\mathrm{w}}) = n\mathrm{Tr}_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,\rho}_{\mathrm{cl}(\xi,x)} \text{ with } \rho \in \mathbb{Z}, \rho < -n, \tag{4.19}$$

$$\mathrm{Tr}_e(a^{\mathrm{w}}) = n\mathrm{Tr}_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{\mu,-n}_{\mathrm{cl}(\xi,x)} \text{ with } \mu \in \mathbb{Z}, \mu < -n, \tag{4.20}$$

independently of ω.

Proof. We verify the first inclusion in (4.17). The other cases can be proved in the same way.

Consider first the case of an elliptic operator $L^{\mu,\mu}_{\mathrm{cl}(\xi,x)} \ni A > 0, -n \le \mu < 0$ with real Weyl symbol a, and therefore defining a isomorphism $L^2(\mathbb{R}^n) \to H^{-\mu,-\mu}(\mathbb{R}^n)$ (because $\mathrm{Ind}\, A = 0$). Then its inverse

$A^{-1} : H^{-\mu,-\mu}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfies the hypotheses of Theorem 4.1 (possibly after the addition of a multiple of the identity operator), so that for its counting function we have the formula

$$N_{A^{-1}}(\lambda) \sim \tilde{C}_\mu \lambda^{-\frac{n}{\mu}} \log \lambda, \tag{4.21}$$

with

$$\tilde{C}_\mu = -\frac{(2\pi)^{-n}}{n\mu} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \sigma^{\mu,\mu}_{\psi,e}(a)^{-\frac{n}{\mu}}\, d\theta\, d\theta'.$$

Standard arguments (cf. Shubin [30], Proposition 13.1) show that (4.21) is equivalent to the following formula for the eigenvalues λ_k of A^{-1}:

$$\lambda_k^{-\frac{n}{\mu}} \log \lambda_k \sim \tilde{C}_\mu^{-1} k,$$

which, by Lemma 4.6, implies

$$\lambda_k \sim g_{-\frac{n}{\mu}}(\tilde{C}_\mu^{-1} k) \sim (-nk/(\mu \tilde{C}_\mu \log k))^{-\mu/n}.$$

For the eigenvalues of A, that are λ_k^{-1}, we obtain the formula

$$\lambda_k^{-1} \sim (-nk/(\mu \tilde{C}_\mu \log k))^{\mu/n}. \tag{4.22}$$

From (4.22) it follows that

$$\begin{aligned} \sum_{k=1}^{N} \lambda_k^{-1} &\sim \left(-\frac{n}{\mu}\tilde{C}_\mu^{-1}\right)^{\frac{\mu}{n}} \int_1^N \left(\frac{\log x}{x}\right)^{-\frac{\mu}{n}} dx \\ &\sim \begin{cases} \frac{n}{n+\mu}\left(-\frac{n}{\mu}\tilde{C}_\mu^{-1}\right)^{\frac{\mu}{n}} N^{1+\frac{\mu}{n}} (\log N)^{-\frac{\mu}{n}} & \textit{for } -n<\mu<0 \\ \frac{1}{2}\tilde{C}_{-n} (\log N)^2 & \textit{for } \mu=-n. \end{cases} \end{aligned} \tag{4.23}$$

Hence $A \in \mathcal{L}^{(-n/\mu,\infty)}_{\log}(L^2(\mathbb{R}^n))$. As $\mathcal{L}^{(-n/\mu,\infty)}_{\log}(L^2(\mathbb{R}^n))$ is an ideal of $\mathcal{B}(H)$ the first inclusion in (4.17) follows, since one can write $P \in S^{\mu,\mu}_{\mathrm{cl}(\xi,x)}$ as $P = (PA^{-1})A$ where PA^{-1} is bounded in $L^2(\mathbb{R}^n)$.

Now we come to the relations (4.18), (4.19), (4.20) between the traces $\mathrm{Tr}_{\psi,e}, \mathrm{Tr}_\psi, \mathrm{Tr}_{\mathrm{e}}$ and the Dixmier traces. We limit ourselves to prove (4.18).

It follows from (4.23) that (4.18) holds for an elliptic operator $A > 0$ with real Weyl symbol. By linearity then it suffices to prove that such operators span $S^{-n,-n}_{\mathrm{cl}(\xi,x)}$, up to trace class operators (on which both $\mathrm{Tr}_{\psi,e}$ and Tr'_ω vanish). Now, every operator can be written as sum of two self-adjoint operators, and if $P = p^{\mathrm{w}}$ is self-adjoint but not elliptic we write $p = (p + Cq) - Cq$ where $q(x,\xi) = (1 + |x|)^\mu (1 + |\xi|)^\mu$ and

$C = -\inf p/q + 1$. So P is seen as difference of two elliptic operators with positive Weyl symbol. Therefore, by Lemma 3.2 of Hörmander [14], we can limit ourselves to consider $P \geq 0$ elliptic. Then, as by Fredholm theory $V = \mathrm{Ker}P$ is a finite dimensional subspace of $\mathcal{S}(\mathbb{R}^n)$, the orthogonal projection P_V on V is regularizing. Now $P = (P+P_V)-P_V$, and $P+P_V$ is elliptic, strictly positive, with real Weyl symbol. This concludes the proof. □

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