Local solvability for $\Box_b$ on degenerate CR manifolds and related systems

Fabio Nicola

Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi, 24 - 10129 Torino, Italy
e-mail: fabio.nicola@polito.it

Abstract

We show that the Kohn Laplacian is locally solvable in the Sobolev spaces $H^k$, $k \geq 0$, on any degenerate CR manifold whose Levi form has kernel of constant dimension. A similar result is indeed proved for a more general class of systems of linear partial differential operators.

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1 Introduction and discussion of the results

This note is mostly motivated by recent results obtained by Nagel, Ricci and Stein [11] and Peloso and Ricci [15, 16] (see also Treves [20]). One of the main problems treated there is the (local) solvability of the Kohn Laplacian on quadratic CR manifolds.

To be definite, consider a Hermitian map $\Phi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^m$ and the associated quadratic manifold

$$S = \{(z, t + iu) \in \mathbb{C}^n \times \mathbb{C}^m : u = \Phi(z, z)\}.$$

One then defines the $\overline{\partial}_b$-complex on $S$ and the corresponding Kohn Laplacian $\Box_b^{(q)} = \overline{\partial}_b \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b$ acting on $(0, q)$-forms on $S$. For any linear form $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}$ we define $\Phi^\lambda(z, z') = \lambda(\Phi(z, z'))$ (upon complexifying $\lambda$), which can be identified with the scalar valued Levi forms of the CR manifold $S$. One of the results that we are interested in here is the following (Corollary 3 of [16]):

Let us suppose that the scalar Levi forms $\Phi^\lambda$ are degenerate for all $\lambda$. Then the operator $\Box_b^{(q)}$ is solvable for any $q$.

In proving such a result the authors take advantage of the natural group structure carried by any quadratic CR manifold, where techniques from Harmonic Analysis then apply. In particular, they make use of explicit representations.

We think that the result above holds for more general CR manifolds. Indeed the aim of this note is to prove such a generalization for any abstract CR manifold of arbitrary codimension, under the additional hypothesis that the Levi form
has kernel of locally constant dimension. This further condition simplifies the symplectic geometry of the characteristic set of $\Box_b$. It is clearly satisfied, for example, by quadratic CR manifolds of the hypersurface type, where the Levi form is then independent of the base point.

We begin with the following result.

**Theorem 1.1.** Let $M$ be a CR manifolds of CR-dimension $n$ and real codimension $1$; let $\Box^{(q)}_b$ be the Kohn Laplacian with respect to a fixed Riemannian metric on $M$, acting on $(0, q)$-forms. Let us suppose that the Levi form is degenerate and has kernel of locally constant dimension.

Then for every point $x_0 \in M$ and any integer $k \geq 0$ there exists a neighborhood $\Omega_k$ of $x_0$ such that the system $\Box_b^{(q)} u = f$ has a solution $u \in H^k_{(0,q)}(\Omega_k)$ for every $f \in H^k_{(0,q)}(\Omega_k)$, $q = 0, \ldots, n$.

Under the hypotheses of Theorem 1.1 the condition $Y(q)$ may of course be violated. For example, the condition $Y(n)$ does not hold on any pseudoconvex manifold of CR-dimension $n$ (see e.g. [2]). The condition given is therefore sufficient for local solvability in absence of hypoellipticity (in general).

Theorem 1.1 indeed extends to CR manifolds of higher codimension as follows (see e.g. Shaw and Wang [18] or Section 3 below for terminology).

**Theorem 1.2.** Let $M$ be a CR manifolds of CR-dimension $n$ and real codimension $h \geq 1$; let $\Box^{(q)}_b$ be the Kohn Laplacian with respect to a fixed Riemannian metric on $M$, acting on $(0, q)$-forms. Let us suppose that the Levi form $\mathcal{L}(\rho)$ is degenerate and has kernel of locally constant dimension when $\rho$ varies in the characteristic bundle $N^*(M)$ with the 0-section removed.

Then for every point $x_0 \in M$ and any integer $k \geq 0$ there exists a neighborhood $\Omega_k$ of $x_0$ such that the system $\Box_b^{(q)} u = f$ has a solution $u \in H^k_{(0,q)}(\Omega_k)$ for every $f \in H^k_{(0,q)}(\Omega_k)$, $q = 0, \ldots, n$.

Theorem 1.2 will be shown to follow from a more general result we are going to establish, concerning a class of systems with double characteristics whose principal symbol is a scalar multiple of the identity matrix.

More precisely, let $X$ be an open subset of $\mathbb{R}^n$ and consider a $N \times N$ system $P$ of linear partial differential operators in $X$ of order $m$. We will assume the following:

($H_1$) The principal symbol of $P$ has the form $p_m \text{Id}_{N \times N}$, where $p_m(x, \xi)$ is homogeneous of degree $m$ with respect to $\xi$ and vanishes exactly to second order on a manifold $\Sigma \subset T^* X \setminus 0$ (transversal ellipticity), namely $\text{Ker } F_{p_m}(\rho) = T_\rho \Sigma$, $\forall \rho \in \Sigma$.

Here we used the standard notation for the fundamental matrix (or symplectic Hessian) $F_{p_m}(\rho)$ associated with $p_m$, defined by

$$\sigma(v, F_{p_m}(\rho) w) = \frac{1}{2} \langle \text{Hess } p_m(\rho) v, w \rangle, \quad \forall v, w \in T_\rho T^* X,$$

where $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the canonical symplectic 2-form on $T^* X$. The definition (1.1) has an invariant meaning at points of $\Sigma$, because $p_m$ vanishes to second order there, see Section 21.5 of [7]. Moreover, since $p_m$ is non negative then the spectrum of $F_{p_m}$ consists of the eigenvalue 0 and of eigenvalues $\pm i\mu_j$, with $\mu_j > 0$. One then
sets $\text{Tr}^+ F_{pm} = \sum_j \mu_j$.

Suppose

$(H_2)$ $\Sigma$ is a manifold with locally constant symplectic rank, namely $\dim(T_\rho \Sigma \cap T_\rho \Sigma^\sigma) = \text{const}$ when $\rho$ varies in the connected components of $\Sigma$, where $T_\rho \Sigma^\sigma$ is the symplectic orthogonal of $T_\rho \Sigma$.

$(H_3)$ $\sigma|_\Sigma$ is degenerate, namely $\dim(T_\rho \Sigma \cap T_\rho \Sigma^\sigma) \geq 1$, $\forall \rho \in \Sigma$.

$(H_4)$ For every $\rho \in \Sigma$, $T_\rho \Sigma \cap T_\rho \Sigma^\sigma \not\subset \ker d\pi(\rho)$, where $\pi : T^* X \to X$ is the projection.

Before stating our result, we recall that the subprincipal symbol $p^\rho_{m-1}$ of $P$ is defined by

$p^\rho_{m-1} = p_{m-1} + \frac{i}{2} (\partial_x, \partial_t)p_m \text{Id},$

($p_{m-1}$ denotes the matrix of the homogeneous terms of degree $m - 1$), which is invariantly defined at points of $\Sigma$ as well.

For any given complex matrix $Q$ we set $\text{Re} Q := (Q + Q^*)/2$.

**Theorem 1.3.** Let $P$ be a $N \times N$ system of linear partial differential operators in $X$ of order $m$, satisfying $(H_1)-(H_4)$. Let us suppose, in addition, that

$$\text{Re} p^\rho_{m-1}(\rho) + \text{Tr}^+ F_{pm}(\rho) \text{Id} \geq 0, \quad \forall \rho \in \Sigma,$$

(1.2)
as Hermitian matrix.

Then for every $x_0 \in X$ and any integer $k \geq 0$, there exists a neighborhood $\Omega_k$ of $x_0$ such that the system $Pu = f$ has a solution $u \in H^{k+m-2}(\Omega_k; \mathbb{C}^N)$ for every $f \in H^k(\Omega_k; \mathbb{C}^N)$.

As a model in the scalar case ($N = 1$), the reader may think to the Baouendi-Goulaouic type operator

$$P = D_1^2 + x_1^2 D_2^2 + D_3^2 - D_2$$
in $\mathbb{R}^3$ (as usual $D_j = -i \partial/\partial x_j$). We can write $P = MM^* + D_3^2$, where $M = D_1 + ix_1 D_2$ is the Mizohata operator, and therefore solvability in $L^2(B(0, \epsilon))$, $\epsilon > 0$, immediately follows from the estimate

$$\langle Pu, u \rangle \geq \|D_3 u\|_0^2 \geq 2\epsilon^{-2} \|u\|_0^2,$$for smooth $u$ with supp $u \subset B(0, \epsilon)$.

When $N = 1$, $k = 0$, and $T_\rho \Sigma^\sigma \subset T_\rho \Sigma$ for every $\rho \in \Sigma$, namely $\Sigma$ is involutive, Theorem 1.3 reduces to Theorem 3(2D) by Popivanov [17]. Notice that $\text{Tr}^+ F_{pm} \equiv 0$ in that case.

Let us now discuss the hypotheses of Theorem 1.3 in more detail.

For a formally self-adjoint second order system satisfying $(H_1)$ and $(H_2)$, condition (1.2) is actually equivalent to saying that $P$ is bounded from below in $L^2$ (see Theorem 3.1 below), i.e. $(Pu, u) \geq -C_K \|u\|_0^2$, for any compact subset $K \subset X$ and every $u \in C^\infty_0 (K; \mathbb{C}^N)$, which is of course true for $\Box_b$.

As regards Hypothesis $(H_4)$, it is easy to see that it is really essential; for example, the operator $P = (x_2 D_1 - x_1 D_2)^2$ satisfies all the remaining hypotheses (with
Next, it is not locally solvable at the origin, because $P^* u = 0$ for every $u \in C^\infty_0(\mathbb{R}^2)$ which is rotation invariant.

Also, if the symplectic form is non degenerate then $P$ may be non locally solvable, as $\Box_b(0)$ on the Heisenberg group $\mathbb{H}_n$ (see Lewy [8], Folland and Stein [3]), or the operator $P = D_1^2 + x_2^2 D_2^2 = D_2^2$ in $\mathbb{R}^2$ (see Grushin [6], Gilioli and Treves [4], cf. also Müller [10]). However, if we replace (1.2) by a strict inequality then the system is known to be hypoelliptic and locally solvable with loss of one derivative (without assuming $(H_2) = (H_4)$, and the fact the principal symbol is diagonal), as shown by Boutet de Monvel, Grigis and Helffer [1]. For the Kohn Laplacian on a CR manifold this corresponds to the case in which the condition $Y(q)$ is satisfied, see Grigs [5] and Parmeggiani [13]. Instead, under the hypotheses of Theorem 1.3 the system $P$ is not hypoelliptic in general; one may think, for example, to the operator $P = D_1^2$ in $\mathbb{R}^2$, $n \geq 2$. Finally we observe that in Theorem 1.3 the radial vector field is allowed to be symplectically orthogonal to the tangent space to $\Sigma$ at some characteristic point.

Theorem 1.3 and Theorem 1.2 are proved in Section 2 and Section 3 respectively. As regards other recent results about local solvability of $\Box_b$ and $\partial_b$ we refer to the numerous references in Chen and Shaw [2].

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2 Proof of Theorem 1.3

First we prove, for any second order classical and properly supported pseudodifferential system $P \in \text{OPS}^2(X; \mathbb{C}^N)$ satisfying the hypotheses of Theorem 1.3, the following key estimate:

For every $\epsilon > 0$ there exists a neighborhood $\Omega_\epsilon$ of $x_0$ such that

$$\|u\|_0^2 \leq \epsilon \text{Re}(Pu, u), \quad \forall u \in C^\infty_0(\Omega_\epsilon; \mathbb{C}^N).$$

(2.1)

We will use the following result due to Hörmander, that the reader can extract from the proof of Theorem 22.3.2 of [7].

Lemma 2.1. Consider a smooth non negative function $p_2(x, \xi)$ which is positively homogeneous of degree 2 (with respect to $\xi$), and vanishes exactly to second order on a manifold $\Sigma \subset T^{*}X \setminus 0$ satisfying $(H_2)$. Fix any $\rho_0 \in \Sigma$ and let $l = \dim(T_{\rho_0} \Sigma \cap T_{\rho_0} \Sigma^\sigma)$, and $2\nu = \dim T_{\rho_0} \Sigma^\sigma / (T_{\rho_0} \Sigma \cap T_{\rho_0} \Sigma^\sigma)$, for $\rho$ near $\rho_0$. Then there exist independent real functions $f_j$, $j = 1, \ldots, 2\nu + l$, defined in a conic neighborhood $V$ of $\rho_0$, and homogeneous of degree 1, such that

$$p_2(x, \xi) = \sum_{j=1}^{2\nu+l} f_j^2,$$

and, upon defining

$$\begin{cases}
g_j = f_{2j-1} + if_{2j} & j = 1, \ldots, \nu \\
g_{\nu+j} = f_{2\nu+j} & j = 1, \ldots, l,
\end{cases}$$

(2.2)
satisfying \( \sum_{j=1}^{\nu} \{g_j, g_j\} = -2i \text{Tr}^+ F_{p_2} \) at \( \Sigma \cap V \) and
\[
\{g_j, g_k\} = 0 \text{ at } \Sigma \cap V, \quad \forall j = \nu + 1, \ldots, \nu + l, \quad \forall k = 1, \ldots, \nu + l.
\]  

(2.3)

Consider then a point \( x_0 \in X \) and apply Lemma 2.1 to any characteristic point \( \rho_0 = (x_0, \xi) \) in the fibre over \( x_0 \). Let \( X_j \in \text{OPS}^1(X; \mathbb{C}^N) \) be pseudodifferential systems whose principal symbols coincide with \( g_j \text{Id} \) in \( V \). Microlocally near \( \rho_0 \) we have
\[
Q := P - \sum_{j=1}^{\nu+l} X_j^* X_j \in \text{OPS}^1.
\]

Moreover the principal symbol of \( Q \) is given, at \( \Sigma \cap V \), by
\[
q_1 = p_1^* + \frac{i}{2} \sum_{j=1}^{\nu} \{g_j, g_j\} \text{Id} = p_1^* + \text{Tr}^+ F_{p_2} \text{Id},
\]
so that \( \text{Re} q_1 \geq 0 \) on \( \Sigma \cap V \) by (1.2). Now we take any Hermitian matrix \( \tilde{q}(x, \xi) \), defined in \( V \), non negative and positively homogeneous of degree 1, such that \( \tilde{q} = \text{Re} q_1 \) on \( \Sigma \cap V \). Then
\[
\begin{align*}
\text{Re} q_1 &= \tilde{q} + \sum_{j=1}^{\nu+l} r_j^* g_j + \tilde{g}_j r_j \in V,
\end{align*}
\]

for convenient \( N \times N \) matrices \( r_j(x, \xi) \) homogeneous of degree 0. Let \( \tilde{Q} \in \text{OPS}^1(X; \mathbb{C}^N) \) and \( R_j \in \text{OPS}^0(X; \mathbb{C}^N) \) be pseudodifferential systems whose principal symbols coincide with \( \tilde{q} \) and \( r_j \) in \( V \), respectively, and let \( \psi \in S^0(X \times \mathbb{R}^n) \) be any real symbol supported in \( V \). Upon setting \( \Psi = \psi(x, D) \otimes \text{Id} \), by the Sharp Gårding inequality for systems applied to \( \tilde{Q} \) (see e.g. Theorem 18.6.14 of [7]) we have
\[
\begin{align*}
\text{Re}(P \Psi u, \Psi u) &\geq \sum_{j=1}^{\nu+l} \| (X_j + R_j) \Psi u \|_0^2 - C\|u\|_0^2 \\
&\geq \sum_{j=1}^{\nu+l} \| (X_j + R_j) \Psi u \|_0^2 - C\|u\|_0^2 \\
&\geq \sum_{j=\nu+1}^{\nu+l} \| X_j \Psi u \|_0^2 - C\|u\|_0^2, \quad \forall u \in C_0^\infty(X; \mathbb{C}^N).
\end{align*}
\]

(2.4)

Now we are going to prove the following estimate:

For some \( j_0 \in \{\nu + 1, \ldots, \nu + l\} \) and every \( \epsilon > 0 \) there exists a neighborhood \( \Omega_\epsilon \) of \( x_0 \) such that
\[
\|X_{j_0}\Psi u\|_0 \geq \frac{1}{\epsilon} \|\Psi u\|_0^2 - C_\epsilon \|u\|_{-1}, \quad \forall u \in C_0^\infty(\Omega_\epsilon; \mathbb{C}^N).
\]

(2.5)

To this end, we observe that, by (2.3), the vector bundle \( T\Sigma \cap T\Sigma' \) on \( \Sigma \) of rank \( l \) is generated by the Hamilton vector fields \( H_{g_{\nu+1}}, \ldots, H_{g_{\nu+l}} \), on \( \Sigma \cap V \). By Hypothesis \((H_4)\) it follows that for some \( j_0 \in \{\nu + 1, \ldots, \nu + l\} \) it turns out \( d_x g_{j_0} \neq 0 \) at \( \rho_0 \).
Possibly after shrinking \( V \), by the implicit function theorem and Taylor’s formula
we can then write, say, \( g_{\beta_0}(x, \xi) = e(x, \xi)(\xi_1 - \lambda(x, \xi')) \) in \( V \), with \( \xi' = (\xi_2, \ldots, \xi_n) \), for suitable real functions \( \lambda \) and \( e \) homogeneous of degree 1, and \( e \neq 0 \). The elliptic factor \( e \) is of course irrelevant, so that we can assume that \( g_{\beta_0}(x, \xi) = \xi_1 - \lambda(x, \xi') \) in \( V \). Now, there exists a canonical transformation \( \psi = \chi(x, \xi) \) from \( V \) into a conic neighborhood of \((x_0, \epsilon), \epsilon = (0, \ldots, 0, 1) \in \mathbb{R}^n \), with \( y_1 = x_1 \) and \( \eta_1 = g_{\beta_0} \).

Let \( F \) be any properly supported unitary Fourier integral operator associated with \( \chi \). By Egorov’s Theorem (see e.g. Theorem 6.2 of [19]) we have

\[
\|X_{\beta_0} \Psi u\|_0 = \|D_{\psi_1} F \Psi u\|_0 + O(\|\Psi u\|_0).
\]  

(2.6)

Now, for any given \( \epsilon \), we take \( \psi_1 \in C^\infty_0(B(x_0, \epsilon)) \), \( \psi_2 \in C^\infty_0(B(x_0, \epsilon/3)) \), with \( \psi_2 = 1 \) on \( B(x_0, \epsilon/4) \), and \( \phi_1 = 1 \) in \( B(x_0, \epsilon/2) \), so that \( \psi_2 u = u \) if \( \supp u \subset B(x_0, \epsilon/4) \).

Then the operator \((1 - \phi_1) F\Psi \phi_2\) is regularizing, i.e. it maps \( D'(X) \rightarrow C^\infty(X) \), because of the choice of \( \chi \), see for example Section 4.1 in [9]. It follows from the Poincaré inequality that

\[
\|D_{\psi_1} F \Psi u\|_0 \geq \frac{\sqrt{2}}{\epsilon} \|\phi_1 F\Psi \phi_2 u\|_0 - C_\epsilon \|u\|_{-1}
\]

\[
\geq \frac{\sqrt{2}}{\epsilon} \|\Psi u\|_0 - C'\epsilon \|u\|_{-1}, \quad \forall u \in C^\infty_0(B(x_0, \epsilon/4)).
\]

(2.7)

From (2.6) and (2.7) we obtain (2.5). In view of (2.4) we therefore deduce the following estimate:

Any given point \( \rho_0 = (x_0, \xi) \) has a conic neighborhood \( V \) such that, for every real symbol \( \psi \in S^0(X \times \mathbb{R}^n) \) supported in \( V \) and any \( \epsilon > 0 \) there exists a neighborhood \( \Omega_{\epsilon} \) of \( x_0 \) for which

\[
\Re(P\Psi u, \Psi u) \geq \frac{1}{\epsilon} \|\Psi u\|_0^2 - C_\epsilon \|u\|_{-1}^2, \quad \forall u \in C^\infty_0(\Omega_{\epsilon}; \mathbb{C}^N).
\]

(2.8)

Indeed, when \( \rho_0 \not\in \Sigma \) this estimate is of course satisfied.

Now we are going to patch together the microlocal estimates (2.8). We take real symbols \( \psi_j \in S^0(X \times \mathbb{R}^n) \), \( j = 1, \ldots, J \), with so small support such that (2.8) holds for each of them and \( \sum_{j=1}^{J} \psi_j(x, \xi)^2 = 1 \) for \( x \) in a neighborhood \( \Omega \) of \( x_0 \). Then we observe that, with \( \Psi_j = \psi_j(x, D) \otimes \Id \), we have \( \Re(Pu, u) = \sum_{j=1}^{J} \Re(P\Psi_j u, \Psi_j u) + O(\|u\|_0^2) \) and \( \|u\|_0^2 = \sum_{j=1}^{J} \|\Psi_j u\|_0^2 + O(\|u\|_2^2) \) for \( u \in C^\infty_0(\Omega; \mathbb{C}^N) \). Hence for any \( \epsilon' > 0 \) we can find an open neighborhood \( \Omega_{\epsilon'} \) such that

\[
\Re(Pu, u) \geq \frac{1}{\epsilon'} \|u\|_0^2 - C\|u\|_0^2 - C'\epsilon\|u\|_{-1}^2, \quad \forall u \in C^\infty_0(\Omega_{\epsilon'}; \mathbb{C}^N),
\]

where the constant \( C \) is independent of \( \epsilon' \). Given \( \epsilon > 0 \), we therefore choose \( \epsilon' \leq \min\{C/2, \epsilon/4\} \) and we take any neighborhood \( \Omega_{\epsilon} \subset \Omega_{\epsilon'} \) such that \( C'\epsilon\|u\|_{-1}^2 < \frac{1}{\epsilon'} \|u\|_0^2 \) for every \( u \in C^\infty_0(\Omega_{\epsilon}; \mathbb{C}^N) \). Then (2.1) is verified.

**Proof of Theorem 1.3**

It is clear (see e.g. the proof of Theorem 5 of [12]) that the result follows from the following estimate (applied for \( s = -k \)):

For every \( \epsilon > 0 \), \( s \in \mathbb{R} \), and any point \( x_0 \in X \) there exists a neighborhood \( \Omega_{\epsilon,s} \) of \( x_0 \) such that

\[
\|u\|_s \leq \epsilon\|Pu\|_{s+2-m} + C_{\epsilon,s}\|u\|_{s-1}, \quad \forall u \in C^\infty_0(\Omega_{\epsilon,s}; \mathbb{C}^N).
\]

(2.9)
In fact, Theorem 1.3 applies to $P$ if and only if it applies to $P^*$. It suffices to prove (2.9) for any second order classical and properly supported pseudodifferential system $P \in \text{OPS}^2(X; \mathbb{C}^N)$ satisfying the hypotheses of Theorem 1.3. Hence, by Cauchy-Schwarz’ inequality it is enough to prove that

$$\|u\|_s^2 \leq \epsilon \text{Re}(Pu, u)_s + C_{\epsilon, s}\|u\|_{s-1}^2, \quad \forall u \in C_c^\infty(\Omega_{\epsilon, s}; \mathbb{C}^N),$$

(2.10)

where $(\cdot, \cdot)_s$ denotes the scalar product in $H^s(\mathbb{R}^n; \mathbb{C}^n)$. Let $\Lambda(\xi) = (1 + |\xi|^2)^{1/2}$ and $\Lambda^* = \Lambda(D)^s \otimes \text{Id}$, $s \in \mathbb{R}$. We have

$$(Pu, u)_s = (\Lambda^* Pu, \Lambda^* u) = ((P + [\Lambda^*, P]\Lambda^{-s}) \Lambda^* u, \Lambda^* u).$$

Now the operator $[\Lambda^*, P]\Lambda^{-s}$ is an operator from $\mathcal{E}^0, q(M) \to \mathcal{E}^0, q+1(M)$.

The Kohn Laplacian $\Box_b$ on a CR manifold of codimension $h \geq 1$; see [18] for details. Let $T^{0,1}M$ be the CR structure of $M$ and $\mathcal{E}^q(M) := \Gamma(M, A^qM)$, $\mathcal{E}^0, q(M) := \Gamma(M, A^{0, q}M)$ be the spaces of $q$-forms and $(0, q)$-forms respectively, on $M$. Let $\pi_q : \mathcal{E}^q(M) \to \mathcal{E}^0, q(M)$ be the projection and define $\partial_b^{(q)} := \pi_q \circ d : \mathcal{E}^{0, q}(M) \to \mathcal{E}^0, q + 1(M)$. Fix a Hermitian metric on $\mathbb{C}TM$ such that $T^{0,1}M \perp T^{1,0}M$ and define

$$\Box_b^{(q)} = \partial_b^{(q-1)} \partial_b^{(q-1)*} + \partial_b^{(q)*} \partial_b^{(q)} : \mathcal{E}^0, q(M) \to \mathcal{E}^0, q(M),$$

for $q = 0, 1, \ldots n$. The adjoint is taken with respect to the induced inner product. Consider a local basis $\bar{1}, \ldots, \bar{n}$ of sections of $T^{0,1}M$ and real vector fields $T_1, \ldots, T_h$ such that $L_1, \ldots, L_n, \bar{1}, \ldots, \bar{n}, T_1, \ldots, T_h$ is a local orthonormal basis of $\mathbb{C}TM$. We recall that the characteristic bundle $N^*(M) \subset T^*M$ consists of the covectors which are conormal to $T^{1,0}M \oplus T^{0,1}M$.

Let $\omega_1, \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n, \tau_1, \ldots, \tau_h$, be the dual basis to $L_1, \ldots, L_n, \bar{1}, \ldots, \bar{n}, T_1, \ldots, T_h$, and let $\phi = \sum_I \phi_I \omega^I$ be a $(0, q)$-form, with $I = (i_1, \ldots, i_q)$, $1 \leq i_1 < \ldots < i_q \leq n$, and $\omega^I = \omega^{i_1} \wedge \ldots \wedge \omega^{i_q}$. An easy computation shows that the Kohn Laplacian $\Box_b^{(q)}$ reads as

$$\Box_b^{(q)} \phi = -\frac{1}{2} \sum_I \sum_{j=1}^n (L_j L_j + \bar{L}_j \bar{L}_j) \phi_I \omega^I + \text{lower order terms}.$$
We therefore have

\[ \Sigma = \{ \rho \in T^*M \setminus 0 : v_j := \sigma_1(L_j)(\rho) = 0, \ j = 1, \ldots, n \}, \]

which is exactly \( N^*(M) \) with the 0-section removed.

Moreover, we observe that if \( \alpha \) is such that \( H \) and therefore Hypotheses \( (H_1) \) is satisfied.

For \( \rho \in N^*(M) \) the Levi form \( \mathcal{L}(\rho) \) is then defined as the Hermitian matrix whose entries are

\[ \mathcal{L}(\rho)_{jk} := i\langle \rho, [L_j, T_k] \rangle = \sigma_1([L_j, T_k])(\rho) = i\{v_j, v_k\}(\rho), \ j, k = 1, \ldots, n. \]

Let \( \rho \in \Sigma = N^*(M) \setminus 0 \); we claim that the map

\[ \text{Ker} \mathcal{L}(\rho) \ni \alpha \mapsto u = \sum_{j=1}^n \alpha_j H_{v_j} + \alpha_j H_{v_j} \in T_\rho \Sigma \cap T_\rho \Sigma^\sigma \]  

(3.1)
is an isomorphism of vector spaces on \( \mathbb{R} \), as it was first observed by Parmeggiani [13, 14] in the case of codimension 1. Indeed, the Hamilton vector fields \( H_{v_j} \) and \( H_{v_j} \), \( j = 1, \ldots, n \), generate \( \mathcal{C}T\Sigma^\sigma \), so that if \( u \in T_\rho \Sigma^\sigma \) if and only if \( u \) has the form in (3.1), for some \( \alpha \in \mathbb{C}^n \). Now, by transversal ellipticity we have \( T_\rho \Sigma = \text{Ker} F_{p_2}(\rho) \) with \( p_2 = \sum_{j=1}^n |v_j|^2 \), and moreover \( F_{p_2}u = \frac{1}{2} \sum_{j=1}^n \sigma(u, H_{v_j})H_{v_j} + \sigma(u, H_{v_j})H_{v_j} \) (see [13]). Hence \( u \in T_\rho \Sigma \cap T_\rho \Sigma^\sigma \) if and only if \( u \) has the form in (3.1) with \( \alpha \in \text{Ker} \mathcal{L}(\rho) \).

It follows that

\[ \dim(T_\rho \Sigma \cap T_\rho \Sigma^\sigma) = 2\dim_{\mathbb{C}} \text{Ker} \mathcal{L}(\rho), \]  

(3.2)
and therefore Hypotheses \( (H_2) \) and \( (H_3) \) are fulfilled as well.

Moreover, we observe that if \( \alpha \neq 0 \), the vector \( u \) in (3.1) satisfies

\[ d\pi(u) = i \sum_{j=1}^n (\alpha_j L_j - \alpha_j T_j) \neq 0, \]

so that \( (H_4) \) is verified.

We now come to (1.2). Since \( \Box_b^{(q)} \) is formally self-adjoint, its subprincipal symbol \( p^*_1 \) is indeed a Hermitian matrix. We are going to use the necessity part of the following lower bound.

**Theorem 3.1.** Let \( X \) be an open subset of \( \mathbb{R}^n \) and \( P = P^* \in \text{OPS}^m(X; \mathbb{C}^N) \) be a \( N \times N \) matrix of classical pseudodifferential operators satisfying \( (H_1) \) and \( (H_2) \). Then the following properties are equivalent:

\[ p^*_m(\rho) + \text{Tr} F_{p_m}(\rho) \text{Id} \geq 0, \ \forall \rho \in \Sigma; \]  

(3.3)
For every compact subset \( K \subset X \) there exists a constant \( C_K > 0 \) such that

\[ (Pu, u) \geq -C_K \|u\|^{2(m-2)/2}, \ \forall u \in C_0^\infty(K; \mathbb{C}^n). \]  

(3.4)

This theorem is a slight generalization of Theorem 22.3.2 and Proposition 22.4.1 of [7], to systems with a principal symbol which is a scalar multiple of the identity matrix. We do not repeat the proof here, which goes exactly as the one given in [7].

Instead, we observe that the Kohn Laplacian certainly satisfies (3.4) \( (m = 2) \), because \( (\Box_b^{(q)} \phi, \phi) \geq 0 \) for every smooth \((0, q)\)-form \( \phi \) with compact support. It follows from Theorem 3.1 that (3.3) is verified by \( \Box_b^{(q)} \), and this concludes the proof.
References


