Mathematical Models for Mesenchymal Motion

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(1) Movement in Fibre Networks
Outline

- (1) Movement in Fibre Networks
- (2) Modelling Tissue Changes
Outline

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- (3) The 1-D Case
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- (4) Drift-Diffusion Limit
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- (5) Discussion, Applications
Outline

- (1) Movement in Fibre Networks
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- (3) The 1-D Case
- (4) Drift-Diffusion Limit
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- (6) Exercise 3
(1) Movement in Fibre Networks

- Fibre network in $\mathbb{R}^n$. 

Distribution of fibres:

$$ q(x; S_n) = \Pr_{S_n}(x) = \int_{S_n} q(x; S_n) \, dS = 1 $$
(1) Movement in Fibre Networks

- Fibre network in $\mathbb{R}^n$.
- Orientation $\sigma \in S^{n-1}$.
(1) Movement in Fibre Networks

- Fibre network in $\mathbb{IR}^n$.
- Orientation $\sigma \in S^{n-1}$.
- Distribution of fibres:

$$q(x, \sigma) = \Pr \left( \Sigma(x) = \sigma | X = x \right).$$

$$\int_{S^{n-1}} q(x, \sigma) d\sigma = 1.$$
Cell Velocities

Set of all possible cell velocities $V$:

$$V = [s_1, s_2] \times S^{n-1}, \quad 0 \leq s_1 \leq s_2 < \infty.$$
Cell Velocities

Set of all possible cell velocities $V$:

$$V = [s_1, s_2] \times S^{n-1}, \quad 0 \leq s_1 \leq s_2 < \infty.$$ 

$$\hat{v} := \frac{v}{\|v\|}$$
$q$ versus $g$

$q$ is a distribution on $S^{n-1}$. To make this into a distribution on $V$ we define

$$g(x, v) := \frac{q(x, \hat{v})}{\omega}$$

where

$$\omega = \int_V q(x, \hat{v}) dv = \begin{cases} \frac{s_2^n - s_1^n}{s^{n-1}}, & s_1 < s_2, n > 1 \\ s_1 = s_2 = s. & \end{cases}$$
Transport Equation

\( p(t, x, v) \): cell distribution at time \( t \), location \( x \), velocity \( v \).

\[
\begin{align*}
  p_t(t, x, v) + v \cdot \nabla p(t, x, v) &= -\mu p(t, x, v) \\
  &\quad + \mu \int_V g(x, v)p(t, x, v')dv'
\end{align*}
\]

\( \mu > 0 \) constant turning rate.

\( g(x, v) \): probability distribution of new chosen directions.
\[ p_t + v \cdot \nabla p = \mu (g\bar{p} - p) = Lp \]

\[ \bar{p} = \int_V p(t, x, v) dv. \]
Timely Varying Tissue

\[ p_t + v \cdot \nabla p = \mu (g(t, x, v)\bar{p} - p) \]
Fibre orientation: $\sigma \in S^{n-1}$. 
(2) Modelling Tissue Changes

Fibre orientation: \( \sigma \in S^{n-1} \).

Undirected fibres (such as collagen), \( \sigma \approx -\sigma \).
(2) Modelling Tissue Changes

Fibre orientation: $\sigma \in S^{n-1}$.

Undirected fibres (such as collagen), $\sigma \approx -\sigma$.

Directed fibres (such as tubulin or actin), $\sigma \neq -\sigma$. 
Tissue equation

In general we are looking for an equation of the form

\[ g_t(t, x, v) = G(v, g, p). \]

with the condition

\[ G(-v, g, p) = G(v, g, p) \]

for undirected fibres.
Fibre Variables

- $Q(t, x, \sigma)$: fibre density.
Fibre Variables

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- $q(t, x, \sigma)$: directional distribution of the fibre network.
Fibre Variables

- $Q(t, x, \sigma)$: fibre density.
- $q(t, x, \sigma)$: directional distribution of the fibre network.
- $g(t, x, v)$: distribution of newly chosen velocities.
Assumptions: Undirected Fibres

\( \sigma \): fibre direction, \( v \): cell velocity.

- (i) If \( \sigma \) is orthogonal to \( v \) then there is a high probability of cutting. (\( |\sigma \cdot \hat{v}| \approx 0 \)).
Assumptions: Undirected Fibres

\( \sigma \): fibre direction, \( \nu \): cell velocity.

- (i) If \( \sigma \) is orthogonal to \( \nu \) then there is a high probability of cutting. \((|\sigma \cdot \hat{\nu}| \approx 0)\).
- (ii) If \( \sigma \) is parallel to \( \nu \) then there is no cutting \((|\sigma \cdot \hat{\nu}| \approx 1)\).
Assumptions: Undirected Fibres

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- (i) If \(\sigma\) is orthogonal to \(v\) then there is a high probability of cutting. (\(|\sigma \cdot \hat{v}| \approx 0\)).

- (ii) If \(\sigma\) is parallel to \(v\) then there is no cutting (\(|\sigma \cdot \hat{v}| \approx 1\)).

Define a projection operator:

\[ \Pi_u(t, x, \sigma) = \frac{1}{\bar{p}(t, x)} \int_V |\sigma \cdot \hat{v}| p(t, x, v) \, dv, \]

which satisfies

\[ 0 \leq \Pi_u \leq 1. \]
Assumptions: Undirected Fibres

- (iii) Frequency of encounter of cell and fibre is proportional to $\bar{p}Q$. 

$$Q_t(t;x) = u(t;x) 1_{\mathbb{R}^n} p Q$$
Assumptions: Undirected Fibres

- (iii) Frequency of encounter of cell and fibre is proportional to \( \bar{p}Q \).
- (iv) \( \kappa > 0 \): protease cutting efficiency.
Assumptions: Undirected Fibres

- (iii) Frequency of encounter of cell and fibre is proportional to $\bar{p}Q$.
- (iv) $\kappa > 0$: protease cutting efficiency.

$$Q_t(t, x, \sigma) = \kappa \left( \Pi_u(t, x, \sigma) - 1 \right) \bar{p} \ Q$$
We define $q$ by

$$q(t, x, \sigma) = \frac{Q(t, x, \sigma)}{\int_{S^{n-1}} Q(t, x, \sigma) d\sigma}.$$
Equation for $q(t, x, \sigma)$

We define $q$ by

$$q(t, x, \sigma) = \frac{Q(t, x, \sigma)}{\int_{S^{n-1}} Q(t, x, \sigma) \, d\sigma}.$$  

Then

$$q_t(t, x, \sigma) = \kappa(\Pi_u(t, x, \sigma) - A_u(t, x))\bar{\rho}q$$

$$A_u(t, x) = \int_{S^{n-1}} \Pi_u(t, x, \sigma)q(t, x, \sigma) \, d\sigma$$
\[
q_t(t, x, \sigma) = \kappa (\Pi_u(t, x, \sigma) - A_u(t, x)) \bar{p}q
\]
\[
A_u(t, x) = \int_{S^{n-1}} \Pi_u(t, x, \sigma) q(t, x, \sigma) d\sigma
\]
The above equation for $q$ can be reformulated in terms of the probability distribution of new chosen velocities

$$g(t, x, v) = \frac{q(t, x, \hat{v})}{\omega}.$$
Equation for $g$

The above equation for $q$ can be reformulated in terms of the probability distribution of new chosen velocities

$$g(t, x, v) = \frac{q(t, x, \hat{v})}{\omega}.$$

$$g_t(t, x, v) = \kappa(\Pi_u(t, x, \hat{v}) - B_u(t, x))\bar{p}(t, x)g(t, x, v)$$

$$B_u(t, x) = \int_V \Pi_u(t, x, \hat{v})g(t, x, v)dv$$
Assumptions: Directed Fibres

\(\sigma\): fibre direction, \(\nu\): cell velocity.

- (i) If \(\sigma\) is orthogonal to \(\nu\) then there is a high probability of cutting. \((\sigma \cdot \hat{\nu} \approx 0)\).
Assumptions: Directed Fibres

$\sigma$: fibre direction, $\nu$: cell velocity.

- (i) If $\sigma$ is orthogonal to $\nu$ then there is a high probability of cutting. ($\sigma \cdot \hat{\nu} \approx 0$).
- (ii) If $\sigma$ is parallel to $\nu$ then there is no cutting ($\sigma \cdot \hat{\nu} \approx 1$).
Assumptions: Directed Fibres

\( \sigma \): fibre direction, \( v \): cell velocity.

- (i) If \( \sigma \) is orthogonal to \( v \) then there is a high probability of cutting. \( (\sigma \cdot \hat{v} \approx 0) \).
- (ii) If \( \sigma \) is parallel to \( v \) then there is no cutting \( (\sigma \cdot \hat{v} \approx 1) \).
- (iii) If \( \sigma \) points opposite to \( v \), then the probability of cutting is even higher \( (\sigma \cdot \hat{v} \approx -1) \).
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- (i) If \(\sigma\) is orthogonal to \(v\) then there is a high probability of cutting. \((\sigma \cdot \hat{v} \approx 0)\).
- (ii) If \(\sigma\) is parallel to \(v\) then there is no cutting \((\sigma \cdot \hat{v} \approx 1)\).
- (iii) If \(\sigma\) points opposite to \(v\), then the probability of cutting is even higher \((\sigma \cdot \hat{v} \approx -1)\).

Define a projection operator:

\[
\Pi_d(t, x, \sigma) = \frac{1}{\bar{p}(t, x)} \int_V \sigma \cdot \hat{v} p(t, x, v) dv,
\]

which satisfies \(-1 \leq \Pi_d \leq 1\).
Assumptions: Directed Fibres

(iv) Frequency of encounter of cell and fibre is proportional to $\bar{p}Q$. 
Assumptions: Directed Fibres

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(v) $\kappa > 0$: protease cutting efficiency.
Assumptions: Directed Fibres

- (iv) Frequency of encounter of cell and fibre is proportional to $\bar{p}Q$.
- (v) $\kappa > 0$: protease cutting efficiency.

\[ Q_t(t, x, \sigma) = \kappa \left( \Pi_d(t, x, \sigma) - 1 \right) \bar{p} Q \]
Equation for $q(t, x, \sigma)$

$$q_t(t, x, \sigma) = \kappa \left( \Pi_d(t, x, \sigma) - A_d(t, x) \right) \bar{p} q$$

$$A_d(t, x) = \int_{S^{n-1}} \Pi_d(t, x, \sigma) q(t, x, \sigma) d\sigma$$
Equation for \( q(t, x, \sigma) \)

\[
q_t(t, x, \sigma) &= \kappa (\Pi_d(t, x, \sigma) - A_d(t, x)) \bar{p}q \\
A_d(t, x) &= \int_{S^{n-1}} \Pi_d(t, x, \sigma) q(t, x, \sigma) d\sigma
\]

Equation for \( g \):

\[
g_t(t, x, v) &= \kappa (\Pi_d(t, x, \hat{v}) - B_d(t, x)) \bar{p}(t, x) g(t, x, v) \\
B_d(t, x) &= \int_V \Pi_d(t, x, \hat{v}) g(t, x, v) dv
\]
\[
\begin{align*}
pt + v \cdot \nabla p &= \mu(g \bar{p} - p) \\
g_t(t, x, v) &= G(\hat{v}, p, g)
\end{align*}
\]
The 1-D Case: Directed

\[ S^0 = \{+1, -1\}, \quad V = \{+s, -s\} \]
The 1-D Case: Directed

\[ S^0 = \{+1, -1\}, \quad V = \{+s, -s\} \]

\[ p^+ := p(t, x, +s) \quad p^- := p(t, x, -s) \]

\[ g^+ := g(t, x, +s) \quad g^- := g(t, x, -s) \]
The 1-D Case: Directed

\[ S^0 = \{+1, -1\}, \quad V = \{+s, -s\} \]

\[ p^+ := p(t, x, +s) \quad p^- := p(t, x, -s) \]
\[ g^+ := g(t, x, +s) \quad g^- := g(t, x, -s) \]

\[ p_t^+ + s p_x^+ = -\mu p^+ + \mu g^+ (p^+ + p^-) \]
\[ p_t^- - s p_x^- = -\mu p^- + \mu g^- (p^+ + p^-) \]
\[ \Xi_d(t, x, \sigma) = \frac{1}{\bar{p}} \int_V \sigma \cdot \hat{v} p(t, x, v) dv \]
\[ = \frac{1}{p^+ + p^-} (\sigma (+1)p^+ + \sigma (-1)p^-) \]
\[ \Pi_d(t, x, \sigma) = \frac{1}{\bar{p}} \int_V \sigma \cdot \hat{v} p(t, x, v) dv \]

\[ = \frac{1}{p^+ + p^-} (\sigma (+1)p^+ + \sigma (-1)p^-) \]

\[ \Pi_d^+ := \Pi_d(t, x, +1) = \frac{p^+ - p^-}{p^+ + p^-} \]
\[ \Pi_d^- := \Pi_d(t, x, -1) = \frac{p^- - p^+}{p^+ + p^-} \]
\[ B_d(t, x) = \int \Pi_d g dv \]
\[ = \Pi_d^+ g^+ + \Pi_d^- g^- \]
\[ = \frac{p^+ - p^-}{p^+ + p^-} (g^+ - g^-) \]
\[ g_t^+ = \kappa (\Pi_d^+ - B_d) \bar{p} g \]
\[ = \kappa \left( \frac{p^+ - p^-}{p^+ + p^-} - \frac{p^+ - p^-}{p^+ + p^-} (g^+ - g^-) \right) (p^+ + p^-) g^+ \]
\[ = \kappa (p^+ - p^-) (g^- - g^+ + 1) g^+ \]
\[ g_t^+ = \kappa (\Pi_d^+ - B_d) \bar{p} g \]
\[ = \kappa \left( \frac{p^+ - p^-}{p^+ + p^-} - \frac{p^+ - p^-}{p^+ + p^-} (g^+ - g^-) \right) (p^+ + p^-) g^+ \]
\[ = \kappa (p^+ - p^-) (g^- - g^+ + 1) g^+ \]

\[ g_t^- = \kappa (p^+ - p^-) (g^- - g^+ - 1) g^- \]
1-D Model, Directed

\[
\begin{align*}
\frac{p_t^+}{p_t^-} + sp_x^+ & = -\mu p^+ + \mu g^+(p^+ + p^-) \\
\frac{p_t^-}{p_t^-} - sp_x^- & = -\mu p^- + \mu g^-(p^+ + p^-) \\
g^+ & = \kappa(p^+ - p^-)(g^- - g^+ + 1)g^+ \\
g^- & = \kappa(p^+ - p^-)(g^- - g^+ - 1)g^-
\end{align*}
\]
1-D Case, Undirected

\[ \Pi^\pm_u = \frac{1}{p^+ + p^-}(|\sigma|p^+ + |\sigma|p^-) = 1 \]
1-D Case, Undirected

\[ \Pi_u^\pm = \frac{1}{p^+ + p^-}(|\sigma|p^+ + |\sigma|p^-) = 1 \]

Hence

\[ g_t^\pm = 0. \]
1-D Model, Undirected

\[
\begin{align*}
pt^+ + sp_x^+ &= -\mu p^+ + \mu g^+ (p^+ + p^-) \\
pt^- - sp_x^- &= -\mu p^- + \mu g^- (p^+ + p^-) \\
g^+ &= 0 \\
g^- &= 0
\end{align*}
\]
(3) Drift-Diffusion Limits

- (a) moment expansions, (Hillen, Preziosi)
(3) Drift-Diffusion Limits

- (a) moment expansions, (Hillen, Preziosi)
- (b) parabolic scaling, (Othmer, Hillen and others)
(3) Drift-Diffusion Limits

- (a) moment expansions, (Hillen, Preziosi)
- (b) parabolic scaling, (Othmer, Hillen and others)
- (c) hydrodynamic scaling
  (see also Dolak + Schmeiser 2005)
Hydrodynamic Scaling for Directed Tissue

\[ \theta = \varepsilon t, \quad \xi = \varepsilon x, \quad \varepsilon p_{\theta} + \varepsilon u \cdot \nabla \xi p = \mathcal{L} \]

\[ \text{Hence the kernel of } \mathcal{L} \text{ is spanned by } g. \]

\[ p(\theta; \xi; v) = p(\theta; x) g(\xi; v) + g^* \]
Hydrodynamic Scaling for Directed Tissue

\[ \theta = \varepsilon t, \quad \xi = \varepsilon x, \quad \varepsilon p_\theta + \varepsilon v \cdot \nabla_\xi p = \mathcal{L}p \]

kernel of \( \mathcal{L} \):

\[ \mathcal{L}\phi = 0 \iff \phi = \overline{\phi}g. \]

Hence the kernel of \( \mathcal{L} \) is spanned by \( g \).
\[ \theta = \varepsilon t, \quad \xi = \varepsilon x, \quad \varepsilon p_\theta + \varepsilon v \cdot \nabla_\xi p = \mathcal{L}p \]

kernel of \( \mathcal{L} \):

\[ \mathcal{L}\phi = 0 \iff \phi = \bar{\phi}g. \]

Hence the kernel of \( \mathcal{L} \) is spanned by \( g \).

We use the Chapman Enskog expansion:

\[ p(\theta, \xi, v) = \bar{p}(\theta, \xi)g(\theta, \xi, v) + \varepsilon g^\perp(\theta, \xi, v) \]

with

\[ \int_V g^\perp dv = 0. \]
We substitute the Chapman-Enskog expansion into the scaled transport equation and get:

\[ \varepsilon \bar{p}_\theta g + \varepsilon^2 g_\theta \perp + \varepsilon (\mathbf{v} \cdot \nabla)(\bar{p}g) + \varepsilon^2 (\mathbf{v} \cdot \nabla)g \perp = \varepsilon \mathcal{L}g \perp. \]
We substitute the Chapman-Enskog expansion into the scaled transport equation and get:

\[ \varepsilon \bar{p}_\theta g + \varepsilon^2 g^\perp_\theta + \varepsilon (v \cdot \nabla) (\bar{p} g) + \varepsilon^2 (v \cdot \nabla) g^\perp = \varepsilon \mathcal{L} g^\perp. \]

We integrate to obtain

\[ \bar{p}_\theta + \nabla \cdot \left( \int_V v g \, dv \, \bar{p} + \varepsilon \int_v v g^\perp \, dv \right) = 0 \]
We substitute the Chapman-Enskog expansion into the scaled transport equation and get:

\[ \varepsilon \bar{p}_\theta g + \varepsilon^2 g'_\theta + \varepsilon (v \cdot \nabla)(\bar{p}g) + \varepsilon^2 (v \cdot \nabla)g' = \varepsilon \mathcal{L}g'. \]

We integrate to obtain

\[ \bar{p}_\theta + \nabla \cdot \left( \int_V vg \, dv \bar{p} + \varepsilon \int V g' \, dv \right) = 0 \]

Which can be written as

\[ \bar{p}_\theta + \nabla \cdot \left( u_c \bar{p} + \varepsilon \int V q' \, dv \right) = 0 \]

\[ u_c = \int_V vg \, dv \]
Leading Order

\[ \bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = 0. \]

\[ u_c(\theta, \xi) = \int_V v_g(\theta, \xi, \hat{v}) dv \]
The drift velocity $u_c$ is the mean value of $g$ over $V$. 

\[ \bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = 0. \]

\[ u_c(\theta, \xi) = \int_V v g(\theta, \xi, \hat{v}) dv \]
Leading Order

\[ \bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = 0. \]

\[ u_c(\theta, \xi) = \int_V v_g(\theta, \xi, \hat{v}) dv \]

The drift velocity \( u_c \) is the mean value of \( g \) over \( V \).

Note that \( u_c = 0 \) for undirected tissue.
Correction term $g^\perp$

We find

$$
    g^\perp = - \frac{1}{\mu} \left( g(v - u_c) \cdot \nabla \bar{p} + (v \cdot \nabla g - g \nabla \cdot u_c + g_{\theta}) \bar{p} \right) + \mathcal{O}(\varepsilon).
$$
Correction term $g^\perp$

We find

$$g^\perp = -\frac{1}{\mu} \left( g(v - u_c) \cdot \nabla \bar{p} + (v \cdot \nabla g - g \nabla \cdot u_c + g_\theta)\bar{p} \right) + \mathcal{O}(\varepsilon).$$

which we substitute into

$$\bar{p}_\theta + \nabla \cdot \left( \int_V vg \, dv \, \bar{p} + \varepsilon \int vg^\perp \, dv \right) = 0$$

and we obtain after rearrangements ...
\[
\bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = \frac{\varepsilon}{\mu} \nabla \left[ \nabla (D_c \bar{p}) + \left( u_c (\nabla \cdot u_c) + u_{c,\theta} \right) \bar{p} \right]
\]

\[
D_c(\theta, \xi) = \int_V (v - u_c)(v - u_c)^T g(\theta, \xi, v) dv
\]

\[
u_c(\theta, \xi) = \int_V v g(\theta, \xi, v) dv
\]
With Diffusion Correction

$$\bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = \frac{\varepsilon}{\mu} \nabla \left[ \nabla (D_c \bar{p}) + \left( u_c (\nabla \cdot u_c) + u_{c,\theta} \right) \bar{p} \right]$$

$$D_c(\theta, \xi) = \int_V (v - u_c)(v - u_c)^T g(\theta, \xi, v) dv$$

$$u_c(\theta, \xi) = \int_V vg(\theta, \xi, v) dv$$

The diffusion tensor $D_c$ is the variance-covariance matrix of the distribution $g$ over $V$. 
The $g$-equation

\[ g_t = \kappa(\Pi_d(\theta, x, \sigma)) - B_d)^\rho g, \]

with

\[ \Pi_d(t, x, \sigma) = \frac{1}{\bar{p}(t, x)} \int V \sigma \cdot \hat{v}p(t, x, v) dv \]
The $g$-equation

\[ g_t = \kappa(\Pi_d(\theta, x, \sigma)) - B_d)\bar{p}g, \]

with

\[
\Pi_d(t, x, \sigma) = \frac{1}{\bar{p}(t, x)} \int_V \sigma \cdot \hat{\nu}p(t, x, v)dv
\]

Use again the Chapman-Enskog expansion $p = \bar{p}g + \varepsilon g^\perp$:

\[
\Pi_d(t, x, \sigma) = \gamma \sigma \cdot u_c
\]

\[
B_d(t, x) = \gamma^2 u_c^2,
\]

where

\[
\gamma^{-1} = \begin{cases} 
(s_{n+1}^2 - s_{n+1}^2)/n\omega & \text{for } s_2 < s_1 \\
s^n & \text{for } s_1 = s_2 = s.
\end{cases}
\]
Tissue Equation

\[ \varepsilon g_\theta = \kappa (\gamma \sigma \cdot u_c - \gamma^2 u_c^2) \bar{p} g \]
\[
\bar{p}_\theta + \nabla \cdot (u_c \bar{p}) = \frac{\varepsilon}{\mu} \nabla \left[ \nabla (D_c \bar{p}) + \left( u_c (\nabla \cdot u_c) + u_{c,\theta} \right) \bar{p} \right]
\]

\[
\varepsilon g_\theta = \kappa (\gamma \sigma \cdot u_c - \gamma^2 u_c^2) \bar{p} q
\]

\[
D_c(\theta, \xi) = \int_V (v - u_c)(v - u_c)^T g(\theta, \xi, v) dv
\]

\[
u_c(\theta, \xi) = \int_V v g(\theta, \xi, v) dv
\]
Macroscopic Model, Undirected Tissue

\[ u_c = 0 \]
Macroscopic Model, Undirected Tissue

\[ u_c = 0 \]

\[
\bar{p}_\theta = \frac{\varepsilon}{\mu} \nabla \left[ \nabla (D_c \bar{p}) \right]
\]

\[
\varepsilon g_\theta = \kappa (\Pi_u (v, \bar{p}, g) - B_u) \bar{p} q
\]

\[
D_c (\theta, \xi) = \int_V \nu \nu^T g(\theta, \xi \nu) d\nu
\]

\[
\Pi_d (\theta, \xi, \sigma) = \int_V |\sigma \cdot \hat{\nu}| g(\theta, \xi, \nu) d\nu
\]

\[
B_d = \int \Pi_d (\theta, \xi, \hat{\nu}) g(\theta, \xi, \nu) d\nu.
\]
Discussion

- *Dickinson + Tranquillo 1996, Dickinson 2000:* They study the case of $q$ independent of time, or $q$ varies slowly over time. They use projection methods.
Discussion

- **Dickinson + Tranquillo 1996, Dickinson 2000:** They study the case of \( q \) independent of time, or \( q \) varies slowly over time. They use projection methods.

- **Barocas + Tranquillo, 1997:** Anisotropic Biphasic Theory (ABT). Use of mass and momentum balance equations for the cell-tissue phase and an interstitial fluid phase.
Discussion

Dickinson + Tranquillo 1996, Dickinson 2000: They study the case of $q$ independent of time, or $q$ varies slowly over time. They use projection methods.


Dallon, Sherratt, Maini, 1997: Wound healing and tissue formation.
Example: Spatially Homogeneous, directed

Assume

\[ q(\theta, \xi, \sigma) = q(\sigma). \]

Then

\[ \bar{\rho}_\theta + \nabla \cdot (u_c \bar{\rho}) = \frac{\varepsilon}{\mu} \nabla \cdot D_c \nabla \bar{p}, \]

and \( D_c \) is a constant matrix.
Exercise 3: Directed, Uni-directional Tissue

Assume

\[ q(\theta, \xi, \sigma) := \begin{cases} 
0.5 & \text{for } \sigma = e_1, \\
0.5 & \text{for } \sigma = -e_1, \\
0 & \text{otherwise.} 
\end{cases} \]

and assume that \( V = sS^{n-1} \).

Find the drift diffusion limit.
(Note: You need to calculate \( \omega, u_c, D_c \).)