Transport Equations

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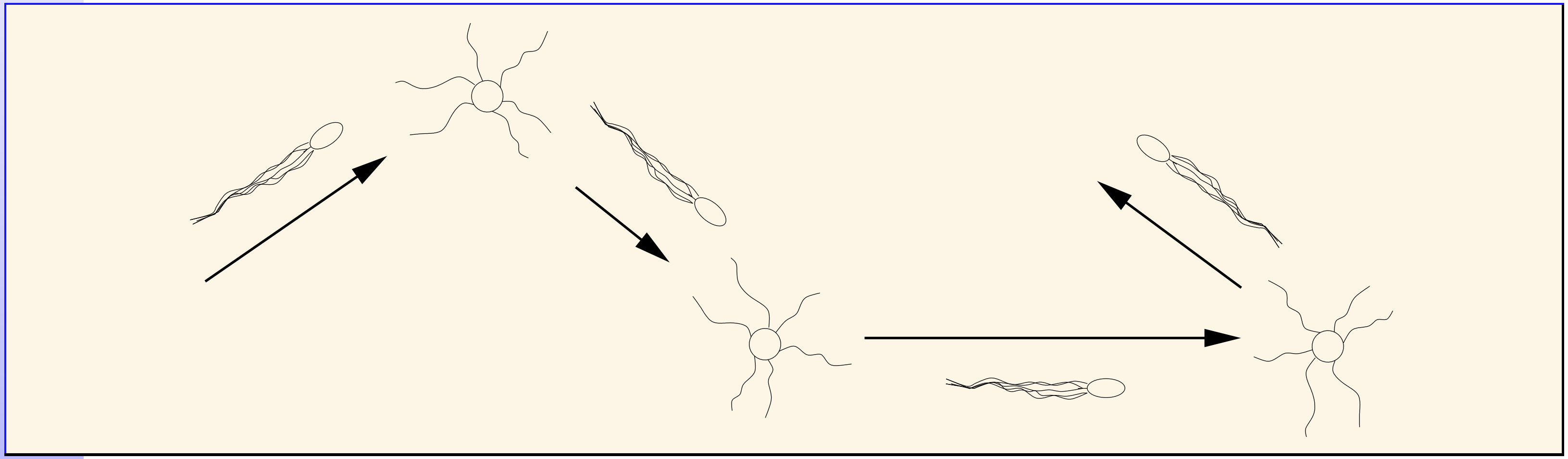
Run and Tumble



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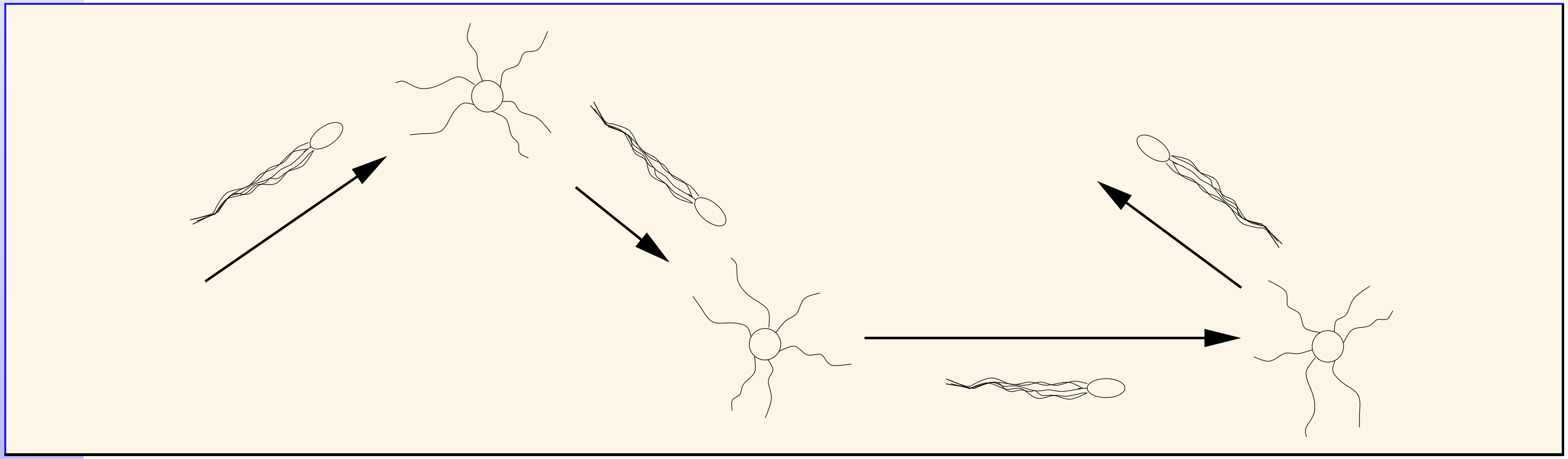

$p(t, x, v)$: density of cells at time $t \geq 0$, location $x \in \Omega$ and velocity $v \in V$,

V compact and symmetric.

Directed Movement

The equation

$$p_t(t,x,v) + v \cdot \nabla p(t,x,v) = 0$$

is solved by

$$p(t,x,v) = \varphi(x - vt),$$

which describes movement in direction of v.

With Directional Changes

μ: turning rate.

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Transport equation:

$$
\begin{aligned}
p_t(t,x,v) + v \cdot \nabla p(t,x,v) &= -\mu p(t,x,v) \\
&\quad + \mu \int_V T(v,v') p(t,x,v') dv'
\end{aligned}
$$

Stroock, 1974, Alt 1980, 1981,

Othmer, Dunbar, Alt, 1988, Dickinson 2000

Hillen, Othmer, 2000, 2001

Notation

For $\varphi \in L^2(V)$ we write

$$\mathcal{T}\varphi(v) := \int_V T(v, v')\varphi(v')dv'$$

$$\mathcal{L}\varphi := -\mu(I - \mathcal{T})\varphi \quad \text{turning operator}$$

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Then the transport equation reads

$$p_t + v \cdot \nabla p = \mathcal{L}p$$

Some General Assumptions

(Hillen and Othmer, SIAP 2000)

- $\mathcal{T}$ is a compact Hilbert-Schmidt operator.

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- $\mathcal{T}$ is u_0-positive (Krasnoselskii).
- $L^2(V) = \langle 1 \rangle \oplus \langle 1 \rangle^\perp$,

$$\|\mathcal{T}|_{\langle 1 \rangle^\perp}\| < 1, \quad \text{dissipativity}$$

Spectral Theorem

(Hillen, Othmer, SIAP 2000)

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- For all $\psi \in \langle 1 \rangle^\perp$ we have

$$\int_V \psi \mathcal{L} \psi dv \leq -\nu \|\psi\|_2^2,$$

with

$$\nu = \mu(1 - \|\mathcal{T}|_{\langle 1 \rangle^\perp}\|).$$

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- There exists a pseudo inverse

$$\mathcal{F} = \left(\mathcal{L}|_{\langle 1 \rangle^\perp}\right)^{-1}.$$

Example 1: Pearson Walk

$$V = sS^{n-1}, \qquad T(v, v') = \frac{1}{|V|}$$

$$\begin{aligned} p_t + v \cdot \nabla p &= \mathcal{L}p \\ &= \mu \left(\frac{\bar{p}}{|V|} - p \right) \\ \bar{p}(t, x) &= \int_V p(t, x, v)\, dv. \end{aligned}$$

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Then

$$y = -\frac{1}{\mu}\psi.$$

and

$$\mathcal{F} = \left(\mathcal{L}|_{\langle 1 \rangle^\perp}\right)^{-1} = -\frac{1}{\mu}.$$

(1) The Parabolic Scaling

$$\tau = \varepsilon^2 t \qquad \xi = \varepsilon x$$

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Hilbert expansion:

$$p(\tau, \xi, v) = p_0(\tau, \xi, v) + \varepsilon p_1(\tau, \xi, v) + \varepsilon^2 p_2(\tau, \xi, v) + \dots$$

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Compare orders of ε:

$$\begin{aligned}
\mathcal{O}(1): &\qquad 0 = \mathcal{L}p_0 \\
\mathcal{O}(\varepsilon): &\qquad v \cdot \nabla p_0 = \mathcal{L}p_1 \\
\mathcal{O}(\varepsilon^2): &\qquad p_{0,\tau} + v \cdot \nabla p_1 = \mathcal{L}p_2 \\
\vdots &
\end{aligned}$$

$$\mathcal{O}(1): \qquad 0 = \mathcal{L}p_0$$

Hence $p_0(\tau, \xi)$ does not depend on $v \in V$.

$$\mathcal{O}(\varepsilon): \qquad v \cdot \nabla p_0 = \mathcal{L} p_1$$

Can be solved for p_1 if $v \cdot \nabla p_0 \in \langle 1 \rangle^{\perp}$.

I.e.

$$0 = \int_V v \cdot \nabla p_0 dv = \int_V v dv \cdot \nabla p_0.$$

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Solvability:

$$\int_V p_{0,\tau} dv + \int_V v \cdot \nabla \left(\mathcal{F} v \cdot \nabla p_0 \right) dv = 0$$

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Solvability:

$$\int_V p_{0,\tau} dv + \int_V v \cdot \nabla \left(\mathcal{F} v \cdot \nabla p_0 \right) dv = 0$$

$$|V| p_{0,\tau} + \nabla \cdot \int_V v \mathcal{F} v^T dv \cdot \nabla p_0 = 0$$

Diffusion Matrix

Define a diffusion matrix

$$D := -\frac{1}{|V|}\int_V v\mathcal{F}v^T dv.$$

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$$p_{0,\tau} = \nabla D \nabla p_0.$$

Pearson walk

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We can show that for $V = sS^{n-1}$ we have

$$\int_V vv^T dv = \frac{|V|s^2}{n} I_n,$$

where I_n is the $(n \times n)$-identity.

Person walk

Hence

$$D = \frac{s^2}{\mu n} I_n$$

isotropic diffusion.

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The parabolic limit of the Pearson walk is

$$p_{0,\tau} = d\Delta p_0, \qquad d = \frac{s^2}{\mu n}.$$

Approximation Theorem

Assume p_0 solves the diffusion limit equation and

$$\begin{aligned} p_1 &= \mathcal{F} v \cdot \nabla p_0, \\ p_2 &= \mathcal{F} p_{0,\tau} + v \cdot \nabla p_1, \end{aligned}$$

then for all $\vartheta > 0$ there exists a $C > 0$ such that

$$\|p - (p_0 + \varepsilon p_1 + \varepsilon^2 p_2)\|_2 \leq C\varepsilon^3$$

for all

$$0 < \frac{\vartheta}{\varepsilon^2} < t < \infty.$$

(2) The Hydrodynamic (Hyperbolic) Scaling

$$\theta = \varepsilon t \qquad \xi = \varepsilon x$$

$$\varepsilon p_\theta + \varepsilon v \cdot \nabla_\xi p = \mathcal{L} p. \qquad \text{(H1)}$$

(2) The Hydrodynamic (Hyperbolic) Scaling

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kernel $\mathcal{L} = \langle 1 \rangle$, hence we consider the Chapman-Enskog expansion:

$$p(\theta, \xi, v) = \alpha(\theta, \xi) + \varepsilon \alpha^\perp(\theta, \xi, v),$$

with $\alpha^\perp \in \langle 1 \rangle^\perp$.

Substitute into (H1):

$$\begin{aligned} \varepsilon\alpha_\theta + \varepsilon^2\alpha_\theta^\perp + \varepsilon v\cdot\nabla\alpha + \varepsilon^2 v\cdot\nabla\alpha^\perp &= \mathcal{L}(\alpha + \varepsilon\alpha^\perp) \\ &= \varepsilon\mathcal{L}\alpha^\perp \end{aligned}$$

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Integrate (H2):

$$|V|\alpha_\theta + \varepsilon\int_V \alpha_\theta^\perp + \int_V v\cdot\nabla\alpha + \varepsilon\int_V v\cdot\nabla\alpha^\perp = \int_V \mathcal{L}\alpha^\perp.$$

$$\int \alpha^{\perp} dv = 0 \quad \Longrightarrow \int \alpha^{\perp}_{\theta} dv = 0.$$

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Hence

$$|V| \alpha_{\theta} + \varepsilon \int_{V} v \cdot \nabla \alpha^{\perp} dv = 0.$$

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Now we substitute (H3) into (H2) to find

$$\begin{aligned}\mathcal{L}\alpha^\perp &= v \cdot \nabla \alpha \\ &\quad +\varepsilon\left(-\frac{1}{|V|}\int_V v \cdot \nabla \alpha^\perp dv + \alpha_\theta^\perp + v \cdot \nabla \alpha^\perp\right)\end{aligned}$$

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The right hand side is to leading order in $\langle 1 \rangle^\perp$, hence

$$\alpha^\perp = \mathcal{F} v \cdot \nabla \alpha + \mathcal{O}(\varepsilon).$$

We substitute $\alpha^{\perp}$ into (H3) and erase $\mathcal{O}(\varepsilon^2)$-terms:

$$\begin{aligned} \alpha_\theta &= -\frac{\varepsilon}{|V|}\int_V v\cdot\nabla\mathcal{F}(v\cdot\nabla\alpha)\,dv \\ &= -\frac{\varepsilon}{|V|}\nabla\int_V v\mathcal{F}v^T dv\nabla\alpha \\ &= \varepsilon\nabla D\nabla\alpha \end{aligned}$$

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If $\tau = \varepsilon\theta$, then this is identical to the parabolic limit.

Pearson walk

Diffusion equation

$$\alpha_\theta = \varepsilon d \Delta \alpha, \qquad d = \frac{s^2}{\mu n}$$

Summary

- parabolic scaling:

  $\tau = \varepsilon^2 t, \xi = \varepsilon x,$

  Hilbert expansion in ε. Use solvability conditions.

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  Hilbert expansion in ε. Use solvability conditions.

- hydrodynamic scaling

  $\theta = \varepsilon t, \xi = \varepsilon x$

  Chapman-Enskog expansion, where the part from $\langle 1 \rangle^\perp$ is of lower order: $p = \alpha + \varepsilon \alpha^\perp$.

Applications

$\mathcal{O}(1)$ Perturbations

Let b be any given external information. We consider

$$T(v, v') = T_0(v, v') + T_1(v, v', b).$$

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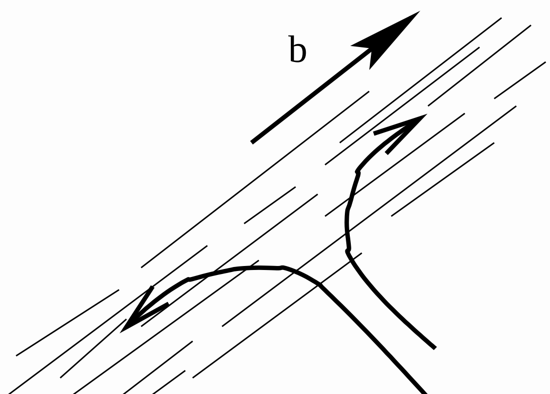

The parabolic limit gives nonisotropic diffusion

$$D = \frac{s^2}{\mu n}\left(I + \frac{|V|s^2}{n}\kappa b b^T \left(I - \frac{|V|s^2}{n}\kappa b b^T\right)^{-1}\right)$$

$\mathcal{O}(\varepsilon)$-Perturbations

$$T(v, v') = T_0(v, v') + \varepsilon T_1(v, v', S(.))$$

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parabolic limit:

$$\begin{aligned} p_{0,\tau} &= \nabla(D\nabla p_0 - u_c p_0) \\ u_c &= -\frac{\mu}{|V|} \int_V c\mathcal{F}\beta_1(v) dv \end{aligned}$$

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The parabolic limit reads:

$$p_{0,\tau} = \nabla(d \nabla p_0 - p_0 \chi(S) \nabla S)$$
$$d = \frac{s^2}{\mu n}, \qquad \chi(S) = \frac{k_1(S)\mu}{|V|}$$

Summary

From the knowledge of individual movement characteristics such as velocity $v \in V$, turning rate μ, directional distribution $T(v, v')$, we can deduce macroscopic quantities, such as diffusion tensor D, and drift velocity u_c.

Exercise 2

In one space dimension with two possible velocities, $V = \{+s, -s\}$, and with $T(v, v') = 1/2$ we obtain the Goldstein-Kac model for correlated random walk.

1. Write $p(t, x, \pm s) = u^{\pm}(t, x)$ and show that the Pearson walk has the form

$$\begin{aligned} u_t^+ + s u_x^+ &= \mu/2(u^- - u^+) \\ u_t^- - s u_x^- &= \mu/2(u^+ - u^-) \end{aligned}$$

2. Define $\mathcal{L}$ on $L^2(V) = \mathbb{R}^2$ and find its kernel and the pseudo-inverse.
3. Find the parabolic limit.
4. Find the hydrodynamic limit.