Invariants and exponential rate of convergence to steady state in the renewal equation

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Abstract We consider the renewal equation (also called McKendrick-VonFoerster) equation that arises as a simple model for structured population dynamics. We use an entropy approach to prove the exponential convergence in long time to the steady state, after renormalization by a damping factor to compensate for the system growth.

Our approach, by opposition with the original method of Feller based on Laplace transform, uses the direct variable. It uses new invariants of the equation, to which we systematically associate a condition for the exponential convergence.

Key-words Renewal equation, Entropy method, Contraction principle, long time asymptotic.

1 Main result

In this paper, we consider the renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \qquad t \geq 0, x \geq 0, \\ \\ n(t, x=0) = \int_0^\infty B(x) n(t,x) dx, \end{cases} \tag{1}$$

with initial condition $n(t=0,x) = n^0(x)$. This equation is one of the most standard in biology and describes age structured populations thanks to the density $n(t,x)$ of individuals of age x at time t. It is also known as VonFoerster equation but was introduced by McKendrick in 1926. The biological motivations and classical results can be found in [9, 16, 6, 19]. Because it can be reduced to a Volterra integral equation and because it can be solved through the Laplace transform, many results are known on this equation. Especially, it is known that after renormalization by a time exponential factor, the density $n(t,x)$ converges to a steady state with an exponential rate (some kind of spectral gap property). Our result seems however to use much weaker assumptions what is known in the literature after and basically follows Feller's original method [5]. The real target is to study more general models of structured populations. For instance, one can see this equation as a particularly simple case of the general cell division equation

$$\frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + \big(B(x)+d(x)\big)n(t,x) = \int_{y=x}^{\infty} b(y,x)n(t,y)dy.$$

Indeed, when $b(x,y) = B(x)[\delta(y=0)+\delta(y=x)]$ one recovers (1). For this equation including (1), and many others, and under fairly general assumptions on b, the Generalized Relative Entropy Method shows that the solution $n(t,x)$ converges to a steady state, [10, 11]. Our motivation here is to understand how the GRE method can also be used to prove the exponential convergence in time to this steady state. Such a result is known in the case when $B(x)$ is close to a constant and $b(x,y) = 2B\delta(y=x/2)$, [15], but again the method does not seem to apply to more general cases. It is also known under the assumption that the 'smallest mother cell is at least twice larger that the larger daughter cells', and extensions of this assumptions, that reduces the equation to systems of renewal equations [9]. As a first step in this programme we prove the exponential rate of convergence to a steady state for (1). The question of return to steady state is fundamental for biological interpretation and also for comparison with experiments and parameters estimations. See for instance [2, 3].

We would like to point out that the exponential decay to steady state is a classical and important topic in PDEs, linear or nonlinear. To prove that entropy dissipation controls the entropy itself is the standard method, and is related to deep concepts as Logarithmic Sobolev inequalities [1, 18, 17] or requires specific analysis [7, 4, 13] and the references therein. In the nonconservative problems at hand, entropy dissipation does not control the entropy itself and we do not know of general concepts to attack the problem of exponential decay by PDE methods. The method of invariants we introduce here might be one.

Let us assume

$$\begin{cases} B(\cdot) \in L^1_+(\mathbb{R}^+) \qquad (\text{non}-\text{negative integrable functions}), \\ d(\cdot) \geq 0, \quad \int_0^x d(y)\, dy \in C^0(\mathbb{R}^+), \end{cases} \tag{2}$$

$$\exists \lambda_0 > 0, \quad \int_{x\geq 0} B(x) e^{-\int_0^x [\lambda_0 + d(y)]\, dy}\, dx = 1. \tag{3}$$

The first two assumptions are very general and are related to existence of a steady state. They discard for instance $d = 0$, $B(x) = \delta(x = a_\sharp)$ for which periodic solutions n exist.

Last, we need an assumption for the initial data, namely

$$\exists K^0 > 0, \quad |n^0(x)| \leq K^0 N(x) \qquad a.e.. \tag{4}$$

Here $N(x)$ denotes the stationary solution associated with the eigenvalue problem for eigenelement λ_0. That is

$$\begin{cases} \frac{\partial}{\partial x} N(x) + [d(x) + \lambda_0] N(x) = 0, & x \geq 0, \\ N(x=0) = \int_0^\infty B(x) N(x) dx, \\ N(\cdot) \geq 0, \qquad \int_0^\infty N(x) dx = 1. \end{cases} \tag{5}$$

Note that the solution to (5) is simply given by

$$N(x) = \rho e^{-\int_0^x [\lambda_0 + d(y)]\, dy}, \qquad \frac{1}{\rho} = \int_0^\infty e^{-\int_0^x [\lambda_0 + d(y)]\, dy} dx. \tag{6}$$

Notice that our assumptions imply that $N(\cdot)$ is well defined as a L^∞ solution (say in distribution sense), it is non-increasing and therefore BV and thus $[d(x) + \lambda_0] N(x) \in M_{\text{loc},+}(\mathbb{R}^+)$. This formalism contains in particularly a case which is often treated (although very doubtful in terms of its interpretation)

$$\exists a_\sharp, \quad \int_0^{a_\sharp} d(x)\, dx = \infty,$$

and reduces everything to compact support because $N(x) = 0$ for $x > a_\sharp$.

We also consider the dual problem:

$$\begin{cases} -\frac{\partial}{\partial x} \phi(x) + [d(x) + \lambda_0]\, \phi(x) = \phi(0) B(x), & x \geq 0, \\ \phi(\cdot) \geq 0, \qquad \int_0^\infty \phi(x) N(x) dx = 1. \end{cases} \tag{7}$$

The support of ϕ is equal to the smallest interval containing 0 and supp$B$. The equation on ϕ is also better expressed on $N\phi$ because we have

$$\begin{cases} \frac{\partial}{\partial x} [N(x)\phi(x)] = -\phi(0) B(x) N(x), & x \geq 0, \\ N(x)\phi(x) \underset{x\to\infty}{\longrightarrow} 0, \qquad \int_0^\infty \phi(x) N(x) dx = 1. \end{cases} \tag{8}$$

In order to state our main result we also need to rewrite the renewal equation after renormalization by the exponential growth of the system. We define

$$\tilde{n}(t,x) = n(t,x)e^{-\lambda_0 t}.$$

Then, $\tilde{n}(t,x)$ satisfies the equations

$$\begin{cases} \frac{\partial}{\partial t}\tilde{n}(t,x) + \frac{\partial}{\partial x}\tilde{n}(t,x) + [d(x)+\lambda_0]\tilde{n}(t,x) = 0, \qquad t \geq 0, x \geq 0, \\ \tilde{n}(t,x=0) = \int_0^\infty B(x)\tilde{n}(t,x)dx, \end{cases} \tag{9}$$

with initial condition $\tilde{n}(t=0,x) = n^0(x)$. As a consequence of the GRE principle we have the conservation law:

$$\frac{d}{dt}\int_0^\infty \tilde{n}(t,x)\phi(x)dx = 0. \tag{10}$$

Finally, we introduce the function

$$h(t,x) = \tilde{n}(t,x) - \int_0^\infty n^0(y)\phi(y)dyN(x).$$

Then $h(t,x)$ fulfills

$$\begin{cases} \frac{\partial}{\partial t}h(t,x) + \frac{\partial}{\partial x}h(t,x) + [d(x)+\lambda_0]h(t,x) = 0, \qquad t \geq 0, x \geq 0, \\ h(t,x=0) = \int_0^\infty B(x)h(t,x)dx, \end{cases} \tag{11}$$

with initial condition $h(t=0,x) = h^0(x)$.

Our purpose is to prove the long time asymptotic on n through the

Theorem 1.1 *Assume (2)–(4). There is a $y_0 > 0$, a $\mu > 0$ and a bounded function φ (positive on the support of ϕ) such that solutions to (11) satisfy*

$$\int_0^\infty |h(t,x)|\varphi(x)\,dx \leq e^{-\mu(t-y_0)}\int_0^\infty |h(y_0,x)|\varphi(x)\,dx < \infty.$$

In other words, we have obtained the exponential rate of convergence to a steady state of $\tilde{n}$ because the inequality of this Theorem is also

$$\int_0^\infty |mN(x) - \tilde{n}(t,x)|\varphi(x)dx \le C_0 e^{-\mu(t-y_0)} \int_0^\infty |mN(x) - \tilde{n}(y_0,x)|\varphi(x)dx\; dx \quad (12)$$

with $m = \int_0^\infty n^0(x)\phi(x)dx$.

Note also that explicit constants y_0 and μ can be computed in specific cases. An example is given in Section 2 below.

The proof of Theorem 1.1 consists in using a contraction property (a special case of the GRE) combined with invariants for the equation. The classical invariant

$$\int_0^\infty h(t,x)\phi(x)\; dx = 0, \quad (13)$$

was used in [12] and is not enough to cover cases where $B(x)$ can vanish at a point x_0 but be positive for $x > x_0$. We recall the method of [12] and improve the result in Section 2 where we introduce, as an example, a second invariant. The third invariant is introduced in Section 3 and the proof of Theorem 1.1 follows in Section 5.

2 A first result

Before we prove Theorem 1.1, we begin with a simpler result which serves to introduce several notations and concepts. Mostly we wish to find more invariants to equation (11) than the mere quantity (13).

Proposition 2.1 *Assume (2)–(4), then the solutions to (11) satisfy*

$$\int_y^\infty \frac{h(t,x)}{N(x)}\, N(x-y)\phi(x-y)dx = 0 \qquad \forall t \ge y \ge 0. \quad (14)$$

Additionally, if

$$\exists \mu_0 > 0 \quad \text{and} \quad y \ge 0 \quad \text{s.t.} \quad \forall x \in \mathbb{R}_+ \quad B(x+y)N(x+y) \ge \mu_0 \frac{\phi(x)}{\phi(0)} N(x), \quad (15)$$

then

$$\int_0^\infty |h(t,x)|\phi(x)dx \le \min\left\{1; e^{-\mu_0(t-y)}\right\} \int_0^\infty |h^0(x)|\phi(x)dx. \quad (16)$$

Example 2.1 *Consider $d = 0$, $B(x) = \Gamma \mathbb{1}_{\{x>a_*\}} e^{-\gamma x}$ with Γ large enough to fulfill the condition (3) and $a_* > 0$. Then the choice $y = 0$ in [12] is not enough to satisfy the assumption (15). But, taking $y = a_*$, (15) is satisfied and exponential decay (16) holds true.*

Proof of Proposition 2.1 The proof is a generalization of the proof of [12, Proposition. A.3] using a new invariant.

1st step. Contraction property. Since ϕ is the solution to the dual problem, we have

$$\begin{cases} \frac{\partial}{\partial t}[h(t,x)\phi(x)] + \frac{\partial}{\partial x}[h(t,x)\phi(x)] = -\phi(0)B(x)h(t,x) \\ \phi(0)h(t,x=0) = \phi(0)\int_0^\infty B(x)h(t,x)dx. \end{cases} \tag{17}$$

Therefore (see [12, Appendix, Lemma 1])

$$\begin{cases} \frac{\partial}{\partial t}[|h(t,x)|\phi(x)] + \frac{\partial}{\partial x}[|h(t,x)|\phi(x)] = -\phi(0)B(x)|h(t,x)| \\ \phi(0)|h(t,x=0)| = \phi(0)|\int_0^\infty B(x)h(t,x)dx|. \end{cases} \tag{18}$$

After integration in x, we obtain

$$\frac{d}{dt}\int_0^\infty |h(t,x)|\phi(x)dx = -\phi(0)\int_0^\infty B(x)|h(t,x)|dx + \phi(0)\left|\int_0^\infty B(x)h(t,x)dx\right|. \tag{19}$$

Notice that (19) provides the property

$$t \mapsto \int_0^\infty |h(t,x)|\phi(x)dx \quad \text{is monotone nonincreasing.} \tag{20}$$

2nd step. Invariant (14). We have

$$\frac{\partial}{\partial t}\frac{h(t,x)}{N(x)} + \frac{\partial}{\partial x}\frac{h(t,x)}{N(x)} = 0, \tag{21}$$

therefore we also have

$$\frac{h(t,x+z)}{N(x+z)} = \frac{h(t-z,x)}{N(x)}, \quad \text{for } t \geq z, \tag{22}$$

and thus, from (13),

$$0 = \int_0^\infty \frac{h(t-y,x)}{N(x)}N(x)\phi(x)dx = \int_y^\infty \frac{h(t,x)}{N(x)}N(x-y)\phi(x-y)dx \quad \forall t \geq y.$$

Therefore (14) is proved.
3rd step. Exponential decay. Using (14) and (15), for all $t \geq y$, it holds

$$
\begin{aligned}
&\Big|\int_0^\infty \phi(0)B(x)h(t,x)dx\Big| \\
&\qquad = \Big|\int_0^\infty \phi(0)B(x)h(t,x)dx - \int_y^\infty \mu_0 \frac{h(t,x)}{N(x)} N(x-y)\phi(x-y)dx\Big| \\
&\qquad \leq \int_0^\infty [\phi(0)B(x)N(x) - \mu_0 \mathbb{1}_{\{x\geq y\}} N(x-y)\phi(x-y)] \frac{|h(t,x)|}{N(x)} dx.
\end{aligned}
$$

Combining this inequality and (19), we arrive at

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty |h(t,x)|\phi(x)dx &\leq -\mu_0 \int_y^\infty \frac{|h(t,x)|}{N(x)} N(x-y)\phi(x-y)dx \\
&= -\mu_0 \int_0^\infty \frac{|h(t,x+y)|}{N(x+y)} N(x)\phi(x)dx \\
&= -\mu_0 \int_0^\infty \frac{|h(t-y,x)|}{N(x)} N(x)\phi(x)dx,
\end{aligned}
$$

(using (22) again with absolute values). The monotonicity property (20) now gives

$$
\frac{d}{dt}\int_0^\infty |h(t,x)|\phi(x)dx \leq -\mu_0 \int_0^\infty |h(t,x)|\phi(x)dx \qquad \text{for} \quad t \geq y.
$$

The assertion (16) then follows from the Gronwall inequality and (20). ■

3 A convolution invariant

Notice however that the assumption (15) cannot hold if B has a compact support that does not contain 0 because $\phi(0) \neq 0$ imposes to choose $y > 0$ and then $B(x+y)$ can vanish where $\phi(x)$ does not. This motivates to find a more general condition for exponential decay. To do so, we define

$$
Q(x) = B(x)\frac{N(x)}{N(0)}, \qquad \int Q = 1.
$$

Proposition 3.1 *Assume (2)–(4), then the solutions to (11) satisfy*

$$\int_{x=0}^{\infty} \Big[\int_{y=0}^{x} \nu(y) Q(x-y) dy - \nu(x) \Big] \frac{h(t,x)}{N(x)}\, dx = 0, \quad \forall t \geq y_0 \geq 0, \tag{23}$$

for all $\nu \in L^1(\mathbb{R}^+)$ such that $\nu(x)$ is constant for $x \geq y_0$. Additionally, if

$$\int_0^\infty \left| Q(x) - \int_0^x \nu(y)\, Q(x-y) dy + \nu(x) \right| dx < 1, \tag{24}$$

for some L^1 function ν such that $\nu(x)$ is constant for $x > y_0$, then there exist a nonnegative function φ (strictly positive on suppϕ*) and μ, such that solutions to (11) satisfy*

$$\frac{d}{dt} \left[\int_0^\infty |h(t,x)| \varphi(x) dx e^{\mu t} \right] \leq 0, \qquad \forall t \geq y_0. \tag{25}$$

Proof of Proposition 3.1. We use the same notations as before.
1st step. An auxiliary function. Consider the function $c(x)$ to be chosen later such that

$$c(\cdot) \geq 0, \qquad \int_0^\infty c(x) N(x) dx = N(0). \tag{26}$$

Then replace $B(x)$ by $c(x)$ in equation (7) defining ϕ i.e.

$$\begin{cases} -\frac{\partial}{\partial x} \varphi(x) + [d(x) + \lambda_0]\, \varphi(x) = \varphi(0) c(x), & x \geq 0 \\ \varphi(\cdot) \geq 0, \qquad \int_0^\infty \varphi(x) N(x) dx = 1, \end{cases} \tag{27}$$

where $N(x)$ is the solution to equation (5). Equivalently, we have

$$\frac{\partial}{\partial x} [N(x)\varphi(x)] = -\varphi(0) c(x) N(x). \tag{28}$$

Therefore by condition (26) we have $\varphi(x)N(x) \underset{x\to\infty}{\longrightarrow} 0$ and $N\varphi$ is decreasing. Similarly to (18)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) |h(t,x)| + (d(x) + \lambda_0)|h(t,x)| = 0 \quad \text{a.e. in} \quad \mathbb{R}_+ \times \mathbb{R}_+. \tag{29}$$

By multiplying equation (29) with φ and integrating, thanks to (27), we get

$$\frac{d}{dt} \int_0^\infty |h(t,x)| \varphi(x) dx = \varphi(0) \left[|h(t,0)| - \int_0^\infty c(x) |h(t,x)| dx \right].$$

Note that, by equation (29), the trace of $|h(t)|$ at $x = 0$ is well defined in L^1_{loc}. Using the boundary condition for $h(t, 0)$ we get

$$\frac{d}{dt}\int_0^\infty |h(t,x)|\varphi(x)dx = \varphi(0)\left[\left|\int_0^\infty B(x)h(t,x)dx\right| - \int_0^\infty N(x)c(x)\frac{|h(t,x)|}{N(x)}dx\right]. \tag{30}$$

2nd step. Invariant (23). Departing from the invariant (14), let $\mu(y)$ be a smooth function that vanishes for $y \geq y_0$, then

$$\int_{x=0}^\infty \int_{y=0}^x \mu(y)\; N(x-y)\phi(x-y)\frac{h(t,x)}{N(x)}dydx = 0, \quad \text{for } t \geq y_0. \tag{31}$$

We recall that

$$N(x)\phi(x) = \phi(0)\int_x^\infty B(y)N(y)dy.$$

We choose $\mu(y) = \nu'(y)N(0)/\phi(0)$ with $\nu(0) = 0$ and integrating by parts the dy integral in (31), it becomes (23). In this integral in x, and by density, the function ν can be chosen merely L^1 and thus the pointwise condition $\nu(0) = 0$ disappears.

3rd step. Strong decay. In order to choose, in (27), the function

$$c(x) = \mu\varphi(x) + \left|B(x) - \frac{N(0)}{N(x)}\int_0^x \nu(y)\,Q(x-y)dy + \frac{\nu(x)N(0)}{N(x)}\right| \tag{32}$$

for some $\mu > 0$ and some $\nu(x)$, as in the 2nd step, and since c is only constrained via (26), it is enough to be sure that,

$$\int_0^\infty \left|B(x)N(x) - N(0)\left(\int_0^x \nu(y)\,Q(x-y)dy + \nu(x)\right)\right|dx < N(0). \tag{33}$$

With this condition, for $t \geq y_0$ it holds

$$\frac{d}{dt}\int_0^\infty |h(t,x)|\varphi(x)dx \leq -\mu\int_0^\infty \varphi(x)|h(t,x)|dx, \tag{34}$$

and the assertion (25) follows. ■

4 A more general invariant

Assume supp$Q$ contains the interval $[a_1, a_2]$ $(0 < a_1 < a_2 < \infty)$. We define a sequence by

$$\begin{aligned} Q^{(1)}(x) &= Q(x), \\ Q^{(n+1)}(x) &= Q^{(n)}(x) + \int_0^x Q(x-z)\mathbb{1}_{I_n}(z)Q^{(n)}(z)dz - \mathbb{1}_{I_n}(x)Q^{(n)}(x) \end{aligned} \tag{35}$$

where $I_n = [na_1, na_2]$. Observe that the sequence $\{Q^{(n)}\}$ defined above is nonnegative and the integral term provides that supp$Q^{(n)}$ contains the interval $[na_1, na_2]$.

Proposition 4.1 *Let the sequence $\{Q^{(n)}\}$ be defined by (35). Then for all $n \in \mathbb{N}$ $\int_0^\infty Q^{(n)}dx = \int_0^\infty Qdx$.*

Proof An easy change of variables implies

$$\begin{aligned} \int_0^\infty Q^{(n+1)}dx &= \int_0^\infty \int_0^x Q(x-z)\mathbb{1}_{I_n}(z)Q^{(n)}(z)dzdx + \int_0^\infty Q^{(n)}dx - \int_0^\infty Q^{(n)}\mathbb{1}_{I_n}dx \\ &= \int_0^\infty \int_0^\infty Q(y)dy\mathbb{1}_{I_n}(z)Q^{(n)}(z)dz + \int_0^\infty Q^{(n)}dx - \int_0^\infty Q^{(n)}\mathbb{1}_{I_n}dx \\ &= \int_0^\infty Q^{(n)}dx. \end{aligned}$$

■

5 Proof of Theorem 1.1

In order to prove Theorem 1.1, we use a combination of the invariants built in the previous section. We also use the notations of section 4.

We choose

$$\nu^{(n)} = -\left(\mathbb{1}_{I_1}Q^{(1)} + \mathbb{1}_{I_2}Q^{(2)} + \mathbb{1}_{I_3}Q^{(3)} + ... + \mathbb{1}_{I_n-1}Q^{(n-1)}\right).$$

Note that thanks to formula (35) we obtain by iteration

$$Q(x) - \int_0^x \nu^{(n)}(z)Q(x-z)dz + \nu^{(n)}(x) = Q^{(n)}, \tag{36}$$

hence with help of Proposition 4.1 we conclude that

$$\int_0^\infty \left| Q(x) - \int_0^x \nu^{(n)}(z) Q(x-z) dz + \nu^{(n)}(x) \right| dx = 1.$$

We can now find n such that

$$2 \int_{na_1}^{na_2} \phi(x - na_1) N(x - na_1) dx > \int_0^\infty \phi(x) N(x) dx.$$

Then for some strictly positive λ it holds (using formula (8))

$$\int_0^\infty \left| Q^{(n)} - \int_0^x \lambda \mathbb{1}_{\{z \geq na_1\}}(z) Q(x-z) dz + \lambda \mathbb{1}_{\{x \geq na_1\}}(x) \right| dx$$
$$= \int_0^\infty \left| Q^{(n)} - \lambda \mathbb{1}_{\{x \geq na_1\}} \frac{\phi(x - na_1) N(x - na_1)}{\phi(0) N(0)} \right| dx < 1.$$

Hence for $\nu = \nu^{(n)} + \lambda \mathbb{1}_{\{x \geq na_1\}}$ condition (24) holds, which completes the proof applying Proposition 3.1 with $y_0 = \max\{(n-1)a_2; na_1\}$. ■

Remark 5.1 *The proof of Theorem 1.1 remains true if we assume only that* $B \in M_{loc,+}(\mathbb{R}^+)$ *and is not purely singular distributed.*

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