A recursive algorithm for the approximate solution of Volterra integral equations of first kind of convolution type *

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Abstract

In this paper we present an algorithm for the approximate solution of a class of Volterra integral equations of first kind whose kernel depends on the difference of the arguments, i.e. a *deconvolution algorithm.* As well known, this is an ill posed problem, but we give conditions under which the algorithm is robust against noise and sampling of the output.

The class of systems for which we can prove consistency of the algorithm includes Abel equations with square integrable unknown input functions.

Key words: Recursive deconvolution problems, input identification, Abel integral equations.

1 Introduction

In this paper we study a recursive deconvolution problem of the following type: we read the output y of the input-output relation

$$y(t) = \int_0^t K(t-s)u(s)\,\mathrm{d}s \qquad 0 \le t \le T \tag{1}$$

and we want to compute the input u. This classical problem is ill posed and the relevant references are enormous. A key point in the solution of the problem is a device that accumulates the available information on the output y and constructs a function v whose distance from u, in a suitable norm, is below a prescribed tolerance; and this in spite of the presence of unavoidable errors in the measurement of the function y.

Volterra integral equations of the first kind of the form (1) arise in many different applications, both of "geometrical" and "dynamical" type. In the first case it makes sense to accumulate

*This research was supported in part by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica. It fits the program of GNAMPA.

all the available information $y(t)$, $0 \le t \le T$, and then to use them in order to recover u. If however the equation models a dynamical process, and t represents time, we often need to use the information available up to time t in order to obtain an estimate of the value of u precisely at time t. In this way we obtain a *recursive* identification algorithm. The presentation and the analysis of such an algorithm, based on the idea of Tikhonov penalization, is the main goal of this paper.

1.1 Comments on the literature and an example

The literature on the solution of Volterra integral equations of the first kind is enormous since the problem is important for the applications and mathematically interesting and delicate. The role of the deconvolution problem in the solution of inverse problems of several types is well documented in the literature, see for example [27] and [22]. A special instance of the deconvolution problem is the approximate computation of derivatives, which corresponds to the case in which the kernel K is identically 1. See [1, 23, 24, 25] for results and references on this important problem.

Nice books on the solutions of Volterra integral equations of the first kind are [7, 8, 28]. In particular, it is explicitly noted in [7, 28] that the penalization method introduced by Tikhonov (see [29, 30]), although very effective for the solution of ill posed problems, does not seem to be well suited for the special case of Volterra integral equations of the first kind if we want to obtain recursive algorithms. In fact, the standard application of Tikhonov method requires to accumulate all the possible data on the interval $[0, T]$ and only afterwords it is possible to estimate the solution u. Efforts have been devoted to find other solution algorithms which are *recursive* i.e. such that an estimate v of the solution u is constructed at time t only on the basis of the data previously obtained. See for example [11, 12, 14, 21, 26].

As an additional feature of the deconvolution problem, we consider the case when the output signal is sampled at discrete instants. This is natural from the point of view of digital elaboration of data. For completeness we show an example in which this is unavoidable. The example is taken from [9, 20].

Example 1 A heated semiinfinite bar $x \ge 0$ is radiating from the boundary at $x = 0$, in the presence of an external source which supplies heat at the rate $f(t)$. Let the initial temperature be 0. It turns out that the temperature $T(x, t)$ solves the problem

$$\begin{cases} T_t = T_{xx}\,, & x > 0 \\ T(x, 0) = 0 \\ \lim_{x\to+\infty} T(x, t) = 0 \\ T_x(0, t) = cT^n(0, t) - f(t)\,, t > 0\,. \end{cases}$$

The numbers c is nonnegative and $n \ge 1$ is fixed. It is proved in [9] that this problem admits a unique solution and furthermore $T(0, t)$ solves

$$T(0, t) + \frac{c}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} T^n(0, s)\, \mathrm{d}s = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} f(s)\, \mathrm{d}s\,.$$

Now let us assume that the function f is to be identified on the basis of boundary measurements of the temperature taken at $x = 0$. This can be done if we can solve the deconvolution problem

$$y(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} f(s)\, \mathrm{d}s$$

where

$$y(t) = T(0,t) + \frac{c}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} T^n(0,s)\, \mathrm{d}s\,.$$

Now, even if it would be possible to measure the temperature $T(0,t)$ at every time t, the process of numerical integration gives the values of y only at a finite number of steps. Moreover, even in the ideal case that $T(0,t)$ is read without errors, approximations are introduced in the course of the numerical computation of the integral which appears in the definition of y.

This is precisely the kind of problem that we are going to study in this paper. ∎

1.2 The content of this paper

In this paper we show that Tikhonov penalization method can be adapted so to obtain a *recursive* deconvolution algorithm different from those in the cited literature and which works under more general conditions. This algorithm is suggested by the approach taken in [10] for the identification of inputs to control systems described by a class of ordinary differential equations. The key idea is the recursive application of Tikhonov penalization algorithm, as described in Section 2. See [5] for a detailed analysis and [6] for an application to the reduction of the disturbances to a linear control system. However in those papers the Lagrange formula for the representation of the solutions of a linear differential equation is crucially used. On this same line see also [17, 18, 19]. This formula is not available in the more general context studied here. However, as for differential equations, the underline structure behind the algorithm that we are going to present is a singular perturbation now of the Volterra equation. See [2, 15, 16] for this problem.

In Section 2 we present all the technical assumptions made on the kernel K and we introduce the algorithm. Section 3 is devoted to a reformulation of the problem in the frequency domain and to prove a number of technical results needed later. In Section 4 we prove the consistency of our algorithm: Theorem 12 is our main result and shows the convergence of the algorithm when the sampling time, the error on the output measurements, as well the penalization constant go to 0 while respecting certain conditions. In Section 5 we show that under stronger smoothness assumptions on the input signal u we can give explicit estimates (Theorem 18) on the rate of convergence. In Section 6 we present an extension to the case when the system evolves over an infinite time interval. Finally, in Section 7 we show that relevant classes of kernels indeed satisfy the technical assumptions of the paper: among these, C^1 kernels K such that $K(0) \neq 0$ and also Abel type kernels. Simulations and comments on the applications are in sect. 8.

2 The technical assumptions and the algorithm.

We now specify all the technical assumptions we make on the system and on the signals.

1. We assume scalar valued functions u and y defined on an interval $[0,T]$, $T < +\infty$ (in Section 6 we shall also consider an extension to the case $T = +\infty$);

2. We assume that the input u is square integrable and that the kernel K is integrable on $[0,T]$ (further assumptions on K will be described below). These conditions imply, in particular, that the output y is square integrable too.

3. the output y is read at discrete time instants τ_k, equispaced for simplicity , $\tau_k = k\tau$. It $T < +\infty$ then we can choose $\tau = T/N$.

   This makes no difficulty if y is continuous, just read $y_k = y(\tau_k)$ in this case. In general, for each τ we fix a piecewise continuous function φ_τ (the properties of these functions are specified below). Once that such function φ_τ is fixed, we define

$$y_k = (\varphi_\tau * y)(\tau_k)\,. \tag{2}$$

4. We assume that the measurements of the actual values y_k are affected by errors ξ_k, whose tolerance is known. So we assume to have at our disposal the numbers η_k given by

$$\eta_k = y_k + \xi_k\,. \tag{3}$$

   We use the notation ξ for the sequence $\{\xi_k\}$ and we assume that

$$||\xi|| < h$$

   where the above norm may be either the uniform or the square norm: $\sup|\xi_k| < h$ or $\tau \sum \xi_k^2 < h^2$. In the sequel we shall use the square norm since it leads to a more general theory which can also be extended to the case when $T = +\infty$.

The functions φ_τ play the role of an approximate identity, adapted to the step τ. We assume the following properties:

1. $\varphi_\tau(t) \geq 0$ is piecewise continuous and $\varphi_\tau(t)$ is zero for $t > \tau$ and for $t < 0$;
2. there exists a constant $M_0 > 0$ such that, for every τ and t, we have $\varphi_\tau(t) \leq M_0/\tau$;
3. $\int_0^\tau \varphi_\tau(t)\, \mathrm{d}t = 1$.

Now we come to the crucial technical assumptions: we cannot solve the problem in such generality, and we need further conditions on the kernel K. In order to express these conditions we extend K to $[0, +\infty)$ (the extension is still denoted by K). The additional conditions are expressed in terms of the Laplace transform $\hat{K}$ of K. We observe that if we extend K to $[0, +\infty)$, by defining $K(t) = 0$ for $t > T$, then the Laplace transform $\hat{K}$ exists for $\Re e\, \lambda \geq 0$. However, for practical computations, different extensions can be more convenient. For example, it is easier to consider a constant extension of $K(t) = 1$ for $t \in [0, T]$ and to use $\hat{K}(\lambda) = 1/\lambda$. This explains assumption (HP0).

- **(HP0)** The function $K(t)$ extended to $[0, +\infty)$ is $L^1_{\rm loc}(0, +\infty)$ and Laplace transformable.

  Let ν_c be such that $e^{-\lambda t}K(t)$ is integrable for $\Re e\, \lambda > \nu_c$. In the following all Laplace transformations will implicitly be considered in the half plane $\Re e\, \lambda > \nu_c$.

- **(HP1)** There exist *positive* numbers γ_1, M_1 and $R_1 \geq \nu_c$ such that

$$|\hat{K}(\lambda)| \leq \frac{M_1}{|\lambda|^{\gamma_1}}$$

  for $|\lambda| > R_1$.

- **(HP2)** There exist *positive* numbers γ_2, M_2 and $R_2 \geq \nu_c$ such that

$$|\hat{K}(\lambda)| \geq \frac{M_2}{|\lambda|^{\gamma_2}}\,.$$

  for $|\lambda| > R_2$.

- **(HP3)**

  We assume that there is a sector

$$\mathcal{S}_{r,\theta} = \{\lambda \in \mathbb{C}\,,\ |\lambda| < r\,, |\mathrm{Arg}\,\lambda| > \theta\} \tag{4}$$

  ("Arg" denotes the principal argument of a complex number, $-\pi \leq \mathrm{Arg}\lambda < \pi$), and a positive number $\nu_{\mathcal{S}} \geq \nu_C$ such that

$$\Re e\,\lambda > \nu_{\mathcal{S}} \ \Rightarrow\ \hat{K}(\lambda) \notin \mathcal{S}_{r,\theta}$$

  .

There exist important examples of kernels with the prescribed properties. In fact, the conditions are satisfied by every C^1 kernel K such that $K(0) \neq 0$ and by the kernels of Abel type $K(t) = 1/t^{1-\gamma}$, $0 < \gamma \leq 1$ as discussed in Section 7. However, there are important examples which do not satisfy the previous conditions, like the "infinite smoothing" kernel—see [13] for this term—$K(t) = t^{-3/2}e^{-1/t}$, whose Laplace transform decays exponentially fast.

Remark 2 Assumptions (HP1) and (HP2), with $\gamma_1 = \gamma_2 = \gamma$, imply that $K(t)$ is of the order of $t^{\gamma-1}$ for $t \to 0$, see [31, p. 192, Theorem 4.3]. This condition implies

$$\lim_{r\to 0+} \frac{1}{r^\gamma} \int_0^r K(s)\,\mathrm{d}s = L \neq 0\,. \tag{5}$$

We note that condition (5) is satisfied if K is continuous at $t = 0$, with $K(0) = L \neq 0$. In this case $\gamma = 1$. It is also satisfied by kernels of Abel type $K(t) = 1/t^{1-\gamma}$.

Upon replacing u by Lu, it is always possible to assume

$$\lim_{r\to 0} \frac{1}{r^\gamma} \int_0^r K(s)\,\mathrm{d}s = 1\,. \tag{6}$$

We shall use the weaker condition (6) in order to justify a formula for a candidate approximant v of u; but we will be able to prove that v really approximates u only under the stronger conditions (HP0)–(HP3). ∎

2.1 The key idea

Our aim is the construction of an algorithm for the recursive definition of functions v which approximate u in *square mean.* This identification problem is ill posed. Hence, we shall use a penalization approach, so that the constructed approximants v will depend on y, on τ and α (which is the penalization constant), and on the sequence of the errors $\xi = \{\xi_k\}$.

$$v = v_{\tau,\xi,\alpha}$$

but for simplicity we shall simply denote it by v.

For simplicity of presentation of the basic idea, we first assume that y is continuous and that $y_k = y(\tau_k)$. The algorithm will then be studied in the general case.

The key idea is from [10]: we associate to system (1) the "model system"

$$w(t) = \int_0^t K(t-s)v(s)\,\mathrm{d}s\,. \tag{7}$$

We observe that, if y is continuous, then we can write

$$y(\tau_{k+1}) = y(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} K(\tau_{k+1}-s)u(s)\,\mathrm{d}s + \int_0^{\tau_k} F(\tau_k - s)u(s)\,\mathrm{d}s\,,$$
$$F(s) = K(\tau + s) - K(s)\,.$$

An analogous formula holds for w

$$w(\tau_{k+1}) = w(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} K(\tau_{k+1}-s)v(s)\,\mathrm{d}s + \int_0^{\tau_k} F(\tau_k - s)v(s)\,\mathrm{d}s.$$

The rule that we use in order to define v, a candidate approximant of u, is as follows: the function v is piecewise constant, $v(t) = v_k$ for $t \in [\tau_k, \tau_{k+1})$. This implies that w is a continuous function. We construct v_k as follows: at τ_k we consider the number

$$\lambda(v) = w_k + A_0 v\,, \qquad A_0 = \int_{\tau_k}^{\tau_{k+1}} K(\tau_{k+1}-s)\,\mathrm{d}s = \int_0^{\tau} K(s)\,\mathrm{d}s \sim \tau^\gamma\,.$$

Intuitively, we think that we already found a good approximation of u for $t \le \tau_k$ and that the contribution of the integral containing F is small (we can easily quantify this in the special case that K is Lipschitz continuous). This suggests that the value v_k can be constructed as

$$v_k = \arg\min \left\{ |\eta_{k+1} - \lambda(v)|^2 + \alpha\tau^\gamma v^2 \right\}\,.$$

We recall that η_k is the estimate of y_k which is available to us and we note that

$$\tau^\gamma v^2 = \gamma \int_{\tau_k}^{\tau_{k+1}} \frac{1}{[s-\tau_k]^{1-\gamma}} v^2\,\mathrm{d}s\,.$$

We easily compute

$$v_k = -[\alpha\tau^\gamma + A_0^* A_0]^{-1} A_0^* [w_k - \eta_{k+1}] \tag{8}$$

(most often, K will take real values. The previous formula however is also valid for complex K. In this case, the star denotes the conjugate). If $\gamma = 1$, this is essentially the same definition used in [5], in the special case $K(t) = Ce^{As}B$ where the input u was reconstructed with one step delay.

We note that $1/\alpha$ is the principal part, for $\alpha \to 0$, of the function $[\alpha\tau^\gamma + A_0^* A_0]^{-1} A_0^*$. Our previous experience and this observation suggest that we can possibly simplify formula (8) by replacing η_{k+1} with η_k and simplifying the form of the operators. We thus propose to use the simpler definition

$$v_k = \frac{\eta_k - w(\tau_k)}{\alpha} \qquad t \in [\tau_k, \tau_{k+1})\,. \tag{9}$$

The above formula makes sense also in the general case when the output y is merely square integrable, with y_k given by (2). Relations (7) and (9) completely determine a recursive algorithm on which we will focus our attention for the rest of the paper.

We introduce the *error function*

$$e(t) = v(t) - u(t)\,. \tag{10}$$

It depends on τ, α and the errors ξ. Our main goal is to give conditions under which e converges to zero in $L^2(0,T)$ when τ, α and h converge to zero.

In order to obtain a more compact representation for v, we introduce the following notations: if $\{a_k\}$ is a sequence, we define a^τ as a function on $[0,T)$, defined by

$$a^\tau(s) = \sum_{k\geq 0} a_k I_{[\tau_k,\tau_{k+1})}(s) \tag{11}$$

where

$$I_{[\tau_k,\tau_{k+1})}(s) = \begin{cases} 1 & \text{if} \quad s\in[\tau_k,\tau_{k+1}) \\ 0 & \text{if} \quad s\notin[\tau_k,\tau_{k+1}) \end{cases}$$

If $f(t)$ is defined for $t\geq 0$, we use f^τ to denote the piecewise constant function

$$f^\tau(t) = \sum_{k\geq 0} f_k I_{[\tau_k,\tau_{k+1})}(s)$$

where f_k is either $f(\tau_k)$ (of course, if f is continuous) or, more in general, it is given by $f_k = (\varphi_\tau * f)(\tau_k)$, see (2).

It now follows from (7) and from (9) that v is completely determined by the relation:

$$\alpha v = [\varphi_\tau * K * u - K * v]^\tau + \xi^\tau \tag{12}$$

Our strategy is now to study Eq. (12) in order to estimate the error $e(t)$. We shall study the problem in the frequency domain, i.e. we shall use the Laplace transformation. In order to do this we first extend ξ, u and K for $t > T$ (for example with zero, but this is not crucial) and we let Eq. (12) define $v(t)$ for any $t > 0$. In this way the error $e(t)$ is also defined for every $t\geq 0$. The idea is to compute $\hat{e}(\lambda)$. Parseval equality

$$\int_0^{+\infty} |e(t)|^2 \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{e}(i\omega)|^2 \,\mathrm{d}\omega$$

can be used in order to give conditions under which $||e||_2$ converges to zero. This in particular implies that $||e|_{[0,T]}||_2$ converges to zero too. However, when $T < +\infty$, we have also an additional degree of freedom, since the norm

$$||e|_{[0,T]}||_{x_0,2}^2 = \int_0^T e^{-x_0 t}|e(s)|^2 \,\mathrm{d}s$$

is equivalent to the usual norm in $L^2(0,T)$. Hence, in this case we can also estimate

$$||\hat{e}||_{x_0,2}^2 = \int_{-\infty}^{+\infty} |\hat{e}(x_0 + i\omega)|^2 \,\mathrm{d}\omega\,.$$

The number x_0 must be fixed. It should not depend on the unknown perturbation ξ or on the parameters α, τ.

It is instrumental to note that the right hand side of (12) can be manipulated as follows:

$$[K * (u^\tau - v)]^\tau + [\varphi_\tau * K * u - K * u^\tau]^\tau + \xi^\tau \,. \tag{13}$$

Here u^τ is defined in the simplest possible way:

$$u^\tau(t) = \frac{1}{\tau}\int_\tau^{\tau_{k+1}} u(s)\,\mathrm{d}s\,, \qquad t \in [\tau_k, \tau_{k+1})\,.$$

It is seen from this expression that a crucial term is $[K*(u^\tau - v)]^\tau$, the sample of a convolution of K with a piecewise constant function. The Laplace transformations of such functions are studied in the next section.

Remark 3

1. It is important to stress that the argument $\varphi_\tau * K * u - K * u^\tau$ of the bracket in (13) is a *continuous function* whose samples are the values at the points τ_k.

2. As noted by one referee, the simplified formula (9) is reminiscent of Lavrientev regularization, as described in [13, sect. 3]. However, the solution of the singularly perturbed Volterra equation, which appears in the direct use of Lavrientev method, is replaced by a simple recursive procedure thanks to the introduction of the model system.

   Thanks to the use of the model system, it is also natural to introduce sampled measurements from the outset and not just as a final numerical discretization. ∎

3 Formulation of the problem in the frequency domain

From now on the assumptions are those listed in Section 2. The estimated input v is defined through relations (7) and (9), or, more compactly, in (12). In this section we are going to derive a frequency domain representation of the singularly perturbed integral equation (12).

We first need preliminary results on the Laplace transformations of piecewise constant functions and piecewise constant samples of convolutions. The proof of the next lemma is easy:

Lemma 4 *Let $\{a_k\}$ be a sequence and*

$$a^\tau(t) = \sum_{k\geq 0} a_k I_{[\tau_k,\tau_{k+1})}(t)\,, \tag{14}$$

as in (11). Let us assume that

$$\lim_{k\to+\infty} e^{-\sigma\tau_k} a_k = 0 \tag{15}$$

for some exponent $\sigma \geq 0$. Under this condition the Laplace transform of a^τ is defined for $\Re e\,\lambda > \sigma$, and it is given by

$$\widehat{a^\tau}(\lambda) = \frac{1 - e^{-\lambda\tau}}{\lambda}\sum_{k=0}^{+\infty} a_k e^{-\lambda\tau k}\,.$$

Samples at $\tau_k = k\tau$ of a convolution appear in (8). On this subject we have:

Lemma 5 *Let $\{a_k\}$ be a sequence satisfying (15) and let a^τ be defined as in (14). Let*

$$b = K * a^\tau \,. \tag{16}$$

Then, we have:

$$\widehat{b^\tau}(\lambda) = \hat{K}(\tau,\lambda)\widehat{a^\tau}(\lambda)\,, \qquad \Re e\,\lambda > \sigma$$

where

$$\hat{K}(\tau,\lambda) = e^{-\lambda\tau}\sum_{n=0}^{+\infty} K_n e^{-\lambda\tau n}\,, \quad K_n = \int_{\tau n}^{\tau(n+1)} K(s)\,\mathrm{d}s\,.$$

Proof. We note that

$$(K * a^\tau)(\tau_k) = \sum_{j=0}^{k-1} K_{k-j-1} a_j$$

so that

$$\begin{aligned}
\widehat{b^\tau} &= \frac{1-e^{-\lambda\tau}}{\lambda}\sum_{k=0}^{+\infty}\sum_{j=0}^{k-1} K_{k-j-1}a_j e^{-\lambda\tau k} \\
&= \frac{1-e^{-\lambda\tau}}{\lambda}\left[\sum_{k=j+1}^{+\infty} K_{k-j-1}e^{-\lambda\tau(k-j)}\right]\left[\sum_{j=0}^{+\infty} a_j e^{-\lambda\tau j}\right] \\
&= \left[e^{-\lambda\tau}\sum_{r=0}^{+\infty} K_r e^{-\lambda\tau r}\right]\widehat{a^\tau}(\lambda)\,. \quad \blacksquare
\end{aligned}$$

We note that if $K(t) = 0$ for $t > T$ then the sum in the definition of $\hat{K}(\tau,\lambda)$ is finite.

Now we use Lemma 5 and (13) in order to compute the Laplace transformation of both sides of (12). We obtain

$$\alpha\hat{v}(\lambda) = \hat{K}(\tau,\lambda)[\widehat{u^\tau}(\lambda) - \hat{v}(\lambda)] + \mathcal{L}\left([\varphi_\tau * K * u - K * u^\tau]^\tau\right)(\lambda) + \widehat{\xi^\tau}(\lambda)\,.$$

It follows that

$$\hat{e}(\lambda) = -\frac{\alpha}{\alpha + \hat{K}(\tau,\lambda)}\hat{u} + \frac{1}{\alpha + \hat{K}(\tau,\lambda)}N(\tau,\lambda) \tag{17}$$

where

$$N(\tau,\lambda) = \hat{K}(\tau,\lambda)[\widehat{u^\tau}(\lambda) - \hat{u}(\lambda)] + \mathcal{L}\left([\varphi_\tau * K * u - K * u^\tau]^\tau\right)(\lambda) + \widehat{\xi^\tau}(\lambda)\,. \tag{18}$$

In order to study the behavior of (17) for $\tau \to 0$, $\alpha \to 0$, $h \to 0$, we need precise information on the properties of $\hat{K}(\tau,\lambda)$. The rest of this section is devoted to this.

3.1 The properties of $\hat{K}(\tau,\lambda)$

The frequency domain relation between u and y is $\hat{y}(\lambda) = \hat{K}(\lambda)\hat{u}(\lambda)$ so that we expect a formula which relates $\hat{K}(\tau,\lambda)$ and $\hat{K}(\lambda)$. In fact we have the following representation formula:

Theorem 6 *Assume condition (HP0). We have the equality*

$$\hat{K}(\tau,\lambda) = \sum_{n=-\infty}^{+\infty} \left[\frac{1-e^{-\lambda\tau}}{\lambda\tau + 2n\pi i}\right] \hat{K}\left(\lambda + \frac{2n\pi i}{\tau}\right). \tag{19}$$

for any λ *such that* $\Re e\,\lambda > \nu_C$.

Proof. For the proof of this formula we need to recall a result in [3]. First we recall the concept of the z–transform of a sequence of numbers. Let $\{b_k\}$ be a sequence. Its z–transform is the function

$$\mathcal{Z}_b(z) = \sum_{k=0}^{+\infty} \frac{b_k}{z^k}$$

(defined for those complex numbers z such that the series converges). Consider now a continuous function G on $[0,+\infty)$ such that $G(0) = 0$. It is proven in [3, Theorem 4] that, if G is of uniform bounded variation, then

$$\mathcal{Z}_{\{G(\tau_k)\}}(e^{\lambda\tau}) = \frac{1}{\tau} \sum_{k=-\infty}^{+\infty} \widehat{G}\left(\lambda + \frac{2k\pi i}{\tau}\right). \tag{20}$$

Uniform bounded variation is defined as the existence of $\Delta > 0$ such that

$$\sup_{t\geq 0} V_G(x, x+\Delta) < +\infty$$

where $V_G(x, x+\Delta)$ is the total variation of G on $[x, x+\Delta]$.

Assume first that $K \in L^1(0,+\infty)$ and consider $G = K * I_{[0,\tau]}$. A straightforward computation shows that

$$\mathcal{Z}_{\{G(\tau_k)\}}(e^{\lambda\tau}) = \hat{K}(\tau,\lambda). \tag{21}$$

On the other hand,

$$\hat{G}(\lambda) = \frac{1-e^{-\lambda\tau}}{\lambda}\hat{K}(\lambda). \tag{22}$$

From (20), (21), and (22) we obtain the wanted formula. It remains to see that G satisfies the required assumptions. It is clear that G is continuous and that $G(0) = 0$. We must show that G is of uniform bounded variation. This follows from the fact that G is actually in $W^{1,1}(0,+\infty)$. Indeed, Young inequalities show that G is L^1. Moreover, we can write

$$G(t) = \left[\int_0^t K(s)\,\mathrm{d}s\right] \cdot I_{[0,\tau[}(t) + \left[\int_{t-\tau}^t K(s)\,\mathrm{d}s\right] \cdot I_{[\tau,+\infty[}(t)$$

so that

$$G'(t) = K(t) \cdot I_{[0,\tau[}(t) + [K(t) - K(t-\tau)] I_{[\tau,+\infty[}(t)$$

is integrable, since K is integrable.

In the case when K is not in L^1, we must instead consider the function

$$G = K(\cdot)e^{-\nu\cdot} * I_{[0,\tau]}(\cdot)e^{-\nu\cdot}$$

and repeat the previous considerations. ∎

We note that $\lambda \mapsto \hat{K}(\tau, \lambda)$ is periodic of period $2\pi i/\tau$. We now study its properties in the strip $|\Im m\, \lambda \cdot \tau| < \pi$. The idea is that the addendum with $n = 0$ has a dominant role in this strip. This is justified by the following result, which will be crucial. We represent

$$\hat{K}(\tau, \lambda) = \frac{1 - e^{-\tau\lambda}}{\tau\lambda} \hat{K}(\lambda) + \eta(\tau, \lambda) \tag{23}$$

where

$$\eta(\tau, \lambda) = \sum_{n \neq 0} \left[\frac{1 - e^{-\tau\lambda}}{\tau\lambda + 2n\pi i} \right] \hat{K}\left(\lambda + \frac{2n\pi i}{\tau} \right) .$$

We have

Lemma 7 *Assume conditions (HP0) and (HP1). Then, the function $\eta(\tau, \lambda)$ has the following property: there exists $M_3 > 0$ such that for every λ and τ satisfying*

$$\Re e\, \lambda > \nu_C, \quad \tau < \pi/R_1, \quad |\Im m\, \lambda \cdot \tau| < \pi\,,$$

we have

$$|\eta(\tau, \lambda)| \leq M_3 \tau^{\gamma_1} . \tag{24}$$

Proof. Since $|\Im m\, \lambda \cdot \tau| < \pi$, we have that

$$|\lambda\tau + 2n\pi i| \geq (2|n| - 1)\pi \qquad \forall n \neq 0 . \tag{25}$$

In particular, since $\tau < \pi/R_1$, we obtain that

$$\left| \lambda + \frac{2n\pi i}{\tau} \right| > R_1 \qquad \forall n \neq 0 .$$

We can therefore apply condition (HP1) which implies

$$\left| \hat{K}\left(\lambda + \frac{2n\pi i}{\tau} \right) \right| \leq \frac{M_1}{|\lambda + 2n\pi i/\tau|^{\gamma_1}} \tag{26}$$

for every $n \neq 0$. Using now the estimations (25) and (26), we obtain:

$$\begin{aligned} |\eta(\tau, \lambda)| &\leq \sum_{n \neq 0} \left| \frac{1 - e^{-\lambda\tau}}{\lambda\tau + 2n\pi i} \cdot \hat{K}\left(\lambda + \frac{2n\pi i}{\tau} \right) \right| \\ &\leq \sum_{n \neq 0} \frac{2}{|\lambda\tau + 2n\pi i|} \cdot \frac{M_1}{|\lambda + 2n\pi i/\tau|^{\gamma_1}} \leq \tau^{\gamma_1} \cdot \sum_{n \neq 0} \frac{2}{[(2|n| - 1)\pi]^{1+\gamma_1}} = M_3 \tau^{\gamma_1} . \quad \blacksquare \end{aligned}$$

Formula (23) is useful because it gives an explicit link between $\hat{K}(\lambda)$ and $\hat{K}(\tau, \lambda)$. In particular, properties of $\hat{K}(\lambda)$ can be lifted to properties of $\hat{K}(\tau, \lambda)$ on the strip $|\Im m\, \lambda \cdot \tau| < \pi$ by working with the dominant term

$$\frac{1 - e^{-\tau\lambda}}{\tau\lambda} \hat{K}(\lambda) .$$

Assumptions (HP2) and (HP3) are used for this. The first result that we prove is as follows:

Lemma 8 *Assume conditions (HP0) and (HP2). Then, for every $x_0 > R_2$ there exists a number $M(x_0)$ with the following property: if the pair (S, τ) of positive numbers verifies*

$$(x_0 + S)\tau < \pi$$

then we have:

$$\left.\begin{array}{l} R_2 < \Re e\,\lambda \le x_0 \\ |\Im m\,\lambda| \le S \end{array}\right\} \implies \left|\frac{1-e^{-\lambda\tau}}{\lambda\tau}\hat{K}(\lambda)\right| \ge \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}}\,.$$

Proof. As

$$|\lambda\tau| = |(x+iy)\tau| \le |(x_0+S)\tau| < \pi,$$

there exists a number $\rho > 0$ such that

$$\left|\frac{1-e^{-\tau\lambda}}{\tau\lambda}\right| \ge \rho\,.$$

The number ρ does not depend either on x_0 or on S.

Now we use condition (HP2) and the inequality $(x_0+S)^\gamma < 2^\gamma(x_0^\gamma + S^\gamma)$, which holds for every $\gamma \ge 0$. We obtain, if λ is such that $R_2 < \Re e\,\lambda \le x_0$ and $|\Im m\,\lambda| \le S$,

$$\left|\frac{1-e^{-\tau\lambda}}{\tau\lambda}\hat{K}(\lambda)\right| > \rho\frac{M_2}{|\lambda|^{\gamma_2}} \ge \frac{\rho M_2}{1+|\lambda|^{\gamma_2}} > \frac{\rho M_2}{1+(x_0^2+S^2)^{\gamma_2/2}} \ge \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}} \tag{27}$$

where

$$M(x_0) = \frac{\rho M_2}{1+2^{\gamma_2}x_0^{\gamma_2}}\,. \quad \blacksquare \tag{28}$$

Remark 9 It is clear that the number π has no particular role in the previous proof. Any positive number will do. We fixed it for most of clarity, and we chose π for consistency with Lemma 7. ▮

Now we combine Lemma 7 and the estimate just obtained. We get:

Theorem 10 *Assume conditions (HP0)–(HP2). Then, for every $x_0 > R_2$ there exists a number $M(x_0)$ with the following property: if the pair (S, τ) of positive numbers verify*

$$\tau < \pi/R_1, \quad (x_0+S)\tau < \pi$$

then we have

$$\left.\begin{array}{l} R_2 < \Re e\,\lambda \le x_0 \\ |\Im m\,\lambda| < S \end{array}\right\} \implies \left|\hat{K}(\tau,\lambda)\right| \ge \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}} - M_3\tau^{\gamma_1}\,. \tag{29}$$

The right hand side of (29) can be negative, and in this case it is of little use. It is positive if

$$\tau^{\gamma_1}(1+2^{\gamma_2}S^{\gamma_2}) < \frac{M(x_0)}{M_3}\,. \tag{30}$$

Now we examine the consequence on $\hat{K}(\tau,\lambda)$ of the sector property (HP3).

Theorem 11 *Assume conditions (HP0), (HP1), and (HP3). Then, there exist numbers $\tau_0 > 0$, $L > 0$ and a sector $\mathcal{S}_{\tilde{r},\tilde{\theta}}$ such that if $\tau \in (0,\tau_0)$ and if λ is such that $\Re e\,\lambda > \nu_S$, then*

$$\hat{K}(\tau,\lambda) \in \mathcal{S}_{\tilde{r},\tilde{\theta}} \;\Rightarrow\; \Re e\,\hat{K}(\tau,\lambda) > -L\tau^{\gamma_1}\,.$$

Proof. We already noted from the representation formula (19) that $\hat{K}(\tau,\lambda)$ is periodic of period $2\pi i/\tau$. Hence it is sufficient to prove that the stated property holds in the strip $|\mathcal{I}m\,\lambda\tau| < \pi$. The dominant term in this strip is $\frac{1-e^{-\lambda\tau}}{\lambda\tau}\hat{K}(\lambda)$. First of all we prove that this term has the required property.

We introduce $z = \tau\lambda$ and we consider

$$\frac{1-e^{-z}}{z}\hat{K}(z/\tau)\,.$$

Let us choose r_0 such that

$$\left|\frac{1-e^{-z}}{z}\right| > \frac{1}{2}\,, \qquad \left|\mathrm{Arg}\,\frac{1-e^{-z}}{z}\right| < \frac{\pi-\theta}{2} \qquad \text{for } |z| < r_0$$

(θ is the number which appears in condition (HP3)). We fix τ_0 such that the following conditions are satisfied

$$\tau_0 < 1, \quad \tau_0 < \pi/R_1, \quad r_0/\tau_0 > R_1$$

and we impose the condition $0 < \tau < \tau_0$.

Now we distinguish two cases

Case 1: $|z| = |\tau\lambda| \le r_0$. In this case

$$\left|\tfrac{1-e^{-z}}{z}\hat{K}\left(\tfrac{z}{\tau}\right)\right| > \tfrac{1}{2}\left|\hat{K}\left(\tfrac{z}{\tau}\right)\right|\,,$$
$$\left|\mathrm{Arg}\,\tfrac{1-e^{-z}}{z}\hat{K}\left(\tfrac{z}{\tau}\right)\right| \le \left|\mathrm{Arg}\,\hat{K}\left(\tfrac{z}{\tau}\right)\right| + \tfrac{\pi-\theta}{2}\,.$$

It follows that

$$\frac{1-e^{-z}}{z}\hat{K}\left(\frac{z}{\tau}\right) \notin \mathcal{S}_{r/2,(\pi+\theta)/2}\,.$$

Case 2: $|z| = |\tau\lambda| > r_0$. Using condition (HP1) we see that

$$\left|\Re e\,\frac{1-e^{-z}}{z}\hat{K}\left(\frac{z}{\tau}\right)\right| \le \left|\frac{1-e^{-z}}{z}\hat{K}\left(\frac{z}{\tau}\right)\right| < \frac{2}{|z|}\left|\hat{K}\left(\frac{z}{\tau}\right)\right| < \frac{2M_1}{r_0}\frac{1}{|z/\tau|^{\gamma_1}} \le \left(\frac{2M_1}{r_0^{1+\gamma_1}}\right)\tau^{\gamma_1}\,.$$

We sum up: we proved that the required properties hold for the dominant term of $\hat{K}(\tau,\lambda)$. This property is preserved if we add a term of the order τ^{γ_1}. Hence, the result follows from Lemma 7 (notice that λ and τ satisfy the required bounds). ∎

4 Consistency of the algorithm

Now we apply the result of the previous section to the relation (17). We recall that e, which depends on y, also depends on τ and α as well as on the sequence of the errors $\xi = \{\xi_k\}$, of tolerance h: to stress the dependence, in this section we will use the notation $e_{\alpha,\tau,\xi}$. The following result asserts the consistency of the proposed algorithm.

Theorem 12 *Assume conditions (HP0)–(HP3). Then, for every $\epsilon > 0$, there exist α_0, τ_0, h_0 such that*

$$||e_{\alpha_0,\tau,\xi}||_2 < \epsilon, \qquad \forall\tau < \tau_0\,,\ \forall\xi\ :\ ||\xi|| < h_0\,.$$

We now start the proof of the theorem. We fix any $x_0 > \max\{R_2, \nu_S\}$ and we work in the frequency domain with $||\hat{e}(x_0+i\cdot)||_2$. The result will be proven in a number of steps. We impose from now on the condition $\tau < \pi/R_1$ needed in order to apply Theorem 10. This will be assumed and not explicitly repeated.

We first focus on the first term of formula (17). We have the following preliminary result

Lemma 13 *There exists $\delta_0 > 0$ such that*

$$\lim_{\substack{\tau\to 0,\alpha\to 0\\ (\tau^{\gamma_1}/\alpha)<\delta_0}} \left|\left|\frac{\alpha}{\alpha+\hat{K}(\tau,x_0+i\cdot)}\widehat{u^\tau}(x_0+i\cdot)\right|\right|_2 = 0 \tag{31}$$

<u>Proof</u>. We recall that x_0 has been fixed. Fix now a number $\epsilon > 0$. Decompose the square norm in (31) as the sum of the following two terms

$$I_1 = \int_{-S}^{S} \left|\frac{\alpha}{\alpha+\hat{K}(\tau,x_0+iy)}\widehat{u^\tau}(x_0+iy)\right|^2 \,\mathrm{d}y \tag{32}$$

and

$$I_2 = \int_{|y|>S} \left|\frac{\alpha}{\alpha+\hat{K}(\tau,x_0+iy)}\widehat{u^\tau}(x_0+iy)\right|^2 \,\mathrm{d}y \tag{33}$$

Consider I_2 first. It follows from Theorem 11 that

$$|\alpha+\hat{K}(\tau,\lambda)| \geq \min\{\alpha - L\tau^{\gamma_1}, \alpha\sin\tilde{\theta}\} \tag{34}$$

If we impose the condition $\alpha - L\tau^{\gamma_1} \geq \alpha\sin\tilde{\theta}$, equivalently expressed by

$$\frac{\tau^{\gamma_1}}{\alpha} \leq \frac{1}{L}(1-\sin\tilde{\theta}), \tag{35}$$

we have that

$$I_2 \leq \frac{1}{\sin^2\tilde{\theta}} \int_{|y|>S} |\hat{u}(x_0+iy)|^2 \,\mathrm{d}y\,. \tag{36}$$

Fix now $S = S_\epsilon$ so large that the last term in (36) is smaller than ϵ.

At this point both x_0 and S have been fixed. They will not be changed in the following. Consider now I_1. We have:

$$I_1 \leq \sup_{|y|<S} \left|\frac{\alpha}{\alpha+\hat{K}(\tau,x_0+iy)}\right|^2 \cdot ||u||^2_{x_0,2}\,.$$

Using inequality (29), we obtain

$$\left|\alpha+\hat{K}(\tau,x_0+iy)\right| \geq |\hat{K}(\tau,x_0+iy)| - \alpha \geq \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}} - M_3\tau^{\gamma_1} - \alpha\,. \tag{37}$$

Imposing the condition

$$M_3\tau^{\gamma_1} + \alpha < \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}}\,. \tag{38}$$

we obtain that the rightest term in (37) is positive so that

$$I_1 \leq \left[\frac{\alpha}{\frac{M(x_0)}{1+2^{\gamma_2} S^{\gamma_2}} - M_3 \tau^{\gamma_1} - \alpha} \right]^2 ||u||^2_{x_0,2} \,. \tag{39}$$

Since this last term converges to 0 when $\alpha \to 0$ and $\tau \to 0$, we can find $\tau_0 > 0$ and $\alpha_0 > 0$ such that, for any $\alpha < \alpha_0$ and $\tau < \tau_0$, condition (38) is satisfied and the last term in (39) is smaller than ϵ. This completes the proof. ∎

Remark 14 We highlights two facts regarding the proof of the previous lemma which will turn out to be useful in the case when $T = +\infty$. First notice that since $u \in L^2(0, +\infty)$, the choice of S following (36) can be done independently from x_0. Second, we see from (28) that $x_0 \mapsto M(x_0)$ is decreasing. Hence inequality (39) still holds if we replace $M(x_0)$ with $M(\tilde{x})$ with $\tilde{x} \geq x_0$ in both (39) and (38). Moreover, since $||u||_2 \geq ||u||_{x_0,2}$, we can also replace $||u||_{x_0,2}$ with $||u||_2$ in (39). ∎

We now consider the last term $N(\tau, \lambda)/[\alpha + \hat{K}(\tau, \lambda)]$ in (17) whose $||\cdot||_{x_0,2}$–norm is estimated, using (34) and (35), as

$$\begin{aligned}
&\left|\left| \frac{1}{\alpha + \hat{K}(\tau,\lambda)} N(\tau,\lambda) \right|\right|_{x_0,2} \leq \\
&\leq \left\{ \sup_{\mathcal{R}e\lambda = x_0} \left|\left| \frac{\hat{K}(\tau,\lambda)}{\alpha + \hat{K}(\tau,\lambda)} \right|\right| \right\} N_1 + \left\{ \sup_{\mathcal{R}e\lambda = x_0} \left|\left| \frac{1}{\alpha + \hat{K}(\tau,\lambda)} \right|\right| \right\} (N_2 + h) \\
&\leq \left[1 + \frac{1}{\sin \tilde{\theta}} \right] N_1 + \frac{1}{\alpha \sin \tilde{\theta}} (N_2 + h)
\end{aligned} \tag{40}$$

where

$$N_1 = ||\widehat{u^\tau}(\lambda) - \hat{u}(\lambda)||_{x_0,2} \,, \qquad N_2 = ||\mathcal{L}[(\varphi_\tau * K * u - K * u^\tau)^\tau]||_{x_0,2} \,.$$

The following two lemmas now complete the proof of Theorem 12.

Lemma 15 *We have:* $\lim_{\tau \to 0} ||u^\tau - u||_2 = 0$.

<u>Proof</u>. We must prove that, for every $u \in L^2(0, T)$, u is the limit of the piecewise constant functions whose values are

$$\frac{1}{\tau} \int_{\tau_k}^{\tau_{k+1}} u(s) \, \mathrm{d}s \,. \tag{41}$$

The proof is quite standard. The result is easy if u is continuous. Otherwise, we use an approximation argument: given $\epsilon > 0$ we find a continuous u_ϵ such that $||u - u_\epsilon||_2 < \epsilon$ and we estimate

$$||u - u^\tau||_2 \leq ||u - u_\epsilon||_2 + ||u_\epsilon - (\tilde{u}_\epsilon)^\tau||_2 + ||(\tilde{u}_\epsilon)^\tau - u^\tau||_2 \,.$$

Here we use the notation $(\tilde{u}_\epsilon)^\tau$ to stress the fact that the piecewise continuous function $(\tilde{u}_\epsilon)^\tau$ is constructed using integral averages as in (41), in spite of the fact that u_ϵ is continuous.

The L^2 norm of the first addendum is less than ϵ, by construction. The L^2 norm of the third term is estimated as follows

$$||(\tilde{u}_\epsilon)^\tau - u^\tau||_2^2 = \tau \sum_{k=0}^{n-1} \left| \frac{1}{\tau} \int_{\tau_k}^{\tau_{k+1}} [u_\epsilon(s) - u(s)] \, \mathrm{d}s \right|^2$$

$$\leq \frac{1}{\tau} \sum_{k=0}^{N-1} \tau \cdot \int_{\tau_k}^{\tau_{k+1}} [u_\epsilon(s) - u(s)]^2 \, \mathrm{d}s = ||u_\epsilon - u||_2^2 \leq \epsilon^2$$

This estimate does not depend on τ.

The L^2 norm of the second addendum is made less than ϵ by choosing τ small enough. ∎

Lemma 16 *We have:* $\lim_{\tau \to 0} ||[\varphi_\tau * K * u - K * u^\tau]^\tau||_2 = 0$.

Proof. We have that

$$||[\varphi_\tau * K * u - K * u^\tau]^\tau||_2 \leq ||[\varphi_\tau * (K * (u - u^\tau))]^\tau||_2 + ||[\varphi_\tau * K * u^\tau - K * u^\tau]^\tau||_2 \,. \tag{42}$$

We recall that the continuous functions $\varphi_\tau * K * u^\tau - K * u^\tau$ and $\varphi_\tau * (K * (u - u^\tau))$ are sampled by taking the values at the points τ_k, see Remark 3 item 1.

We study the first term

$$||[\varphi_\tau * (K * (u^\tau - u))]^\tau||_2 \,.$$

We note that, for $t \in [\tau_k, \tau_{k+1})$, we have

$$(u^\tau(t) - u(t)) = \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} [u(s) - u(t)] \, \mathrm{d}s$$

(we use $u(t) = 0$ if $t < 0$). Hence, for $t \in [\tau_k, \tau_{k+1})$,

$$[K * (u^\tau - u)](t) = \int_0^t K(t-s)[u^\tau - u](s) \, \mathrm{d}s$$

$$= \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} K(t-s) \left[\frac{1}{\tau} \int_{\tau_{j-1}}^{\tau_j} (u(r) - u(s)) \, \mathrm{d}r \right] \, \mathrm{d}s + \int_{\tau_k}^t K(t-s) \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} [u(r) - u(s)] \, \mathrm{d}r \, \mathrm{d}s \,.$$

Now, since φ_τ has support in $[0, \tau]$ we have that, for $t \in [\tau_k, \tau_{k+1})$,

$$[\varphi_\tau * (K * (u^\tau - u))](\tau_{k+1})$$

$$= \int_{\tau_k}^{\tau_{k+1}} \varphi_\tau(\tau_{k+1} - t) \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} K(t-s) \left(\frac{1}{\tau} \int_{\tau_{j-1}}^{\tau_j} (u(r) - u(s)) \, \mathrm{d}r \right) \, \mathrm{d}s \, \mathrm{d}t$$

$$+ \int_{\tau_k}^{\tau_{k+1}} \varphi_\tau(\tau_{k+1} - t) \int_{\tau_k}^t K(t-s) \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} [u(r) - u(s)] \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t \,.$$

Hence,

$$
\begin{aligned}
&||[\varphi_\tau * (K * (u - u^\tau))]^\tau||_2^2 \\
&\leq 2\tau \sum_{k=0}^{N-1} \left| \int_{\tau_k}^{\tau_{k+1}} \varphi_\tau(\tau_{k+1} - t) \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} K(t-s) \left(\frac{1}{\tau} \int_{\tau_{j-1}}^{\tau_j} [u(r) - u(s)] \,\mathrm{d}r \right) \,\mathrm{d}s \,\mathrm{d}t \right|^2 \\
&+ 2\tau \sum_{k=0}^{N-1} \left| \int_{\tau_k}^{\tau_{k+1}} \varphi_\tau(\tau_{k+1} - t) \int_{\tau_k}^{t} K(t-s) \frac{1}{\tau} \int_{\tau_{k-1}}^{\tau_k} [u(r) - u(s)] \,\mathrm{d}r \,\mathrm{d}s \,\mathrm{d}t \right|^2 \\
&\leq 2\tau \sum_{k=0}^{N-1} \left[\int_{\tau_k}^{\tau_{k+1}} \varphi_\tau(\tau_{k+1} - t) \sum_{j=0}^{k} \int_{\tau_j}^{\tau_{j+1}} |K(t-s)| \left(\frac{1}{\tau} \int_{\tau_{j-1}}^{\tau_j} |u(r) - u(s)| \,\mathrm{d}r \right) \,\mathrm{d}s \,\mathrm{d}t \right]^2 \\
&\leq \frac{1}{\tau^3} \sum_{k=0}^{N-1} \left[\sum_{j=0}^{k} \int_{\tau_j}^{\tau_{j+1}} F_k(s) \int_{\tau_{j-1}}^{\tau_j} |u(r) - u(s)| \,\mathrm{d}r \,\mathrm{d}s \right]^2 \qquad (43)
\end{aligned}
$$

where

$$
\begin{aligned}
&F_k(s) = \int_{\tau_k}^{\tau_{k+1}} \Phi_\tau(\tau_{k+1} - t)|K(t-s)| \,\mathrm{d}t \qquad s \in [0, \tau_{k+1}) \\
&\Phi_\tau(t) = \tau \varphi_\tau \,.
\end{aligned}
$$

The properties of $F_k(s)$ that we are going to use are expressed in the next result :

Lemma 17 *There exists a constant M such that, for every τ, k and $j \leq k$ we have:*

$$
\sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} F_k(s) \,\mathrm{d}s \leq M\tau \,, \qquad \sum_{k=j+1}^{N-1} F_k(s) \leq M \,.
$$

Proof. Notice first that $\Phi_\tau(t)$ is bounded uniformly in τ and t.

We prove the first inequality:

$$
\begin{aligned}
&\sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} F_k(s) \,\mathrm{d}s = \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} \int_{\tau_k}^{\tau_{k+1}} \Phi_\tau(\tau_{k+1} - t)|K(t-s)| \,\mathrm{d}t \,\mathrm{d}s \\
&\leq \int_{\tau_k}^{\tau_{k+1}} \Phi_\tau(\tau_{k+1} - t) \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} |K(t-s)| \,\mathrm{d}s \,\mathrm{d}t \\
&= \int_{\tau_k}^{\tau_{k+1}} \Phi_\tau(\tau_{k+1} - t) \int_0^{\tau_k} |K(t-s)| \,\mathrm{d}s \,\mathrm{d}t \leq M\tau.
\end{aligned}
$$

This proves the first inequality. The second one is seen as follows:

$$
\sum_{k=j+1}^{N-1} F_k(s) = \sum_{k=j+1}^{N-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\tau(\tau_{k+1} - t)|K(t-s)| \,\mathrm{d}t \leq M \int_{\tau_{j+1}}^{T} |K(t-s)| \,\mathrm{d}t \,.
$$

This is bounded. ∎

We are now ready to continue the estimation (43). We represent the integrand as

$$\sum_{j=0}^{k}\left(\int_{\tau_j}^{\tau_{j+1}} F_k(s)\,\mathrm{d}s\right)^{1/2}\left(\int_{\tau_j}^{\tau_{j+1}} F_k(s)\left[\int_{\tau_{j-1}}^{\tau_j}|u(r)-u(s)|\,\mathrm{d}r\right]^2\,\mathrm{d}s\right)^{1/2}.$$

We use the previous Lemma and Schwartz inequality. We obtain

$$||[\varphi_\tau*(K*(u-u^\tau))\,]^\tau||_2^2\le\frac{M}{\tau}\sum_{j=0}^{N-1}\int_{\tau_j}^{\tau_{j+1}}\left[\sum_{k=j+1}^{N-1}F_k(s)\right]\cdot\int_{\tau_{j-1}}^{\tau_j}[u(r)-u(s)]^2\,\mathrm{d}r\,\,\mathrm{d}s$$

$$\le\frac{M^2}{\tau}\sum_{j=0}^{N-1}\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_{j-1}}^{\tau_j}[u(r)-u(s)]^2\,\mathrm{d}r\,\,\mathrm{d}s\le\frac{M^2}{\tau}\int_{-2\tau}^{0}\int_0^T[u(s+\nu)-u(s)]^2\,\,\mathrm{d}s\,\,\mathrm{d}\nu\,.$$

Continuity of the shift shows that the function

$$G(\nu)=\int_0^T[u(s+\nu)-u(s)]^2\,\,\mathrm{d}s$$

is continuous and converges to zero for $\nu\to 0$. Consequently,

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_{-2\tau}^{0}G(\nu)\,\mathrm{d}\nu\,,$$

as wanted.

We now study the term

$$||\,[\varphi_\tau*K*u^\tau-K*u^\tau\,]^\tau||_2=||\,[K*(\varphi_\tau*u^\tau-u^\tau)]^\tau||_2\,.$$

For $t\in[\tau_k,\tau_{k+1})$ we represent

$$(\varphi_\tau*u^\tau)(t)=\int_{t-\tau}^{t}\varphi_\tau(t-s)u^\tau(s)\,\mathrm{d}s=\int_{\tau_k}^{t}\varphi_\tau(t-s)u^\tau(s)\,\mathrm{d}s+\int_{t-\tau}^{\tau_k}\varphi_\tau(t-s)u^\tau(s)\,\mathrm{d}s$$

$$=\frac{1}{\tau}\left[\int_{\tau_k}^{t}\varphi_\tau(t-s)\int_{\tau_{k-1}}^{\tau_k}u(r)\,\mathrm{d}r\,\,\mathrm{d}s+\int_{t-\tau}^{\tau_k}\varphi_\tau(t-s)\int_{\tau_{k-2}}^{\tau_{k-1}}u(r)\,\mathrm{d}r\,\,\mathrm{d}s\right]$$

$$=\frac{1}{\tau}\left\{\int_{t-\tau}^{t}\varphi_\tau(t-s)\,\mathrm{d}s\int_{\tau_{k-1}}^{\tau_k}u(r)\,\mathrm{d}r+\int_{t-\tau}^{\tau_k}\varphi_\tau(t-s)\left[\int_{\tau_{k-2}}^{\tau_{k-1}}u(r)\,\mathrm{d}r-\int_{\tau_{k-1}}^{\tau_k}u(r)\,\mathrm{d}r\right]\,\mathrm{d}s\right\}$$

$$=\frac{1}{\tau}\left\{\int_{\tau_{k-1}}^{\tau_k}u(r)\,\mathrm{d}r+\int_{t-\tau_k}^{\tau}\varphi_\tau(s)\int_{\tau_{k-2}}^{\tau_{k-1}}[u(r)-u(r+\tau)]\,\mathrm{d}r\,\,\mathrm{d}s\right\}\,.$$

It follows that, for $t\in[\tau_k,\tau_{k+1})$, we have

$$(\varphi_\tau*u^\tau-u^\tau)(t)=\frac{1}{\tau}\int_{t-\tau_k}^{\tau}\varphi_\tau(s)\,\mathrm{d}s\int_{\tau_{k-2}}^{\tau_{k-1}}[u(r)-u(r+\tau)]\,\mathrm{d}r\,.$$

The convolution with K gives

$$(K*(\varphi_\tau * u^\tau - u^\tau))(\tau_k) \\ = \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} K(\tau_k - s) \frac{1}{\tau} \int_{s-\tau_j}^{\tau} \varphi_\tau(r) \, \mathrm{d}r \int_{\tau_{j-2}}^{\tau_{j-1}} [u(\xi) - u(\xi+\tau)] \, \mathrm{d}\xi \, \mathrm{d}s \,.$$

We note that $0 \le \int_{s-\tau_j}^{\tau} \varphi_\tau(r) \, \mathrm{d}r \le 1$. We take the absolute values and we compute the $L^2(0,T)$ norm, which is less than a constant multiplied by

$$\tau \sum_{k=0}^{N-1} \left[\sum_{j=0}^{k-1} \int_{\tau_k-\tau_{j+1}}^{\tau_k-\tau_j} |K(r)| \, \mathrm{d}r \frac{1}{\tau} \int_{\tau_{j-2}}^{\tau_{j-1}} |u(\xi) - u(\xi+\tau)] \, \mathrm{d}\xi \right]^2 . \tag{44}$$

Now we use Schwarz inequality and we bound the expression (44) by

$$\tau \sum_{k=0}^{N-1} \left\{ \left(\sum_{j=0}^{k-1} \int_{\tau_k-\tau_{j+1}}^{\tau_k-\tau_j} |K(r)| \, \mathrm{d}r \right) \sum_{j=0}^{k-1} \int_{\tau_k-\tau_{j+1}}^{\tau_k-\tau_j} |K(r)| \, \mathrm{d}r \left[\frac{1}{\tau} \int_{\tau_{j-2}}^{\tau_{j-1}} |u(\xi) - u(\xi+\tau)| \, \mathrm{d}\xi \right]^2 \right\}$$
$$\le \tau \left(\int_0^T |K(r)| \, \mathrm{d}r \right) \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} \int_{\tau_k-\tau_{j+1}}^{\tau_k-\tau_j} |K(r)| \, \mathrm{d}r \, \frac{1}{\tau} \int_{\tau_{j-2}}^{\tau_{j-1}} |u(\xi) - u(\xi+\tau)|^2 \, \mathrm{d}\xi$$
$$\le M \left(\int_0^T |K(r)| \, \mathrm{d}r \right) \left(\int_0^T |u(\xi) - u(\xi+\tau)|^2 \, \mathrm{d}\xi \right) .$$

This converges to zero thanks to the L^2 continuity of the shift. ∎

5 Explicit convergence estimates

Theorem 12 does not give any quantitative information on the rate of convergence of the deconvolution algorithm. If we examine the proofs of the various lemmas used to prove Theorem 12, we see that some additional information on u is needed in order to obtain explicit estimates on e as a function of the parameters α, τ, and ξ. This is particularly evident in the way S is chosen to obtain the inequality (36) in Lemma 13, and in the estimates of the L^2-norm of $|u(\cdot) - u(\cdot + x)|$ needed in Lemmas 15 and 16.

In order to obtain quantitative convergence estimates we here assume that u has been extended to a $W^{1,2}(0,+\infty)$ function. We have the following result

Theorem 18 *Assume conditions (HP0)–(HP3) and let $u \in W^{1,2}(0,+\infty)$. Then, there exist $M \ge 0$, $\delta_1, \delta_2 > 0$ such that*

$$||e_{\alpha,\tau,\xi}||_2 \le M \left[\alpha^{\frac{1}{1+2\gamma_2}} + \frac{\sqrt{\tau}}{\alpha} + \frac{h}{\alpha} \right], \qquad \forall \tau, \alpha, \xi \ : \ \tau^{\gamma_1}/\alpha < \delta_1 \,, \ \alpha < \delta_2, \ ||\xi|| < h.$$

Proof. It follows from the regularity assumption made on u that both u and u' are Laplace transformable and

$$\hat{u}(\lambda) = \frac{1}{\lambda} u(0) + \frac{1}{\lambda} \widehat{u'}(\lambda) \qquad \Re e \, \lambda > 0 \,.$$

Moreover, we have that $|\widehat{u'}(\lambda)| \leq ||u'||_1$. This yields

$$|\hat{u}(\lambda)| \leq \frac{M}{|\lambda|} \,. \tag{45}$$

We now reconsider formula (17) using the same notation as in Section 4. In particular, we assume that we have fixed a number $x_0 > \min\{R, \nu_S\}$. Consider the first term in formula (17) whose convergence was studied in Lemma 13. Following the same lines of proof and using the estimation (45), we can see that, if τ^{γ_1}/α and $S^{\gamma_2}\alpha$ are sufficiently small, then

$$\begin{aligned}
&\left|\left|\frac{\alpha}{\alpha + \hat{K}(\tau, x_0 + i\cdot)}\widehat{u^\tau}(x_0 + i\cdot)\right|\right|_2 \leq \\
&\leq \frac{\alpha}{\frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}} - M_3\tau^{\gamma_1} - \alpha}||u||^2_{x_0,2} + \frac{2M}{\sin\tilde{\theta}}\left(\int_S^{+\infty}\frac{1}{x_0^2 + y^2}\,\mathrm{d}y\right)^{1/2} \leq \\
&\leq M\left[\alpha S^{\gamma_2} + \frac{1}{S^{1/2}}\right] .
\end{aligned}$$

If we choose

$$S = \left(\frac{1}{\alpha}\right)^{\frac{2}{1+2\gamma_2}}$$

we obtain

$$\left|\left|\frac{\alpha}{\alpha + \hat{K}(\tau, x_0 + i\cdot)}\widehat{u^\tau}(x_0 + i\cdot)\right|\right|_2 \leq M\alpha^{\frac{1}{1+2\gamma_2}} \,. \tag{46}$$

Notice that

$$S^{\gamma_2}\alpha = \alpha^{\frac{1}{1+2\gamma_2}}$$

so that, since $\gamma_2 < 2$, inequality (46) holds true if τ^{γ_1}/α and α are sufficiently small.

The second term in formula (17) can be estimated using (40) and the fact that since $|u(t) - u(s)| \leq M|t-s|^{1/2}$, we have that $N_1 \leq M_1\sqrt{\tau}$ and $N_2 \leq M_2\sqrt{\tau}$. We thus obtain:

$$\left|\left|\frac{1}{\alpha + \hat{K}(\tau, \lambda)}N(\tau, \lambda)\right|\right|_{x_0,2} \leq M\left[\frac{\sqrt{\tau}}{\alpha} + \frac{h}{\alpha}\right] . \tag{47}$$

This proves the result. ∎

6 The case when $T = +\infty$

In this section we shortly consider the case when $T = +\infty$. We assume that $K \in L^1(0, +\infty)$ so that $K * u \in L^2(0, +\infty)$ for any $u \in L^2(0, +\infty)$. The abscissa of convergence is now $\nu_C = 0$. Actually, $\hat{K}$ is a well defined continuous function in the closed half plane $\Re e\,\lambda \geq 0$. We fix a step $\tau > 0$ and we sample the output at $\tau_k = \tau k$ where $k \in \mathbb{N}$. We still assume $\tau < \pi/R_1$. We introduce now a stronger version of (HP2):

- **(HP2+)** There exist *positive* numbers γ_2 and $\tilde{M}_2$ such that

  $$|\hat{K}(\lambda)| \geq \frac{\tilde{M}_2}{1 + |\lambda|^{\gamma_2}}$$

  for any λ such that $\Re e\,\lambda > 0$.

We have the following extension of Theorem 10

Theorem 19 *Let $K \in L^1(0,+\infty)$ and assume conditions (HP0), (HP1), and (HP2+). Then, for every $x_0 > 0$, there exists a number $M(x_0)$ with the following property: if the pair (S,τ) of positive numbers verify*

$$\tau < \pi/R_1, \quad (x_0 + S)\tau < \pi$$

then we have

$$\left.\begin{array}{l} 0 < \Re e\,\lambda \le x_0 \\ |\Im m\,\lambda| \le S \end{array}\right\} \implies \left|\hat{K}(\tau,\lambda)\right| > \frac{M(x_0)}{1+2^{\gamma_2}S^{\gamma_2}} - M_3\tau^{\gamma_1}\,. \tag{48}$$

Proof. It follows from the analogous extension of Lemma 8 to every $x_0 > 0$. Notice indeed that in Lemma 8 the assumption $x_0 > R_2$ is only used in the first inequality of estimation (28) in order to be able to apply condition (HP2). However, condition (HP2+) allows to obtain in one step the first two inequalities in (28). Hence, everything follows. ∎

Notice now that Theorem 11 does not need any modification to be used in the new set up (R_1 and R_2 do not show up in the statement)and notice that the proofs of the technical Lemmas 15 and 16 can be repeated without modification (the summations from 0 to $N-1$ appearing in Lemma 16 are replaced by the corresponding series). We prove the following extension of Theorem 12.

Theorem 20 *Let $K \in L^1(0,+\infty)$ and assume conditions (HP0), (HP1), (HP2+), and (HP3). Assume moreover, $\nu_S = 0$. Then, for every $\epsilon > 0$, there exist α_0, τ_0, h_0 such that*

$$||e_{\alpha_0,\tau,\xi}||_{L^2(0,+\infty)} < \epsilon, \qquad \forall \tau < \tau_0\,, \;\; \forall \xi \,:\, ||\xi||_2 < h_0\,.$$

Proof. Fix $\epsilon > 0$. We fix any $T > 0$ and we show that it is possible to find α_0, τ_0, h_0 (independent of T) such that

$$||(e_{\alpha_0,\tau,\xi})_{|[0,T]}||_2 < \epsilon, \qquad \forall \tau < \tau_0\,, \;\; \forall \xi \,:\, ||\xi||_2 < h_0\,.$$

We notice that

$$||(e_{\alpha,\tau,\xi})_{|[0,T]}||_2 \le ||\widehat{e_{\alpha,\tau,\xi}}||_{x_0,2} \cdot e^{x_0 T}\,.$$

We fix $x_0 \in (0,1)$ such that $e^{x_0 T} < 2$. Now the fundamental thing to be noticed is that the limit in Lemma 13 is uniform in $x_0 \in (0,1)$. This can be seen noticing that the choice of S following (36) can be done independently on x_0 while in estimation (39) we can get rid of x_0 using Theorem 19 and replacing $M(x_0)$ with $M(1)$ and $||u||^2_{x_0,2}$ with $||u||^2_2$ (see the Remark 14). Finally we already noticed that Lemmas 15 and 16 can be proved also when $T = +\infty$. In this way the result is proven. ∎

7 Classes of admissible kernels

It is easy to see that $K(t) = e^{\beta t}$ (for $\beta \in \mathbb{C}$), $K(t) = t^{-\gamma}$ (for $0 \le \gamma < 1$) all satisfy conditions (HP0)–(HP3). Moreover if $\Re e\,\beta \le 0$, then for $K(t) = e^{\beta t}$ we have that $\nu_C = \nu_S = 0$ and also (HP2+) holds true. The same happens for $K(t) = t^{-\gamma}$ (for $0 \le \gamma < 1$). In this section we will present a larger class of kernels satisfying the required conditions, encompassing these particular examples.

It is convenient to extend our terminology as follows: we say that a function $r(\lambda)$ defined on a half plane $\Re e\,\lambda > \nu$ satisfies the condition (HPi) (for $i = 1, 2, 3, 4$), when it satisfies the corresponding property even if $r(\lambda)$ is not a Laplace transform.

We shall use the following lemma:

Lemma 21 *Let*

$$r(\lambda) = \frac{M + \phi(\lambda)}{\lambda^\gamma}$$

where $M > 0$, $0 \le \gamma < 2$, and $\phi(\lambda)$ is defined and bounded on a half plane $\Re e\,\lambda > \nu$ and satisfies the condition

$$\lim_{\Re e\,\lambda \to +\infty} \phi(\lambda) = 0 \tag{49}$$

Then, conditions (HP1)-(HP3) hold for $r(\lambda)$.

Proof. It is clear that properties (HP1) and (HP2) hold. We prove (HP3). Notice that as $\gamma < 2$ property (HP3) holds for M/λ^γ. In fact

$$\Re e\,\lambda > 0 \;\Rightarrow\; \left|\text{Arg}\,\frac{M}{\lambda^\gamma}\right| \le \frac{\gamma\pi}{2} < \pi\,.$$

Now,

$$|\text{Arg}\,[r(\lambda)]| \le \left|\text{Arg}\,\frac{M}{\lambda^\gamma}\right| + \left|\text{Arg}\,\left[1 + \frac{\phi(\lambda)}{M}\right]\right|.$$

Condition (49) implies that, if $\Re e\,\lambda$ is large enough, then,

$$\left|\text{Arg}\,\left[1 + \frac{\phi(\lambda)}{M}\right]\right| < \frac{\pi - \frac{\gamma\pi}{2}}{2}\,.$$

This proves the result. ∎

The previous lemma can be applied in the case

$$r(\lambda) = \frac{M}{\lambda} + \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda s} g(s)\,\mathrm{d}s$$

for some $M > 0$ and some function $g : [0, +\infty[\to \mathbb{R}$ such that $e^{-\nu\cdot} g(\cdot) \in L^p(0, +\infty)$ with $0 \le p \le \infty$ for some $\nu \in \mathbb{R}$. In fact,

$$\begin{aligned}
p = \infty \quad &\Rightarrow \quad \left|\int_0^{+\infty} e^{-\lambda s} g(s)\,\mathrm{d}s\right| \le \frac{||e^{-\sigma\cdot} g(\cdot)||_\infty}{|\lambda - \sigma|} \\
1 < p < \infty \quad &\Rightarrow \quad \left|\int_0^{+\infty} e^{-\lambda s} g(s)\,\mathrm{d}s\right| \le \frac{||e^{-\sigma\cdot} g(\cdot)||_p}{(|\lambda - \sigma| q)^{1/q}} \qquad q = \frac{p}{p-1} \\
p = 1 \quad &\Rightarrow \quad \lim_{\Re e\;\lambda + \infty} \int_0^{+\infty} e^{-\lambda s} g(s)\,\mathrm{d}s\, = 0\,.
\end{aligned}$$

The last statement follows from the dominated convergence theorem.

We can now prove the following result

Theorem 22 *If $K \in W^{1,p}(0, +\infty)$ and $K(0) > 0$, then conditions (HP1)-(HP3) hold.*

Proof. If $K \in W^{1,p}(0,+\infty)$ we have that

$$\hat{K}(\lambda) = \frac{1}{\lambda}K(0) + \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda s}K'(s)\,\mathrm{d}s\,.$$

The result then follows from the previous considerations and from Lemma 21. ∎

We can weaken the above result in the following way. Assume that K is a piecewise continuous function with jumps at the points t_n. Assume that $\{t_n\}$ is an increasing sequence with $t_1 > 0$ and put

$$\Delta_n = K(t_n+) - K(t_n-)\,.$$

Assume moreover that on each interval (t_{n-1}, t_n) the function K is of class $W^{1,p}$ and denote by K' its derivative. If we have that $K' \in L^p(0,+\infty)$ and $\sum_{n=1}^{+\infty}|\Delta_n| < +\infty$ we say that K is *piecewise* $W^{1,p}(0,+\infty)$. We have the following:

Theorem 23 *If K is piecewise $W^{1,p}(0,+\infty)$ and $K(0) > 0$ then conditions (HP1)-(HP3) hold.*

Proof. We compute

$$\hat{K}(\lambda) = \frac{1}{\lambda}K(0) + \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda s}K'(s)\,\mathrm{d}s + \frac{1}{\lambda}\sum_{n=1}^{+\infty} e^{-\lambda t_n}\Delta_n\,.$$

Since,

$$\lim_{\Re e\,\lambda\to+\infty}\sum_{n=1}^{+\infty} e^{-\lambda t_n}\Delta_n = 0$$

we can again apply Lemma 21. ∎

In particular, the previous theorem shows that if K is $W^{1,p}(0,T)$, then conditions (HP1)-(HP3) hold for any piecewise $W^{1,p}(0,+\infty)$ extension of K; in particular, for its extension by zero.

Remark: The statements concerning properties (HP0)–(HP3) in Theorems 22 and 23 still hold true if we assume that $e^{-\nu\cdot}K(\cdot)$ satisfies the corresponding conditions.

We have now a rich class of convolution kernels satisfying the required properties. However, it is not immediately clear whether a kernel like

$$K(t) = \int_0^t e^{-(t-s)}\frac{1}{s^\gamma}\,\mathrm{d}s$$

would also satisfies properties (HPi). The answer is contained in the next result.

Theorem 24 *Let K and G be two convolution kernels satisfying properties (HP1)-(HP3) and let the corresponding sectors of K and G be, respectively, $\mathcal{S}_{r_K,\theta_K}$ and $\mathcal{S}_{r_G,\theta_G}$. If $\theta_K + \theta_G < \pi$, then $H = K * G$ enjoys properties (HP1)-(HP3).*

Proof. As $\hat{H}(\lambda) = \hat{K}(\lambda)\hat{G}(\lambda)$ properties (HP1) and (HP2) are clear. Fix now

$$\nu > \max\{\nu_{\mathcal{S}}(K), \nu_{\mathcal{S}}(G)\}.$$

If $\Re e\,\lambda > \nu$ and if

$$|\hat{K}(\lambda)| < r_K, \qquad |\hat{G}(\lambda)| < r_G \tag{50}$$

then,

$$|\mathrm{Arg}\,\hat{K}(\lambda)| < \theta_K, \qquad |\mathrm{Arg}\,\hat{G}(\lambda)| < \theta_G\,. \tag{51}$$

We observe now that we can achieve conditions (50) by further increasing the value of ν, thanks to condition (HP1) which holds both for K and for G. Hence, we can assume that (50) holds true. Condition (51) now implies

$$|\mathrm{Arg}\,\hat{H}(\lambda)| \le |\mathrm{Arg}\,\hat{K}(\lambda)| + |\mathrm{Arg}\,\hat{G}(\lambda)| \le \theta_K + \theta_G < \pi\,.$$

This means that condition (HP3) holds for H. ∎

The results established up to now are of little use in the case when $T = +\infty$. For instance, even in the case $K \in W^{1,1}(0,+\infty)$ we know that $\nu_C = 0$ for $\hat{K}$, but there is no guarantee that $\nu_{\mathcal{S}} = 0$ in property (HP3). For this, we need a different formulation of Lemma 21.

Lemma 25 *Let $r(\lambda)$ be a continuous function on the half plane $\Re e\,\lambda \ge 0$ such that*

$$r(\lambda) \ne 0 \ \text{ for } \ \Re e\,\lambda \ge 0$$

and such that we can write

$$r(\lambda) = \frac{M + \phi(\lambda)}{\lambda^\gamma} \tag{52}$$

where $M > 0$, $0 \le \gamma < 2$, and $\phi(\lambda)$ satisfies the condition

$$\lim_{\lambda \to +\infty} \phi(\lambda) = 0\,. \tag{53}$$

Then, conditions (HP1), (HP2+), and (HP3) hold for $r(\lambda)$ with $\nu_{\mathcal{S}} = 0$.

Proof. (HP1) follows from (52) while (HP2+) follows from (52) and the fact that $r(\lambda)$ is continuous and does not have zeros. As to (HP3), we can repeat the arguments used in the proof of the Lemma 21: we see that

$$|\mathrm{Arg}\,[r(\lambda)]| \le \frac{\pi + \frac{\gamma\pi}{2}}{2}$$

if $|\lambda| > l$ for some l. On the other hand, $|r(\lambda)| \ge \delta > 0$ for λ such that $|\lambda| \le l$. This yields property (HP3) with $\nu_{\mathcal{S}} = 0$. ∎

We can now state the following final result, easy derivation of Lemma 25.

Theorem 26 *If $K \in W^{1,p}(0,+\infty)$, $K(0) > 0$, and $\hat{K}(\lambda)$ does not have zeroes in $\Re e\,\lambda \ge 0$, then conditions (HP1), (HP2+), and (HP3) hold with $\nu_{\mathcal{S}} = 0$.*

8 Few simulations and final comments

Finally, we present few simulations. In many practical applications we can expect smooth and slowly varying signals, but for simulation pourpouses it is also interesting to see the behaviour of the algorithm when the input has an abrupt change. Hence we consider the case that the kernel is

$$k(t) = \frac{1}{\sqrt{t}}$$

and the inputs are respectively

$$\textbf{a)} \quad u(t) = \frac{1}{t+1}\,, \qquad \textbf{b)} \quad u(t) = \begin{cases} 1 & \text{if } t \le 5 \text{ or } t > 10 \\ 0 & \text{otherwise.} \end{cases}$$

The plot of case **a)** is in fig. 1 while the plot of case **b)** is in fig. 2

In case **b)** it is easy to compute the convolution in closed form, and this fact was used in the algorithm. In case **a)** the convolution was computed at every step with a numerical integration.

In each one of the plots below the error tolerance is $h = .01$ (i.e. 1% of the maximum value of the input.)

The relevant quantites to be considered in the plots are the frequency

$$f = T/N = 1/\tau\,,$$

i.e. the number of steps for unity of time, α and h.

We choose to keep h constant so that the relative error h/α (which appear in the convergence estimate) increase when α decreses. The influence of this fact is clearly seen from the plots.

We observe that convergence is slow for $t \sim 0$. It would be fast if we had $u(0) = 0$. This is due to the fact that, in the absence of any *a priori* information, the function v was initialized at 0. If more information are available then it is possible to impose a different value to $v(0)$ and this can improve convergence. We note moreover that the input u is reconstructed with a delay, as it is clearly seen from the plots.

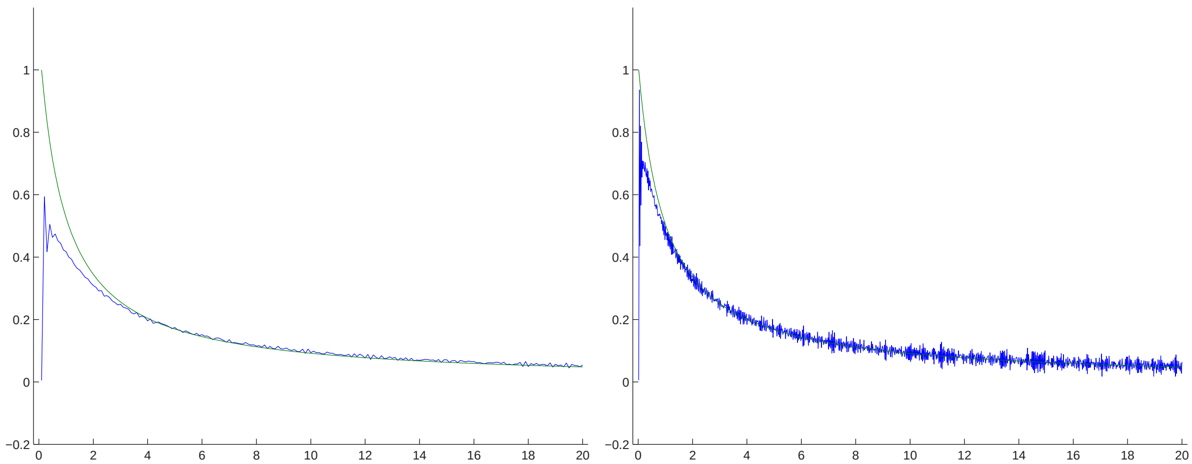
Convergence for large t is faster in case **a)** because in that case the signal is square integrable on $[0, +\infty)$.

8.1 Conclusions

As a final comment we repeat that conditions (HP1)–(HP3) are modeled on the properties of Abel kernels. The solution of Abel differential equations is a "mildly ill posed problem" in the classification of [4, p. 40]. Extension of the method presented here to more general convolution equations, in particular the case when K has a finite order zero at $t = 0$ as in [13], is reserved for future research. Kernels of the type $K(t) = e^{-1/t}/t^{3/2}$ also excape our analysis and it is easy to expect that a frequency domain analysis of this case is much more difficult, because $\hat{K}(\lambda)$ decays exponentially.

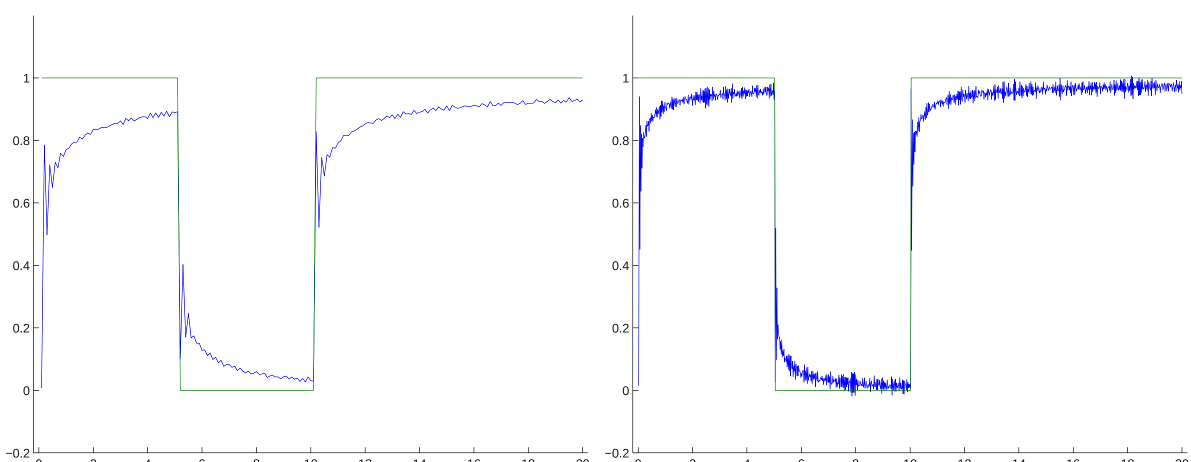
Finally we note an application of *recursive* deconvolution to control problems. If v is a disturbance which affects a linear input-output system, and if it is possible to approximately estimate it in a recursive way, it is then possible to construct an "adaptive" control that removes the effect of the disturbance from the output of the system. This problem has been studied in details in [6] in the context of linear finite dimensional systems even when the kernel is a matrix (in this case the kernel is smooth and moreover we can use the corresponding linear ordinary

Figure 1: $u(t) = 1/(t+1)$, $h = 0.01$.



$f = 10\,, \quad \alpha = 1$ $\qquad$ $f = 50, \quad \alpha = 0.3$

Figure 2: $u(t) = 1$ if $t \leq 5$ or $t > 10$, $u(t) = 0$ otherwise, $h = 0.01$.



$f = 10\,, \quad \alpha = .8$ $\qquad$ $f = 50, \quad \alpha = 0.3$

differential equation to compute the required quantities so that convergence of the algorithm is faster).

The idea goes as follows: let the system be described by

$$\dot{x} = Ax + B(d - u) + f_0 \,, \quad x(0) = x_0 \,, \qquad y = x$$

where now d is the disturbance and u is a control which should kill its effect and y is the output. The signal f_0 is the nominal input to the system.

If d would be known in advance, we could take $u = -d$ but of course a disturbance is by its nature unknown. The idea studied in [6] is that d can be estimate by using a deconvolution methd. Once that an estimate $\hat{v}$ has been obtained it is then possible to feed it back so to (approximately) reduce the effect of the disturbance.

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