Riesz systems and moment method in the study of heat equations with memory in one space dimension

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Abstract

In this paper we reduce observability and controllability of a heat equation with memory to the solution of a moment problem. We prove that this moment problem is solvable by proving that a suitable sequence of functions, associated with the heat equation with memory, is a Riesz system.

Key words: Heat equation with memory, controllability, observability, moment problem.

1 Introduction

In this paper we consider the following heat equation with memory, where $\theta = \theta(t,x)$, $x \in (0,\pi)$:

$$\theta'(t) = \theta_t = 2\alpha\theta(t) + \int_0^t N(t-s)\left[(a(x)\theta_x(s))_x - b(x)\theta(s)\right] \, \mathrm{d}s \qquad (1)$$

with initial and boundary conditions

$$\theta(0,x) = \xi(x)\,, \qquad \theta(t,0) = u(t)\,, \quad \theta(t,\pi) = 0\,. \qquad (2)$$

Standing assumptions in this paper: the kernel $N(t)$ is of class C^3 with $N(0) > 0$ (so, we can assume $N(0) = 1$), the functions $a(x)$ and $b(x)$ are of class C^2 and $a(x)$ is strictly positive while $\xi \in L^2(0,\pi)$, $u(t) \in L^2_{\rm loc}(0,+\infty)$ (positivity of $b(x)$ has a physical interest, but is not used in this paper).

Note that the prime will always be used to denote time derivative while the index denotes derivative with respect to the space variable.

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Equations of this form have been proposed to model several diffusion processes which are expected to display a "hyperbolic behavior", see comments to the literature below. In fact, Eq. (1) reduces to the usual wave equation when $\alpha = 0$, $a(x) = 1$, $b(x) = 0$ and $N(t) = 1$.

We shall study observability of Eq. (1). This means that we assume $u = 0$ and $\xi \in H_0^1(0, \pi)$. We shall see that under these conditions the output $y(t) = \theta_x(t, 0)$ is locally square integrable,

$$y(t) = \theta_x(t, 0) \in L^2(0, T) \qquad \forall T > 0\,. \tag{3}$$

We shall prove the following observability result:

Theorem 1 *There exists a number T_0 such that if If $y(t) = 0$ on $[0, T_0]$ then $\xi = 0$. Furthermore, the inverse transformation: $y(\cdot) \to \xi$: $L^2(0, T_0) \to L^2(0, \pi)$ is continuous.*

The number T_0 is identified in Section 6.

In fact, the focus of our interest is the way used in the proof of this theorem, which identifies a suitable sequence $\{z_n(t)\}$ of functions, which turns out to be a Riesz system in $L^2(0, T_0)$. Using this fact we then prove the following controllability result (see Sect. 7):

Theorem 2 *Let a target $\eta \in L^2(0, \pi)$ (as well as the initial conditions $\xi \in L^2(0, \pi)$) be given. There exists a control $u(t) \in L^2(0, T_0)$ such that the solution $\theta(t)$ of problem (1)-(2) satisfies $\theta(T_0) = \eta$.*

In order that the previous definitions and results make sense, we need a teorem on existence/unicity of solutions to problem (1)-(2), which is as follows:

Theorem 3 *Let T be an arbitrary positive number. Problem (1)-(2) has a unique solution $\theta(t) \in C(0, T; L^2(0, \pi))$. Furthermore, the transformation $(\xi, u(\cdot)) \to \theta(\cdot)$ is continuous from $L^2(0, \pi) \times L^2(0, T)$ to $C(0, T; L^2(0, \pi))$.*

It will be natural to prove this theorem with the methods of our paper but it could be derived from existing results, obtained with more abstract methods, even in the case $x \in \mathbb{R}^n$, see for example [15].

Remark 4 In this paper we need asymptotic estimates of sequences of numbers or functions defined on an interval $[0, T]$. In order to simplify the notations, we use the symbol M to denote a function of T (not the same function at every occurrence) *which does not depend on the index of the sequences involved.* As T is considered as fixed, M will be called "a constant". Furthermore we introduce the following notations: $f_n \asymp g_n$ (for $n \to +\infty$) when the inequalities $M_1||g_n|| \leq ||f_n|| \leq M_2||g_n||$ hold with $M_1 > 0$ and n large enough. Instead, we write $f_n = \mathrm{O}(g_n)$ when there exists a constant M such that $||f_n|| \leq M||g_n||$.

For each n, f_n and g_n may belong to suitable normed spaces (in particular, we might have sequences of functions.) ∎

2 Literature

Equations of the form (1) have been independently proposed long time ago in different applications: applications to viscoelastic matherials go back to J.C. Maxwell while a special instance of Eq. (1) as a "heat equation with finite diffusion speed" was first proposed in [2] and then in its general form in [8] at the end of the sixth'ies. After that, study of Volterra equations in Banach spaces with unbounded kernels fourished, and we refer to [15] for an overview of the most important among the first results on the subject, which is still very active, in particular for the study of stability/stabilization properties, a point we are not going to consider (see for example [3] and references therin).

In this paper we are going to identify several Riesz systems naturally related to Eq. (1). It seems that the first paper to note the existence of Riesz systems naturally associated to Eq. (1) is [10]. See [23] for a Riesz basis associated to a special instance of Eq. (1).

3 Preliminaries

We introduce the following selfadjoint negative operator A:

$$\operatorname{dom} A = H^2(0,\pi) \cap H^1_0(0,\pi)\,, \qquad A\phi = (a(x)\phi_x)_x - b(x)\phi\,. \tag{4}$$

It is well known that this operator has compact resolvent, and that it has a sequence of negative eigenvalues $\{-\lambda_n^2\}_{n\geq 1}$, with the following asymptotic estimate [20, p. 227)]

$$\lambda_n = n + \frac{H_n}{n}\,, \qquad |H_n| < M \tag{5}$$

Remark 5 In fact, these asymptotic formulas (and the formulas below for the eigenfunctions) should have the coefficient π/L, $L = \int_0^\pi \left(1/\sqrt{a(x)}\,\mathrm{d}x\right)$, in front of n. This coefficient can be reduced to 1 with a changment in the time scale. So, we assume the coefficient to be 1 for simplicity. However, we should take into account these transformations if we were to compute the *optimal* observability/controllability time. This problem is not considered in this paper. ∎

The corresponding normalized eigenfunctions $\phi_n(x)$ have the following estimates, see [20, p. 230-231]:

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\left\{\sin nx + \frac{k_n(x)}{n}\right\}\,, \qquad |k_n(x)| < M \tag{6}$$

$$\phi_n'(x) = \sqrt{\frac{2}{\pi}}\left\{n\cos nx + \tilde{k}_n(x)\right\}\,, \qquad |\tilde{k}_n(x)| < M\,. \tag{7}$$

In section 4 we shall need a more precise estimate of $\phi_n(x)$ (see [20, formulas (84"') p. 230)]:

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\left\{\sin nx - \frac{1}{n}K(x)\cos nx\right\} + \frac{M_n(x)}{n^2}\,,$$
$$|K(x)| < M\,, \quad |K'(x)| < M\,, \quad |M_n(x)| < M\,. \tag{8}$$

Note that the previous formulas have been taken from [20] but this reference has $n+1$, with $n \geq 0$, where we wrote n (and we intend $n \geq 1$). More precise estimates are known, see [20, formula (82 p. 227)] for the eigenvalues and [20, formula(85) p. 231)] for the derivatives of the eigenfunctions, but we don't need this.

Note that the normalized eigenfunctions above are real.

The proof of Theorem 3 will show that for "smooth" initial conditions and null control, the solution $\theta(t)$ is differentiable so that the following computation makes sense for such initial conditions. Let $\theta(t)$ solve Eq. (1) and let $\tilde{\theta}(t) = e^{2\beta t}\theta(t)$. The function $\tilde{\theta}(t)$ satisfies:

$$\begin{aligned}
&\tilde{\theta}'(t) = 2\beta e^{2\beta t}\theta(t) \\
&+e^{2\beta t}\left\{2\alpha\theta(t) + \int_0^t N(t-s)\left[(a(x)\theta_x(s))_x - b(x)\theta(s)\right]\,\mathrm{d}s\right\} \\
&= 2(\beta+\alpha)\tilde{\theta}(t) + \\
&\int_0^t \tilde{N}(t-s)\left[(a(x)\tilde{\theta}_x(s))_x - b(x)\tilde{\theta}(s)\right]\,\mathrm{d}s
\end{aligned} \tag{9}$$

and now $\tilde{N}(t) = e^{2\beta t}N(t)$. The previous computations make sense only if the initial condition is smooth but, by continuity, $\tilde{\theta}(t)$ solves Eq. (9) in the "weak" sense specified in section 4 for every initial condition.

The parameter β here is a "free" parameter whose value does not affect neither the observability/controllability properties nor the Riesz basis property. So, once the previous formal computation has been justified, it can be choosen at will, so to simplify the subsequent manipulations. For example, if we choose $\beta = -\alpha$ we can cancel the non integrated term. However, the most convenient choice is

$$\beta = -N'(0)/2\,.$$

With this choice we get an equation of the same form as (1) with α replaced by $\alpha+\beta$ and with the new kernel $\tilde{N}(t)$ which has the property that $\tilde{N}'(0) = 0$.

This point being understood, we shall drop the notation ˜ for the new kernel and for θ, and the coefficient $\alpha+\beta$ will be again denoted α.

The main goal of this paper is the proof that suitable sequences are Riesz sequences in $L^2(0,T_0)$, where T_0 will be identified below. We recall that a sequence of function is a Riesz sequence when it is the image of an orthonormal sequence under a bounded boundedly invertible transformation. Riesz sequences are called L-sequences in [1].

The two crucial tests of Riesz basicity are the following ones. The first, a minor modification of [24, Theorem 13 p. 41], is an adaptation to Hilbert spaces of Paley-Wiener theorem:

Theorem 6 *Let H be a Hilbert space and let $\{\epsilon_n\}$ be a Riesz system in H. Let $\{z_n\}$ be a sequence in H such that*

$$\sum_{n=1}^{+\infty} ||z_n - \epsilon_n||^2 < +\infty\,.$$

There exists N such that $\{z_n\}_{n\geq N}$ is a Riesz system too.

<u>Proof.</u> Let $\{e_n\}$ be an orthonormal sequence such that $\epsilon_n = T^{-1}e_n$ with T and T^{-1} bounded. Let $\zeta_n = Tz_n$. It is clear that

$$\sum_{n=1}^{+\infty} ||\zeta_n - e_n||^2 < +\infty$$

and that it is sufficient to prove that $\{\zeta_n\}_{n>N}$ is a Riesz system for N large enough.

We choose N such that

$$\sum_{n\geq N} ||\zeta_n - e_n||^2 < q < 1$$

and we define the operator S from $\operatorname{cl}\operatorname{span}\{e_n\}_{n\geq N}$ to H:

$$S\sum_{n\geq N}\alpha_n e_n = \sum_{n\geq N}\alpha_n\zeta_n\,.$$

We prove that S is bounded and boundedly invertible. It is bounded since

$$\left|\left|S\sum_{n\geq N}\alpha_n e_n\right|\right|^2 \leq \left|\left|\sum_{n\geq N}\alpha_n[Se_n - e_n]\right|\right|^2 + \left|\left|\sum_{n\geq N}\alpha_n e_n\right|\right|^2 \leq \left\{\sum_{n\geq N}|\alpha_n|^2\right\}(1+q)\ .$$

Its inverse is bounded since $S = (S-I)+I$ and

$$\left|\left|(S-I)\sum_{n\geq N}\alpha_n e_n\right|\right|^2 = \left|\left|\sum_{n\geq N}\alpha_n(\zeta_n - e_n)\right|\right|^2 \leq q\left[\sum_{n\geq N}|\alpha_n|^2\right] \qquad \text{and } q<1\,. \quad \blacksquare$$

Now we give the following definition: a sequence $\{f_n\}$ in a Hilbert space is *ω-independent* when

$$\sum_{n=1}^{+\infty}|\alpha_n|^2 < +\infty\,,\quad \sum_{n=1}^{+\infty}\alpha_n f_n = 0 \implies \sum_{n=1}^{+\infty}|\alpha_n|^2 = 0$$

i.e. every α_n has to be zero.

Now we state the following result (see [24, p. 45] and [6, p. 322])

Theorem 7 (Bari Theorem) *Let $\{\epsilon_n\}$ be a Riesz system and let $\{z_n\}$ be an ω-independent system such that*

$$\sum ||\epsilon_n - z_n||^2 < +\infty\,.$$

Then, $\{z_n\}$ is a Riesz system too.

The statement in [24] assumes that the sequences $\{\epsilon_n\}$ is an orthonormal sequence, and the statement in [6] assumes that it is a Riesz basis, but the statements are easily adapted to the case that $\{\epsilon_n\}$ is merely a Riesz system.

4 Observability

Observability is studied in this section. In the course of this study, we prove a part of Theorem 3. Furthermore, and this is the focus of our interest, we identify the Riesz system associated to Eq. (1).

We proceed formally: let $\theta(t) = \theta(t,x)$ solve problem (1)-(2) with $u(t) = 0$ and any $\xi \in L^2(0,\pi)$. Then, we can represent

$$\theta(t) = \sum_{n=1}^{+\infty}\tilde{z}_n(t)\phi_n(x)\,,\qquad \tilde{z}_n(t) = \int_0^\pi \theta(t,x)\phi_n(x)\,\mathrm{d}x\,. \tag{10}$$

We differentiate formally under the integral and, using the definition of the operator A, we see that

$$\tilde{z}_n'(t) = 2\alpha\tilde{z}_n(t) - \lambda_n^2 \int_0^t N(t-s)\tilde{z}_n(s)\,\mathrm{d}s\,,$$
$$\tilde{z}_n(0) = \xi_n = \int_0^\pi \xi(x)\phi_n(x)\,\mathrm{d}x\,.$$

This suggests the introduction of the sequence of functions

$$z_n'(t) = 2\alpha z_n(t) - \lambda_n^2 \int_0^t N(t-s)z_n(s)\,\mathrm{d}s\,, \qquad z_n(0) = 1 \tag{11}$$

so that

$$\tilde{z}_n(t) = z_n(t)\xi_n\,. \tag{12}$$

The following lemma will be proved below:

Lemma 8 *Let T be fixed. The sequence $\{z_n(t)\}$ is bounded on $[0,T]$, $|z_n(t)| < M$ for every $n \geq 1$ and every $t \in [0,T]$.*

This and (12) show that the series (10) converges in $C(0,T;L^2(0,\pi))$ for every T. So, we use the series (10) as our definition of the solution of problem (1)-(2) with $u(t) = 0$. Furthermore, this solution depends continuously on $\xi(x) \in L^2(0,\pi)$. This proves a part of Theorem 3.

We now use formula (8) in order to to give an estimate of the coefficients ξ_n when $\xi \in H_0^1(0,\pi) \cap H^2(0,\pi)$ (estimate which can also be derived since in this case $\xi \in \operatorname{dom} A$).

Lemma 9 *Let $\xi(x) \in H_0^1(0,\pi)$. Then*

$$|\xi_n| < \frac{\gamma_n}{n}\,, \qquad \{\gamma_n\} \in l^2\,.$$

Let $\xi \in H_0^1(0,\pi) \cap H^2(0,\pi)$ Under this condition we have

$$|\xi_n| \leq \frac{M}{n^2}\,.$$

Proof. Using the asymptotic formula (8), we have

$$\sqrt{\frac{\pi}{2}}\xi_n = \sqrt{\frac{\pi}{2}}\int_0^\pi \xi(x)\phi_n(x)\,\mathrm{d}x = \int_0^\pi \xi(x)\sin nx\,\mathrm{d}x$$
$$-\frac{1}{n}\int_0^\pi \xi(x)K(x)\cos nx\,\mathrm{d}x + \frac{M_n(x)}{n^2}\,.$$

If $\xi \in H_0^1(0,\pi)$ we integrate the first integral by parts once and we get the required estimate.

Let furthermore $\xi \in H^2(0,\pi)$. We integrate by parts the first integral twice and the second integral once (using the fact that the function $K(x)$ is differentable, see [20, formulas (76) p. 222 and (84"') p. 230]). This gives the result. ∎

We now state a property of the sequence $\{z_n(t)\}$, which will be proved in Section 5.1.

Lemma 10 *The sequence* $\{z'_n(t)/n\}$ *is bounded.*

Using Lemmas 7 and 10, we see that when $\xi \in H^1_0(0,\pi) \cap H^2(0,\pi)$ then the series in (10) can be differentiated termwise and we have

Lemma 11 *Let* $\xi \in H^1_0(0,\pi) \cap H^2(0,\pi)$. *Then, the solution* $\theta(t)$ *of Eq. (1) given by (10) belongs to* $W^{1,2}(0,T;L^2(0,\pi))$ *for every* $T > 0$.

This is the result we need in order to prove that we can impose the condition $N'(0) = 0$, without restriction.

Remark 12 The previous results depend on the boundedness of the sequences $\{z_n(t)\}$ and $\{z'_n(t)/n\}$, still to be proved. This is proved in Lemma 14. The condition $N'(0) = 0$ will simplify the arguments in that section but it will not be used in the proof of this lemma. ∎

With the sole condition $\xi(x) \in L^2(0,\pi)$ the observation (3) needs not be locally square integrable. So, we now assume $\xi(x) \in H^1_0(0,\pi)$. We first prove (compare [15, Theorem 24]).

Lemma 13 *Let* $T > 0$ *be fixed and let* $y(t)$ *be given by (3). The transformation* $\xi \to y(\cdot)$ *is continuous from* $H^1_0(0,\pi) \to L^2(0,T)$

Proof. We introduce the operator $D_0 : \mathbb{R} \to L^2(0,\pi)$, such that the function $g(x) = (D_0 u)(x)$ solves

$$g''(x) = 0\,, \qquad g(0) = u\,, \quad g(\pi) = 0 \qquad \text{i.e.} \qquad g(x) = D_0 u = \frac{1}{\pi}(\pi - x)u\,.$$

Then,

$$\xi'(0) = -D_0^* A_0 \xi \qquad \forall \xi \in H^1_0(0,\pi)$$

where $\operatorname{dom} A_0 = H^1_0(0,\pi) \cap H^2(0,\pi)$, $A_0\xi = \xi''(x)$.

Let us consider now

$$D_0^* A_0 \left[\sum_{n=1}^{M} z_n(t)\xi_n\phi_n(x)\right] = \sum_{n=1}^{k} z_n(t)\xi_n\phi'_n(0) \tag{13}$$

and, using the estimate for the sequence $\{\xi_n\}$ in Lemma 9 and (7) we get

$$\{\xi_n(t)\phi'_n(0)\} \in l^2\,.$$

We shall prove, and this is the crucial result of this paper, that $\{z_n(t)\}$ is a Riesz sequence in $L^2(0,T_0)$ for a certain time T_0 identified in Section 6, and for every larger time. This proves that

1) The series (13) converges in $L^2(0,T)$, and that $y(\cdot) \in L^2(0,T)$ depends continuously on $\xi \in H^1_0(0,\pi)$. This is clear for $T > T_0$, thanks to the Riesz property of $\{z_n(t)\}$. The result holds also for $T < T_0$ since $||y(t)||_{L^2(0,T)} \le ||y(t)||_{L^2(0,T_0)}$ if $T < T_0$.

2) It is possible to have zero output on $(0,T_0)$, $y(t) = 0$ a.e. on $(0,T_0)$ if and only if $\xi - n\phi'_n(0) = 0$, i.e. $\xi_n = 0$ for every n, since $\phi'_n(0) \neq 0$. Furtermore,

$$||\{n\xi_n\}||_{l^2} \asymp ||y(t)||_{L^2(0,T_0)}$$

thanks to the Riesz property of the sequence $\{z_n(t)\}$ in $L^2(0,T_0)$ and $\phi'_n(0) \asymp n$.

This is the required observability result. So, in order to complete the arguments in this section, we prove that $\{z_n(t)\}$ is a Riesz sequence in $L^2(0,\pi)$. ∎

5 The properties of the sequence $\{z_n(t)\}$

Now we are going to study the sequence $\{z_n(t)\}$.

We compute the derivatives of both sides of Eq. (11). We find

$$z_n'' = -\lambda_n^2 z_n(t) + 2\alpha z_n'(t) - \lambda_n^2 \int_0^t N'(t-s) z_n(s)\,\mathrm{d}s\,, \qquad \left\{ \begin{array}{l} z_n(0) = 1 \\ z_n'(0) = 2\alpha\,. \end{array} \right. \tag{14}$$

We introduce

$$\sigma_n = \alpha \pm i\beta_n\,, \qquad \beta_n = \sqrt{\lambda_n^2 - \alpha^2}\,.$$

Note that $\beta_n \neq \beta_k$ for $n \neq k$ and that β_n is real for large n. We see, from (5)

$$|\beta_n - \lambda_n| < \frac{M}{n}\,, \qquad \beta_n \asymp \lambda_n \asymp n\,, \qquad |\beta_n - n| < \frac{M}{n}\,. \tag{15}$$

Eq. (14) leads to the following formula for $z_n(t)$:

$$\begin{aligned} z_n(t) &= g_n(t) - \frac{\lambda_n^2}{\beta_n} \int_0^t \sin\beta_n s \left[e^{\alpha s} \int_0^{t-s} N'(t-s-r) z_n(r)\,\mathrm{d}r \right] \mathrm{d}s\,, \\ g_n(t) &= e^{\alpha t} \left[\cos\beta_n t + \frac{\alpha}{\beta_n} \sin\beta_n t \right]. \end{aligned} \tag{16}$$

Formula (16) holds even if β_n is not real, but it takes a different form if $\beta_n = 0$. This is only possible if $\lambda_n = \alpha$ and this is possible for one index n_0 at most. In this case we get the representation

$$z_{n_0}(t) = e^{\alpha t}(1+\alpha t) - \alpha^2 \int_0^t (t-r) e^{\alpha(t-r)} \int_0^r N'(r-s) z_{n_0}(s)\,\mathrm{d}s\;\mathrm{d}r\,. \tag{17}$$

Using that $\sin\beta_n t$ is the derivative of $-(1/\beta_n)\cos\beta_n t$, a partial integration in (16) gives:

$$\begin{aligned} z_n(t) = g_n(t) + \frac{\lambda_n^2}{\beta_n^2} \Big\{ & -\int_0^t N'(t-r) z_n(r)\,\mathrm{d}r \\ & -\int_0^t e^{\alpha s} [\cos\beta_n s] \left[\alpha \int_0^{t-s} N'(t-s-r) z_n(r)\,\mathrm{d}r \right. \\ & \left. -N'(0) z_n(t-s) - \int_0^{t-s} N''(t-s-r) z_n(r)\,\mathrm{d}r \right] \mathrm{d}s \Big\}\,. \end{aligned} \tag{18}$$

Gronwall inequality shows the first statement in the following Lemma:

Lemma 14 *Let $T > 0$ be fixed. The sequence $\{z_n(t)\}$ and $\{z_n'(t)/n\}$ are bounded on $[0,T]$.*

Proof. Boundedness of $\{z_n(t)\}$ follows from (18) and Gronwall Lemma. Now we prove boundedness of the sequence $\{z'_n(t)/n\}$. It is sufficient to consider those indices n such that $\beta_n \neq 0$. We have:

$$z'_n(t) = g'_n(t) - \frac{\lambda_n^2}{\beta_n^2}\left\{ N'(0)z_n(t) + \int_0^t N''(t-r)z_n(r)\,\mathrm{d}r \right.$$
$$-N'(0)e^{\alpha t}\cos\beta_n t - N'(0)\int_0^t e^{\alpha s}[\cos\beta_n s]z'_n(t-s)\,\mathrm{d}s$$
$$+\int_0^t [e^{\alpha s}\cos\beta_n s]\left[[\alpha N'(0) - N''(0)]\, z_n(t-s) \right.$$
$$\left.\left. + \int_0^{t-s} [\alpha N''(t-s-r) - N'''(t-s-r)]\, z_n(r)\,\mathrm{d}r \right] \mathrm{d}s \right\}.$$

We use boundedness of $\{z_n(t)\}$ and the fact that $\{g'_n(t)/n\}$ is bounded. Boundedness of $\{z'_n(t)/n\}$ follows. ∎

Remark 15 Note that the condition $N'(0) = 0$ has not been used in the previous proofs. So, the arguments in section 3 are now completely justified and we can assume $N'(0) = 0$ later on. In particular, *from now on we can use formula (18) with* $N'(0) = 0$. ∎

We are now going to prove that the sequence $\{z_n(t)\}$ and the sequence

$$\left\{ n\int_0^t N(t-s)z_n(s)\,\mathrm{d}s \right\}_{n\geq 1} \tag{19}$$

are Riesz sequences in $L^2(0,T_0)$, with T_0 suitably chosen. We relay on Bari Theorem, so that we prove that these sequences are quadratically close to suitable Riesz sequences, and are ω-independent.

So, with reference to Bari theorem, the proof the Riesz property is in two parts: closeness to some Riesz first and then the proof of it is ω-independence. Theorem 6 is used in this second part of the proof.

5.1 Closeness to a Riesz system

We proceed essentially as in [10]. We recall that we can use formula (18) with $N'(0) = 0$. Consider those indices n, say $n > N_0$, so large that β_n is real and not zero. We define

$$e_n(t) = z_n(t) - g_n(t)\,, \qquad H(t) = \alpha N'(t) - N''(t)$$

so that

$$e_n(t) = -\frac{\lambda_n^2}{\beta_n^2}\left\{ \int_0^t N'(t-r)e_n(r)\,\mathrm{d}r \right.$$
$$\left. + \int_0^t e^{\alpha s}\cos\beta_n s\left[\int_0^{t-s} H(t-s-r)e_n(r)\,\mathrm{d}r \right] \mathrm{d}s \right\}$$
$$-\frac{\lambda_n^2}{\beta_n^2}\left\{ \int_0^t N'(t-r)g_n(r)\,\mathrm{d}r \right. \tag{20}$$
$$\left. + \int_0^t e^{\alpha s}\cos\beta_n s\left[\int_0^{t-s} H(t-s-r)g_n(r)\,\mathrm{d}r \right] \mathrm{d}s \right\} \tag{21}$$

Using that $\cos\beta_n s$ is the derivative of $(\sin\beta_n s)/\beta_n$, we can integrate by parts the last brace. In this way we see that the last brace is of the order of $1/\beta_n$, i.e. it is of the order of $1/n$. Using Gronwall inequality and (15), we get the following estimates, which hold on a fixed interval $[0,T]$:

$$|e_n(t)| = |z_n(t) - g_n(t)| \le \frac{M}{n}$$

and

$$|z_n(t) - e^{\alpha t}\cos\beta_n t| \le \frac{M}{n}\,.$$

Now we observe that $|\cos\beta_n t - \cos nt| < M/n$. So we get

Lemma 16 *Let $T>0$ be fixed. There exists a constant M such that*

$$|z_n(t) - e^{\alpha t}\cos\beta_n t| < \frac{M}{n}\,, \qquad |z_n(t) - e^{\alpha t}\cos nt| \le \frac{M}{n}\,. \tag{22}$$

Consequently, the sequence $\{z_n(t)\}$ is quadratically close to the Riesz sequences $\{e^{\alpha t}\cos nt\}$ on $[0,\pi]$ and to the sequence $\{e^{\alpha t}\cos\beta_n t\}$, which is a Riesz sequence on a suitable interval $[0,T_0]$ identified below. Using Theorem 6, we can state:

Theorem 17 *Let $T \ge \min\{\pi, T_0\}$. There exists $N>0$ such that $\{z_n(t)\}_{n>N}$ is a Riesz system in $L^2(0,T)$.*

In the proof of the controllability property, we shall need a more precise estimate. We consider the integrals at the lines (20)-(21). The integral at the line (20) can be integrated by parts twice, and it is seen to be of the order of $1/\beta_n^2$. Instead, a second integration by parts of the integral at the line (21) gives a term of the order $1/\beta_n^2$ plus the terms

$$\frac{N_0'' t e^{\alpha t}}{2\beta_n}\sin\beta_n t$$

and the integral below, which can be estimated by

$$\frac{a_n(t)}{\beta_n}\,, \qquad \sum_{n=1}^{+\infty}|a_n(t)|^2 < M\,.$$

The integral in question is

$$\frac{1}{\beta_n}\int_0^t \left[\int_0^{t-s}[\alpha H(t-s-r) - H'(t-s-r)]e^{\alpha r}\cos\beta_n r\,\mathrm{d}r\right] e^{\alpha s}\sin\beta_n s\,\mathrm{d}s\,.$$

We get (see [15] for details)

Lemma 18 *Let*

$$\hat{e}_n(t) = e_n(t) - \frac{N''(0)te^{\alpha t}}{2\beta_n}\sin\beta_n t = z_n(t) - e^{\alpha t}\cos\beta_n t - \frac{N''(0)te^{\alpha t}}{2\beta_n}\sin\beta_n t\,.$$

Then we have

$$|\hat{e}_n(t)| \le \frac{a_n(t)}{\beta_n} + \frac{M}{\beta_n^2}\,, \qquad \sum_{n=1}^{+\infty}|a_n(t)|^2 < M\,.$$

Finally let us consider the sequence (19). We prove

Lemma 19 *The sequence in (19) is quadratically close to the sequence $\{e^{\alpha t}\sin\beta_n t\}$ in $L^2(0,T)$ for every T.*

Proof. We insert the expression of $z_n(t)$ from (18) (with $N'(0)=0$) and we get four integrals, which we elaborate as follows. The first integral is

$$\int_0^t N(t-s)g_n(s)\,\mathrm{d}s=\int_0^t N(t-s)e^{\alpha s}\cos\beta_n s\,\mathrm{d}s+\frac{\alpha}{\beta_n}\int_0^t N(t-s)e^{\alpha s}\sin\beta_n s\,\mathrm{d}s\,.$$

Partial integration of both the integrals gives

$$\int_0^t N(t-s)g_n(s)\,\mathrm{d}s=\frac{1}{\beta_n}e^{\alpha t}\sin\beta_n s+\mathrm{O}(1/n^2)\,.$$

We are going to see that the remaining integrals are of the order of $1/\beta_n^2$. We consider first the following integral:

$$\int_0^t N(t-s)\int_0^s N'(s-r)z_n(r)\,\mathrm{d}r\;\mathrm{d}s=\int_0^t N'(r)\int_0^{t-r}N(t-r-\tau)z_n(\tau)\,\mathrm{d}\tau\;\mathrm{d}r$$
$$=-\int_0^t N'(r)\frac{1}{\lambda_n^2}\left\{-2\alpha z_n(t-r)+z_n'(t-r)\right]\,\mathrm{d}r=-\frac{2\alpha}{\lambda_n^2}\int_0^t N'(t-r)z_n(t-r)\,\mathrm{d}r$$
$$+\frac{1}{\lambda_n^2}\left\{N'(t)-\int_0^t N''(r)z_n(t-r)\,\mathrm{d}r\right\}\,.$$

We now consider the last integral in (18) which gives

$$\int_0^t N(t-\tau)\int_0^\tau e^{\alpha s}\cos\beta_n s\left[\int_0^{\tau-s}N''(\tau-s-\nu)z_n(\nu)\,\mathrm{d}\nu\right]\,\mathrm{d}s\;\mathrm{d}\tau\,.$$

We integrated by parts once, and we have the sum of three terms. We first consider

$$\frac{N''(0)}{\beta_n}\int_0^t N(t-\tau)\int_0^\tau e^{\alpha(\tau-s)}\sin\beta_n(\tau-s)z_n(s)\,\mathrm{d}s\;\mathrm{d}\tau$$
$$=\frac{N''(0)}{\beta_n}\int_0^t z_n(s)\int_s^t e^{\alpha(\tau-s)}N(t-\tau)\sin\beta_n(\tau-s)\,\mathrm{d}\tau\;\mathrm{d}s<\frac{M}{\beta_n^2}$$

as it is seen with a last integration by parts. The second integral is

$$\frac{\alpha}{\beta_n}\int_0^t N(t-\tau)\int_0^\tau e^{\alpha s}\sin\beta_n s\int_0^{\tau-s}N''(\tau-s-r)z_n(r)\,\mathrm{d}r\;\mathrm{d}s\;\mathrm{d}\tau\asymp\frac{1}{\beta_n^2}\,.$$

Finally we consider

$$\frac{1}{\beta_n}\int_0^t N(t-\tau)\int_0^\tau \sin\beta_n(\tau-s)e^{\alpha(\tau-s)}\left]\int_0^s N'''(s-r)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s\;\mathrm{d}\tau$$
$$=\frac{1}{\beta_n}\int_0^t\left[\int_s^t N(t-\tau)e^{\alpha(\tau-s)}\sin\beta_n(\tau-s)\,\mathrm{d}\tau\right]\left[\int_0^s N'''(s-r)z_n(r)\,\mathrm{d}r\right]\,\mathrm{d}s\asymp\frac{1}{\beta_n^2}$$

using a last partial integration with respect to τ.

The intermediate term in the brace of (18) gives a contribution which can be treated analogously. ∎

6 The proof of ω-independence and the observability time

We recall

$$\beta_n = \sqrt{\lambda_n^2 - \alpha^2}\,.$$

So, either β_n is purely imaginary (for small n) or it is real. It can be 0 for one index at most, that we denote n_0. In the following arguments we consider this case that $\beta_{n_0} = 0$. If this does not happen, then $\beta_n \neq 0$ for every n and in the formulas below just delete the corresponding term, and replace $\sum_{n\neq n_0}$ with $\sum_{n=1}^{+\infty}$.

It is clear that

$$\begin{aligned} &G_0 = \inf\{|\beta_n - \beta_k|\,,\ n\neq k\} > 0\,,\\ &G = \inf\{|\beta_n - \beta_k|\,,\ n\neq k\,,\ \beta_n,\beta_k\in\mathbb{R}\} > 0\,. \end{aligned}$$

Now we use [13, Ch. 4]. It follows from Ingham theorem (if every β_n is real) and from Ingham theorem combined with Haraux theorem otherwise, that

$$\{1, t, e^{i\beta_n t}\,, e^{-i\beta_n t}\}_{n\neq n_0} \tag{23}$$

is a Riesz system in $L^2(-T_0, T_0)$ provided that $T_0 > \pi/G$.

This is our choice of T_0.

Note the role of the index $n\neq n_0$ in (23). If $\beta_n \neq 0$ for every n then that index has to be disregarded. Otherwise it is needed in order to avoid repetitions of the constant function 1. If $\beta_n\neq 0$ for every n then we can remove the function t from (23).

We first note that our choice of T_0 implies that $\{\cos\beta_n t\}$ is a Riesz system in $L^2(0,T_0)$. In fact, let $\{\alpha_n\}_{n\geq 1}\in l^2$ and define $\alpha_{-n} = \alpha_n$. Then,

$$f(t) = \sum_{n=1}^{+\infty} \alpha_n \cos\beta_n t = \alpha_{n_0} + \frac{1}{2}\sum_{n\neq n_0} \alpha_n e^{i\beta_n t}$$

so that $||f||^2_{L^2(0,T_0)} \asymp ||\{\alpha_n\}||^2_{l^2}$. A similar observation holds for the sequence $\{\sin\beta_n t\}_{n\neq n_0}$.

Now we define the following subspaces of $L^2(0,T_0)$:

$$\begin{aligned} &X = \operatorname{cl\,span}\{1, t, e^{i\beta_n t}\,, e^{-i\beta_n t}\}_{n\neq\pm n_0}\,,\\ &X_0 = \operatorname{cl\,span}\{1, e^{i\beta_n t}\,, e^{-i\beta_n t}\}_{n\neq\pm n_0}\,,\\ &X_1 = \operatorname{cl\,span}\{e^{i\beta_n t}\,, e^{-i\beta_n t}\}_{n\neq\pm n_0}\,. \end{aligned}$$

If $\phi\in X$ then its incremental quotient is in X_0. If furthermore $\phi(t)\in W^{1,2}(0,T_0)$ then it is continuous on the *closed* interval $[0,T_0]$. In this case, $\phi'(t)$ is the limit, in $L^2(0,T_0)$-norm, of the incremental quotient, so that $\phi'(t)\in X_0$. If it happens that $\phi\in W^{2,2}(0,T_0)$ then $\phi''\in X_1$.

We rewrite (18) and (17) in the following forms (we recall that we already performed that transformation which reduces $N'(0)$ to be 0)

$$\begin{aligned}
&\textbf{for } n\neq n_0\textbf{:}\\
&z_n(t)=e^{\alpha t}\cos\beta_n t+\frac{\alpha}{\beta_n}e^{\alpha t}\sin\beta_n t-\frac{\lambda_n^2}{\beta_n^2}\int_0^t K_n(t-s)z_n(s)\,\mathrm{d}s\\
&K_n(t)=N'(t)+\int_0^t[\alpha N'(t-r)-N''(t-r)]e^{\alpha r}\cos\beta_n r\,\mathrm{d}r\,,\\
&\textbf{while for } n=n_0\textbf{:}\\
&z_{n_0}(t)=(1+\alpha t)e^{\alpha t}-\alpha^2\int_0^t K_{n_0}(t-s)z_{n_0}(s)\,\mathrm{d}s\\
&K_{n_0}(t)=\int_0^t(t-r)e^{\alpha(t-r)}N'(r)\,\mathrm{d}r\,.
\end{aligned}$$

Note that

$$\begin{aligned}
&K_n'(t)=N''(t)-N''(0)e^{\alpha t}\cos\beta_n t+\int_0^t\left[\alpha N''(t-r)-N'''(t-r)\right]e^{\alpha r}\cos\beta_n r\,\mathrm{d}r\\
&K_{n_0}'(t)=\int_0^t e^{\alpha(t-r)}[1+\alpha(t-r)]N'(r)\,\mathrm{d}r\,,\\
&K_n(0)=0\,,\quad K_n'(0)=0\quad\text{for every } n.
\end{aligned}$$

The sequences $\{K_n(t)\}$, $\{K_n'(t)\}$, are bounded on $[0,T_0]$.

After these preliminaries, we can prove the following result:

Theorem 20 *The sequence $\{z_n(t)\}$ is ω-independent in $L^2(0,T_0)$, hence it is a Riesz system in $L^2(0,T_0)$.*

The proof is based on Theorem 17.

We assume that a sequence $\{\alpha_n\}\in l^2$ satisfies

$$\sum_{n=1}^{+\infty}\alpha_n z_n(t)=0\,,\tag{24}$$

the convergence being in the norm of $L^2(0,T_0)$. We are going to prove that $\alpha_n=0$ for every n. We introduce the function

$$\phi(t)=-e^{\alpha t}\left\{\sum_{n\neq n_0}\alpha_n\cos\beta_n t+\alpha_{n_0}(1+\alpha t)\right\}\,.\tag{25}$$

If $\beta_n\neq 0$ for every n then the last addendum has to be removed.

Thanks to equality (24), we can write

$$\phi(t)=\sum_{n\neq n_0}\alpha_n\left[z_n-e^{\alpha t}\cos\beta_n t\right]+\alpha_{n_0}\left[z_{n_0}-e^{\alpha t}(1+\alpha t)\right]\,.\tag{26}$$

It is convenient to introduce the notation:

$$e_n(t)=\left[z_n(t)-e^{\alpha t}\cos\beta_n t\right]\,.$$

Note that the functions $e_n(t)$ here are different from those in Section 5.1 but the following inequality follows from the results in Section 5.1:

$$|e_n(t)| < \frac{M}{n} \qquad \forall n\,,\ \forall t \in [0, T_0]\,.$$

We now proceed in three steps to prove that condition (24) implies $\alpha_n = 0$ for every n. The first two steps study the regularity of $\phi(t)$.

Step 1 we prove that $\alpha_n = \tilde{\sigma}_n/\beta_n$, and $\{\tilde{\sigma}_n\} \in l^2$.

We represent the right hand side of (26) as

$$\sum_{n\neq n_0} \alpha \frac{\alpha_n}{\beta_n} e^{\alpha t} \sin \beta_n t - \sum_{n\neq n_0} \alpha_n \frac{\lambda_n^2}{\beta_n^2} \int_0^t K_n(t-s) z_n(s)\, \mathrm{d}s$$

$$-\alpha_{n_0}\alpha^2 \int_0^t K_{n_0}(t-s) z_{n_0}(s)\, \mathrm{d}s$$

$$= \sum_{n\neq n_0} \alpha \frac{\alpha_n}{\beta_n} e^{\alpha t} \sin \beta_n t \tag{27}$$

$$- \sum_{n\neq n_0} \alpha_n \frac{\lambda_n^2}{\beta_n^2} \int_0^t K_n(t-s) e_n(s)\, \mathrm{d}s \tag{28}$$

$$- \sum_{n\neq n_0} \alpha_n \frac{\lambda_n^2}{\beta_n^2} \int_0^t K_n(t-s) e^{\alpha s} \cos \beta_n s\, \mathrm{d}s \tag{29}$$

$$-\alpha_{n_0}\alpha^2 \int_0^t K_{n_0}(t-s) e_{n_0}(s)\, \mathrm{d}s\,. \tag{30}$$

The function in (30) is of class C^2 and it is zero for $t = 0$. Let us consider (28). The series $\sum_{n=1}^{+\infty} \alpha_n e_n(t)$ is uniformly convergent since $|e_n(t)| < M/n$ and $\{\lambda_n^2/\beta_n^2\}$ is bounded. So, the series can be exchanged with the integral. It follows that this series is a $C^1(0, T_0)$ function, which is zero for $t = 0$.

We now consider (29). Using $K_n(0) = 0$, we see that

$$-\int_0^t K_n(t-s) e^{\alpha s} \cos \beta_n s\, \mathrm{d}s = \frac{1}{\beta_n} \int_0^t [\alpha K_n(t-s) - K_n'(t-s)] e^{\alpha s} \sin \beta_n s\, \mathrm{d}s\,.$$

This shows that also in this case we have uniform convergence and we can exchange the series and the integral so that (29) defines a $C^1(0, T_0)$ function, which is zero for $t = 0$.

Finally we consider (27). This series converges uniformly, so that it is a continuous function which is zero for $t = 0$. Furthermore, the series

$$\sum_{n\neq n_0} \alpha_n \cos \beta_n t \tag{31}$$

converges in $L^2(0, T_0)$. So, the series (27) defines a $W^{1,2}(0, T_0)$ function, which is zero for $t = 0$.

In conclusion,

$$-e^{-\alpha t}\phi(t) = \sum_{n\neq n_0} \alpha_n \cos \beta_n t + \alpha_{n_0}(1 + \alpha t) \in W^{1,2}(0, T_0)\,, \qquad \phi(0) = 0$$

and its derivative belongs to X_0:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-e^{\alpha t}\phi(t)\right]=\tilde{\sigma}_{n_0}+\sum_{n\neq n_0}\tilde{\sigma}_n e^{i\beta_n t}\,,\qquad \beta_{-n}=-\beta_n \tag{32}$$

so that

$$-e^{-\alpha t}\phi(t)=\tilde{\sigma}_{n_0}t+\sum_{n\neq\pm n_0}\frac{\tilde{\sigma}_n}{i\beta_n}e^{i\beta_n t}-\sum_{n\neq\pm n_0}\frac{\tilde{\sigma}_n}{i\beta_n}\,.$$

We expand (25) using Euler formulas and we see that

$$\begin{cases}\alpha_n=\dfrac{2\tilde{\sigma}_n}{\beta_n} & \text{if}\quad n\neq n_0\\ \alpha_{n_0}=\displaystyle\sum_{n\neq\pm n_0}\frac{\tilde{\sigma}_n}{i\beta_n} & \\ \tilde{\sigma}_{n_0}=\alpha\alpha_{n_0}=-\alpha\displaystyle\sum_{n\neq\pm n_0}\frac{\tilde{\sigma}_n}{i\beta_n}\,. & \end{cases}$$

This is the result we wanted to achieve. Note that in the following we shall ignore inessential constants and we replace $2i\tilde{\sigma}_n/\beta_n$ with $\tilde{\sigma}_n/\beta_n$ (and the previous equation implies $\alpha_{n_0}=0$.)

Remark 21 Note that we used that $\{1,t,e^{-\beta_n t},e^{-i\beta_n t}\}$ is a Riesz system in $L^2(0,T_0)$. ∎

Step 2 we prove that $\alpha_n=\sigma_n/n^2$, and $\{\sigma_n\}\in l^2$.

We introduce $\alpha_n=\tilde{\sigma}_n/\beta_n$ in (27)-(30) and we get

$$\begin{aligned}\phi(t)=&\\ =&\,\alpha e^{\alpha t}\sum_{n=1}^{+\infty}\frac{\tilde{\sigma}_n}{\beta_n^2}\sin\beta_n t &(33)\\ &-\sum_{n=1}^{+\infty}\frac{\lambda_n^2\tilde{\sigma}_n}{\beta_n^3}\int_0^t K_n(t-s)e_n(s)\,\mathrm{d}s &(34)\\ &-\sum_{n=1}^{+\infty}\frac{\lambda_n^2\tilde{\sigma}_n}{\beta_n^3}\int_0^t e^{\alpha s}K_n(t-s)\cos\beta_n s\,\mathrm{d}s &(35)\\ &-\alpha_{n_0}\alpha^2\int_0^t K_{n_0}(t-s)e_{n_0}(t-s)\,\mathrm{d}s\,. &(36)\end{aligned}$$

We know already that the last term is of class C^2. We prove that the terms (33)-(35) are of class $W^{2,2}(0,T_0)$. We use the convergence of the series of the first derivatives (already proved. Note that we have uniform convergence also of the series (31) since we proved that $\alpha_n=\tilde{\sigma}_n/\beta_n$). So, we prove $L^2(0,T_0)$ convergence of the series of the second derivatives.

- the series in (33): the second derivative computed termwise is

$$-\sum_{n=1}^{+\infty}\tilde{\sigma}_n \sin\beta_n t$$

and this series converges in $L^2(0,T_0)$ since we already know $\{\tilde{\sigma}_n\}\in l^2$.
- the series in (34): we use $K_n(0)=K_n'(0)=0$ and we see that the second derivative of its terms are

$$\frac{\lambda_n^2\tilde{\sigma}_n}{\beta_n^3}\int_0^t K_n''(t-s)e_n(s)\,\mathrm{d}s$$

and

$$\begin{aligned}
&K_n''(t)=\hat{K}_n(t)+\tilde{K}_n(t)\,,\\
&\hat{K}_n(t)=N'''(t)+[\alpha N''(t)-N'''(t)]-\alpha N''(0)e^{\alpha t}\cos\beta_n t\\
&+\int_0^t[\alpha^2N''(r)-\alpha N'''(r)]e^{\alpha(t-r)}\cos\beta_n(t-r)\,\mathrm{d}r\,,\\
&\tilde{K}_n(t)=\beta_n\Big\{N''(0)e^{\alpha t}\sin\beta_n t\\
&-\int_0^t[\alpha N''(r)-N'''(r)]e^{\alpha(t-r)}\sin\beta_n(t-r)\,\mathrm{d}r\Big\}\,.
\end{aligned}$$

The series of each one of the corresponding terms (multiplied by $\lambda_n^2\tilde{\sigma}_n/\beta_n^3$) converges even uniformly, since $|e_n(t)|<M/n$ and $\beta_n\asymp\lambda_n\asymp n$.
- For each n, the second derivative of the terms in (35) is the sum of two terms. The first one is

$$\frac{\lambda_n^2\tilde{\sigma}_n}{\beta_n^3}\int_0^t\hat{K}_n(t-s)e^{\alpha s}\cos\beta_n s\,\mathrm{d}s\,.$$

Boundedness of the sequence $\{\hat{K}_n(t-s)\}$ shows that the series of these terms converges uniformly on $[0,T_0]$.
The second term is

$$\begin{aligned}
&\frac{\lambda_n^2\tilde{\sigma}_n}{\beta_n^2}\int_0^t e^{\alpha(t-s)}\cos\beta_n(t-s)\,\{N''(0)e^{\alpha s}\sin\beta_n s\\
&-\int_0^s[\alpha N''(r)-N'''(r)]e^{\alpha(s-r)}\sin\beta_n(s-r)\,\mathrm{d}r\Big\}\,\mathrm{d}s\\
&=\frac{\lambda_n^2\tilde{\sigma}_n}{2\beta_n^2}N''(0)\int_0^t e^{\alpha t}[\sin\beta_n t-\sin\beta_n(t-2s)]\,\mathrm{d}s\\
&-\frac{\lambda_n^2\tilde{\sigma}_n}{2\beta_n^2}\int_0^t\Big[\int_0^s[\alpha N''(r)-N'''(r)]\cdot\\
&e^{\alpha(t-r)}[\sin\beta_n(t-r)-\sin\beta_n(t-2s+r)]\,\mathrm{d}r\Big]\,\mathrm{d}s\,.
\end{aligned}$$

Let us consider the first series

$$N''(0)\left[te^{\alpha t}\right]\sum_{n=1}^{+\infty}\frac{\lambda_n^2\tilde{\sigma}_n}{2\beta_n^2}\sin\beta_n t$$

which converges in $L^2(0,T_0)$ since $\{\sin\beta_n t\}_{n>N}$ is a Riesz system. So, the series and the integral can be interchanged.

The fact that
$$\frac{\lambda_n^2\tilde\sigma_n}{2\beta_n^2}\int_0^t e^{\alpha t}\sin\beta_n(t-2s)\,\mathrm{d}s\asymp\frac{1}{\beta_n}$$
shows that the corresponding series converges even uniformly and can be interchanged with the integral.

We consider now
$$\frac{\lambda_n^2\tilde\sigma_n}{\beta_n^2}\int_0^t\int_0^s\left[\alpha N''(r)-N'''(r)\right]e^{\alpha(t-r)}\sin\beta_n(t-r)\,\mathrm{d}r\ \mathrm{d}s$$
$$=\int_0^t\left[\alpha N''(t-r)-N'''(t-r)\right]re^{\alpha r}\left[\frac{\lambda_n^2\tilde\sigma_n}{\beta_n^2}\sin\beta_n r\right]\,\mathrm{d}r\,.$$

We have $L^2(0,T_0)$ convergence of the series $\left\{re^{\alpha r}\frac{\lambda_n^2\tilde\sigma_n}{\beta_n^2}\sin\beta_n r\right\}$ since $\{\sin\beta_n r\}_{n>N}$ is a Riesz system in $L^2(0,T_0)$. So, Young inequalities show that also the series of the integrals converge in $L^2(0,T_0)$. The corresponding series can be exchanged with the integral, since $\{\sin\beta_n r\}_{n>N}$ is a Riesz system.

The remaining series are treated analogously (and it is seen to converge even uniformly).

So, we conclude that
$$-e^{-\alpha t}\phi(t)-\alpha_{n_0}(1+\alpha t)=\sum_{n\neq n_0}\alpha_n\cos\beta_n t=\sum_{n\neq n_0}\frac{\tilde\sigma_n}{\beta_n}\cos\beta_n t\in W^{2,2}(0,T_0)\,.$$

Consequently, the second derivative of $e^{-\alpha t}\phi(t)$ belongs to the subspace X_1 and can be expressed as a series
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[e^{-\alpha t}\phi(t)\right]=\sum_{n\neq\pm n_0}\sigma_n e^{i\beta_n t}\,,\qquad\{\sigma_n\}\in l^2$$
so that
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\alpha t}\phi(t)\right]-\left[\alpha\phi(0)+\phi'(0)\right]=\sum_{n\neq\pm n_0}\frac{\sigma_n}{i\beta_n}e^{i\beta_n t}-\sum_{n\neq -n_0}\frac{\sigma_n}{i\beta_n}$$
We compare this expression with (32) and we see that, ignoring inessential multiplicative constants,
$$\tilde\sigma_n=\frac{\sigma_n}{\beta_n}\,,\qquad\alpha_n=\frac{\sigma_n}{\beta_n^2}\,,\qquad\{\sigma_n\}\in l^2\,,$$
as wanted.

Step 3 we prove that $\alpha_n=0$ for every n.

We go back to the equality
$$0=\sum_{n=1}^{+\infty}\alpha_n z_n(t)=\sum_{n\geq k'}\frac{\sigma_n}{\beta_n^2}z_n(t)$$

where k' is the first index such that $\alpha_k \neq 0$. This expression shows that the derivative of the series, which is 0, can be computed termwise:

$$0 = \sum_{n \geq k'} \frac{\sigma_n}{\beta_n^2} \left\{ 2\alpha z_n(t) - \lambda_n^2 \int_0^t N(t-s) z_n(s)\, \mathrm{d}s \right\}$$

$$= \sum_{n \geq k'} \frac{\sigma_n}{\beta_n^2} \lambda_n^2 \int_0^t N(t-s) z_n(s)\, \mathrm{d}s = \int_0^t N(t-s) \left[\sum_{n \geq k'} \frac{\sigma_n}{\beta_n^2} \lambda_n^2 z_n(s) \right] \mathrm{d}s$$

so that we have both the equalities

$$\sum_{n \geq k'} \frac{\sigma_n}{\beta_n^2} z_n(t) = 0\,, \qquad \sum_{n \geq k'} \frac{\sigma_n}{\beta_n^2} \lambda_n^2 z_n(t)\,.$$

This shows that the first term can be cancelled, and we in fact have

$$\sum_{n \geq k'+1} \alpha_n^{(1)} z_n(t) = 0\,, \qquad \alpha_n^{(1)} = \frac{\sigma_n}{\beta_n^2} \left(\lambda_n^2 - \lambda_{k'}^2 \right) = \alpha_n \left(\lambda_n^2 - \lambda_{k'}^2 \right)$$

so that

$$\{\alpha_n^{(1)}\} \in l^2\,.$$

It is important to note that $\alpha_n^{(1)} = 0$ if and only if $\alpha_n = 0$.

The fact that $\{\alpha_n^{(1)}\} \in l^2$ implies that the previous procedure can be repeated. After a finite number of iteration of this process we end up with the equality

$$\sum_{n=N}^{+\infty} \alpha^{(N-k')} z_n(t) = 0\,, \qquad \{\alpha_n^{(N-k')}\} \in l^2\,,$$

and

$$\alpha_n^{(N-k')} = 0 \quad \text{if and only if} \quad \alpha_n = 0\,.$$

We know that N can be so large that $\{z_n(t)\}$ is a Riesz system. Hence we have $\alpha_n^{(N-k')} = 0$ for every $n \geq N$ i.e. also $\alpha_n = 0$ for every $n \geq N$. So, the series in (24) is in fact a finite sum:

$$\sum_{n=1}^{N-1} \alpha_n z_n(t) = 0\,.$$

The proof is now finished thanks to the following lemma:

Lemma 22 *The sequence $\{z_n(t)\}$ is linearly independent.*

Proof. If not, there exists a first element $z_k(t)$ which a is linear combinations of the previous one,

$$\sum_{n=1}^{k} \alpha_n z_n(t) = 0\,, \qquad \alpha_k \neq 0\,.$$

We compute the derivatives of both the sides and we get

$$\int_0^t N(t-s) \left[\sum_{n=1}^{k} \alpha_n \lambda_n^2 z_n(s) \right] \mathrm{d}s = 0 \qquad \text{i.e.} \qquad \sum_{n=1}^{k} \alpha_n \lambda_n^2 z_n(t) = 0\,.$$

It follows that $\sum_{n=1}^{k-1} \alpha_n (\lambda_k^2 - \lambda_n^2) z_n(t) = 0$, in contrast with the definition of k. ∎

Remark 23 A similar result concerning ω-independence was proved in [10] (for the "canonical" case $\alpha = 0$, $a(x) = 1$, $b(x) = 0$). In that paper the property of ω-independence was not proved with a direct argument. Instead, it was deduced from the already known fact that system (1) with boundary control (in the special case $\alpha = 0$, $a(x) = 1$, $b(x) = 0$ there considered) is approximately controllable in time π. Although we believe that the arguments in [21, 22] can be easily adapted to the case under study here (essentially, to the case $b(x) \neq 0$) we gave an independent proof of the property of ω-independence, which will be usefull also in different contexts. ∎

We now study ω-independence of the sequence (19) relaying on the results already proved.

Theorem 24 *Let T_0 be the time identified in Section 6. Sequence (19) is ω-independent in $L^2(0, T_0)$. Hence, it is a Riesz system in $L^2(0, T_0)$.*

<u>Proof.</u> We proceed by contradiction. Let $\{\alpha_n\} \in l^2$ be a sequence such that

$$\sum_{n=1}^{+\infty} \alpha_n \left[n \int_0^t N(t-r) z_n(r) \, \mathrm{d}r \right] = 0 \qquad \text{in } L^2(0, T_0) . \tag{37}$$

Note that if it happens that

$$\alpha_n = \frac{\sigma_n}{n} , \qquad \{\sigma_n\} \in l^2 \tag{38}$$

then we have also

$$0 = \int_0^t N(t-s) \left[\sum_{n=1}^{+\infty} \sigma_n z_n(s) \right] \, \mathrm{d}s = 0 \implies \sum_{n=1}^{+\infty} \sigma_n z_n(s) = 0 \quad \text{i.e. } \{\sigma_n\}{=}0$$

since we already know that $\{z_n(t)\}$ is a Riesz sequence in $L^2(0, T_0)$. So, we prove the representation (38). We proceed as in the first step of the proof of Theorem 20. For simplicity we ignore the term n_0 and we write

$$\begin{aligned} 0 &= \sum_{n=1}^{+\infty} \alpha_n n \int_0^t N(t-r)[z_n(r) - e^{\alpha r} \cos \beta_n r] \, \mathrm{d}r = \sum_{n=1}^{+\infty} \alpha_n n \int_0^t N(t-r) e^{\alpha r} \cos \beta_n r \, \mathrm{d}r \\ &= e^{\alpha t} \left[\sum_{n=1}^{+\infty} \frac{n \alpha_n}{\beta_n} \sin \beta_n t \right] + \sum_{n=1}^{+\infty} \frac{n \alpha_n}{\beta_n} \int_0^t \left[N'(t-r) - \alpha N(t-r) \right] e^{\alpha r} \sin \beta_n r \, \mathrm{d}r . \end{aligned}$$

The last term is of class C^1. Now we represent

$$\begin{aligned} &\alpha_n n \int_0^t N(t-r)[z_n(r) - e^{\alpha s} \cos \beta_n r] \, \mathrm{d}r = \\ &\frac{\alpha_n n}{\beta_n} \int_0^t N(t-r) \alpha e^{\alpha r} \sin \beta_n r \, \mathrm{d}r - n \alpha_n \frac{\lambda_n^2}{\beta_n^2} \int_0^t N(t-r) \int_0^r K_n(r-s) z_n(s) \, \mathrm{d}s \, \mathrm{d}r . \end{aligned}$$

The first integrals on the right hand side sum to a function of classe C^1. The last integral is equal to

$$\int_0^t K_n(s) \left[\int_0^{t-s} N(t-s-\tau) z_n(\tau) \, \mathrm{d}\tau \right] \, \mathrm{d}s .$$

Hence we have

$$\sum_{n=1}^{+\infty} n\alpha_n \frac{\lambda_n^2}{\beta_n^2} \int_0^t K_n(s) \left[\int_0^{t-s} N(t-s-r) z_n(\tau)\, \mathrm{d}\tau \right] \mathrm{d}s =$$
$$\sum_{n=1}^{+\infty} n\alpha_n \int_0^t K_n(s) \left[\int_0^{t-s} N(t-s-r) z_n(\tau)\, \mathrm{d}\tau \right] \mathrm{d}s$$
$$+ \sum_{n=1}^{+\infty} n\alpha_n \left(\frac{\lambda_n^2}{\beta_n^2} - 1 \right) \int_0^t K_n(s) \left[\int_0^{t-s} N(t-s-r) z_n(\tau)\, \mathrm{d}\tau \right] \mathrm{d}s\,.$$

The first series on the right hand side is equal to 0, see (37), and the second one sums to a C^1 function, since $1 - \lambda_n^2/\beta_n^2 = -\alpha^2/(\lambda_n^2 - \alpha^2) \sim 1/n^2$.

In conclusion we have that

$$\sum_{n=1}^{+\infty} \alpha_n \sin \beta_n t = g(t) \in C^1(0, T_0)$$

and its belongs to the space $X_1 \subseteq X_0$ introduced in Section 6. Its derivative must belong to this same space. Now we proceed as in Step 1 of the proof of Theorem 20 and we get equality (38).

The property of ω-independence and Theorem 19 prove that the sequence (19) is a Riesz sequence in $L^2(0, T_0)$. ∎

7 Controllability

In order to give a meaning to the control problem, we have to define the solutions of Eq. (1) with the boundary condition (2) and $u(\cdot) \in L^2_{\mathrm{loc}}(0, +\infty)$. The method used in [15] applies to the problem (1)–(2) (we noted that these methods do not identify the controllability time). Instead, here we define the solution using the Fourier type methods in this paper. So, let

$$\theta(t) = \sum_{n=1}^{+\infty} \theta_n(t)\phi_n(x)\,, \qquad \theta_n(t) = \int_0^\pi \phi_n(x)\theta(t, x)\, \mathrm{d}x \tag{39}$$

be a candidate solution of problem (1)–(2). We differentiate formally $\theta_n(t)$ and we see that this function solves

$$\theta_n'(t) = 2\alpha\theta_n(t) - \lambda_n^2 \int_0^t N(t-s)\theta_n(s)\, \mathrm{d}s + a(0)\phi_n'(0)v(t)\,, \tag{40}$$

$$v(t) = \int_0^t N(t-s)u(s)\, \mathrm{d}s\,, \qquad \theta_n(0) = \xi_n\,. \tag{41}$$

Using the functions $z_n(t)$ in (11) we get the following representation formula:

$$\theta_n(t) = z_n(t)\xi_n + a(0)\phi_n'(0) \int_0^t z_n(s)v(t-s)\, \mathrm{d}s\,. \tag{42}$$

We replace this expression in (39) and we study the convergence of the resulting series. Since $\{\xi_n\} \in l^2$ and we already know boundedness of the sequence $\{z_n(t)\}$ (on every compact interval) we have to study the series

$$\sum_{n=1}^{+\infty} \phi_n(x) \left\{ \phi_n'(0) \int_0^t z_n(s) v(t-s) \, \mathrm{d}s \right\}$$
$$= \sum_{n=1}^{+\infty} \phi_n(x) \left\{ \phi_n'(0) \int_0^t u(t-r) \left[\int_0^r N(r-s) z_n(s) \, \mathrm{d}s \right] \, \mathrm{d}r \right\} . \qquad (43)$$

We recall that $\phi_n'(0) \asymp n$.

Now we study the convergence of the series in (43). We shall use the result in Lemma 19 for this. We have:

$$\sum_{n=1}^{+\infty} \phi_n(x) \phi_n'(0) \int_0^t u(t-r) \left[\int_0^r N(r-s) z_n(s) \, \mathrm{d}s \right] \, \mathrm{d}r$$
$$= \sum_{n=1}^{+\infty} \phi_n(x) \frac{\phi_n'(0)}{n} \int_0^t u(t-r) \left\{ e^{\alpha r} \sin \beta_n r - \left[e^{\alpha r} \sin \beta_n r - n \int_0^r N(r-s) z_n(s) \, \mathrm{d}s \right] \right\} \, \mathrm{d}r$$
$$= \sum_{n=1}^{+\infty} \phi_n(x) \frac{\phi_n'(0)}{n} \int_0^t u(t-r) e^{\alpha r} \sin \beta_n r \, \mathrm{d}r \qquad (44)$$
$$- \sum_{n=1}^{+\infty} \phi_n(x) \frac{\phi_n'(0)}{n} \int_0^t u(t-r) \left[e^{\alpha r} \sin \beta_n r - n \int_0^r N(r-s) z_n(s) \, \mathrm{d}s \right] \, \mathrm{d}r \qquad (45)$$

We prove that the series in (44) and (45) converge in $C(0,T; L^2(0,\pi))$. The sequence $\{\phi_n(x)\}$ being orthonormal in $L^2(0,\pi)$, a series of the form $\sum_{n=1}^{+\infty} \phi_n(x) f_n(t)$ converges in $C(0,T; L^2(0,\pi))$ if the series $\sum_{n=1}^{+\infty} |f_n(t)|^2$ converges uniformly on $[0,T]$. It is easy to prove this fact for the series (45) since, using Lemma 19, the bracket is less then M/n.

Now we consider the series (44). If $u(t) \in W^{1,2}(0,T)$ then the integral is of the order of $1/\beta_n$and the series converges on $C(0,T; L^2(0,\pi))$. We now use an approximation argument: let $\{u_k(t)\} \in W^{1,2}(0,T)$ converge to $u(t)$ in $L^2(0,T)$. We give an estimate of

$$\left\| \sum_{n=1}^{\infty} \frac{\phi_n'(0)}{n} \phi_n(x) \int_0^t \left[u(t-r) - u_k(t-r) \right] e^{\alpha r} \sin \beta_n r \, \mathrm{d}r \right\|_{L^2(0,\pi)}^2 \qquad (46)$$
$$\leq C \sum_{n=1}^{\infty} \left\| \int_0^t \left[u^{(t)}(r) - u_k^{(t)}(r) \right] e^{\alpha r} \sin \beta_n r \, \mathrm{d}r \right\|^2 \qquad (47)$$

since $\{\phi_n\}$ is orthogonal with constant norm. In this formula we denoted $u^{(t)}(r) = u(t-r)$ and $u_k^{(t)}(r) = u_k(t-r)$ so that we have

$$\lim_k u_k^{(t)}(r) = u^{(t)}(r)$$

in $L^2(0,t)$ for every $t > 0$.

Now we use the fact that the sequence $\{\sin\beta_n t\}$ is a Riesz sequence in $L^2(0,T)$ for every $T > T_0$ and we see that the right hand side of (47) is less then

$$M||u_k^{(t)} - u^{(t)}||^2_{L^2(0,T)}\,.$$

This inequality hence holds for $T > T_0$ thanks to the Riesz property of $\{e^{\alpha r}\sin\beta_n r\}$ in $L^2(0,T)$, $T \geq T_0$; it holds also in $L^2(0,T)$, $T \leq T_0$ since the norm in $L^2(0,T)$ is less then the norm in $L^2(0,T_0)$.

Now we complete the previous inequality:

$$||u_k^{(t)} - u^{(t)}||^2_{L^2(0,T)} = \int_0^t |u_k(t-s) - u(t-s)|^2\,\mathrm{d}s$$
$$= \int_0^t |u_k(s) - u(s)|^2\,\mathrm{d}s \leq \int_0^T |u_k(s) - u(s)|^2\,\mathrm{d}s \to 0\,.$$

This shows that the limit is uniform in t.

So, the series (44) is the limit in $C(0,T;L^2(0,\pi))$ of the approximating series obtained with $u(t)$ replaced by $u_k(t)$. This proves that the series in (44) belongs to $C(0,T;L^2(0,\pi))$ for every $T > 0$. Theorem 3 follows.

Now let the initial condition ξ and a target η (both in $L^2(0,\pi)$) be given. We see from (42) that the target η is reachable at time T if and only if there exists a control $u(t) \in L^2(0,T)$ which solves the moment problem

$$\int_0^T \left\{ n \int_0^t N(t-\tau) z_n(\tau)\,\mathrm{d}\tau \right\} u(T-t)\,\mathrm{d}t = c_n \tag{48}$$

where $\{c_n\} \in l^2$,

$$c_n = \frac{n}{\phi_n'(0)a(0)}[\eta_n - z_n(T)\xi_n]$$

(here $\eta_n = \langle \eta, \phi_n \rangle$, $\xi_n = \langle \xi, \phi_n \rangle$).

So, the controllability problem is solvable in time T if and only if the sequence (19) is a Riesz sequence in $L^2(0,T)$. This we already proved in Theorem 24. Theorem teo:SuallaControlla follows.

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