

# Input identification to a class of nonlinear input-output causal systems\*

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## Abstract

In this paper we adapt Lavrent'ev method so to obtain a reconstruction procedure for the input  $u$  to a nonlinear input-output system described by a Volterra integral equation. The proof is based on a monotonicity assumption which does not imply the monotonicity of the operators appearing in the Volterra equation.

## 1 Introduction

In this paper we consider an input-output process described by the following nonlinear integral equation:

$$y(t) = \int_0^t K(t, s)u(s)ds + \int_0^t F(t, s, u(s))ds, \quad t \in [0, T]. \quad (1)$$

We want to reconstruct the signal  $u$  from measures taken on the output  $y$ . Under suitable assumptions, we could redefine  $F$  so as to absorb the contribution of the first integral, but for most of clarity we prefer to keep distinct the roles of the linear and nonlinear terms.

Equation (1) represents a kind of inverse problem that is ill-posed, i.e. it is not generally solvable and, if solvable, the solution  $u$  does not depend continuously on the output  $y$ . So, in order to solve this problem, we need a suitable regularization algorithm.

We shall adapt Lavrent'ev method to identify  $u$ . The version of this method that we use consists in the following one: we introduce

$$z(t) = \varepsilon v(t) + \int_0^t [K(t, s)v(s) + F(t, s, v(s))]ds \quad (2)$$

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\*This paper fits into the programs of GNAMPA-INDAM.

and we impose the equality

$$z(t) = y(t). \quad (3)$$

This equation, under suitable assumptions given in the next section, is solvable on the interval  $[0, T]$  and gives a function  $v_\varepsilon(t)$  which approximates  $u(t)$  in a suitable sense. We shall study both  $L^2(0, T)$  and uniform approximation. See [19] for the introduction of the Lavrent'ev method. Eq. (3) is a singularly perturbed Volterra integral equation for  $v$ . See [5, Sec. 3.4] for this.

In the special case  $F = 0$  and  $K(t, s) = K(t - s)$ , our problem reduces to the deconvolution problem for a causal system, which has been studied by many authors with different methods. Hence, it is essentially impossible to give an overview of the very large literature on our problem and, for the moment, we confine ourselves to cite the survey paper [16] and the references therein. Important relations with concepts from control theory can be found in [7, 8]. Further comments on the literature can be found in the next section, after the introduction of the assumptions of this paper.

The plan of the paper is as follows: we prove convergence of  $v_\varepsilon$  to  $u$  in Sec. 3. For most of clarity we assume in these sections that the output  $y$  is read without errors while the effect of measurement errors is taken into account in Sec. 4. In the final Sec. 5 we present few simulations.

In order to use Lavrent'ev method we shall assume that the input  $u$  is piecewise of class  $W^{1,2}$ , see the definition in assumption 7 of Sec.2, an assumption which is not restrictive for the applications.

## 2 Assumptions and preliminaries

We recall the following definition: let  $f$  be a transformation from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . The norm is the euclidean norm and  $\langle \cdot, \cdot \rangle$  is the usual inner product. The norm of the matrices is the correspondent operator norms. We say that it is *monotonic* if

$$\langle x - y, f(x) - f(y) \rangle \geq 0, \quad \forall x, y.$$

We introduce the following assumptions

### Assumptions

1. The vectors  $y, u$  belong to  $\mathbb{R}^n$ ,  $K(t, s) : \Delta \rightarrow \mathbb{R}^{n \times m}$ ,  $F(t, s, u) : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  where

$$\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}.$$

2.  $F$  is continuous and the partial derivative  $F_t(t, s, u)$  exists for a.e.  $(t, s) \in \Delta$  and for all  $u \in \mathbb{R}^n$ .

3. for each  $u, v$  in  $\mathbb{R}^n$  and a.e.  $(t, s) \in \Delta$  we have

$$\|F(t, s, v) - F(t, s, u)\| \leq N(t, s)\|v - u\|, \quad \|F_t(t, s, v) - F_t(t, s, u)\| \leq L(t, s)\|v - u\|$$

and

$$\sup_{t \in [0, T]} \int_0^t L^2(t, s) ds \leq L, \quad \sup_{t \in [0, T]} \int_0^t N^2(t, s) ds \leq N.$$

4. For every  $t \in [0, T]$ , the function  $u \rightarrow F(t, t, u) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *monotonic*.
5. The kernel  $K(t, s)$  is continuous for  $0 \leq s \leq t \leq T$  and satisfies

$$K(t, t) = I, \quad t \in [0, T].$$

6. The derivative  $K_t(t, s)$  exists a.e. and

$$\sup_{t \in [0, T]} \int_0^t \|K_t(t, s)\|^2 ds \leq B.$$

7. We assume the existence on finitely many points  $t_1 < \dots < t_n$  in  $(0, T)$ , such that the restriction of  $u$  to each interval  $(t_i, t_{i+1})$  belongs to  $W^{1,2}(t_i, t_{i+1})$ . Functions with this property will be call piecewise  $W^{1,2}(0, T)$ .

It will be convenient to introduce the following notation

$$\begin{cases} e(t) = v_\varepsilon(t) - u(t), & \phi(t, u) = F(t, t, u), & \psi(t, s, u) = F_t(t, s, u), \\ \xi(t) = \int_0^t [\psi(t, s, v(s)) - \psi(t, s, u(s))] ds, & \delta(t) = \int_0^t K_t(t, s) e(s) ds. \end{cases} \quad (4)$$

Now we complete our comments on the bibliography. In the contest of non-linear Volterra equations, the assumption of monotonicity is a standard assumption since the paper [3], see for example [4, 6, 17]. Analogously, monotonicity assumptions have been recently used in the analysis of the Lavrent'ev method, see for example [12, 13, 14, 20]. In contrast with this, we do not assume that the operators in the Volterra integral equation (1) have to be monotonic. Monotonicity is only required to the transformation  $u \rightarrow [u + F(t, t, u)]$  as stated in the crucial monotonicity assumption 4. We shall see in Remark 3.1 that also this assumption can be weakened. We note that, under our assumptions, in particular 4, 5, we could achieve monotonicity of the integral in (1) in a norm which is equivalent to the usual norm of  $L^2(0, T)$ . This might be used in order to derive the proof of the  $L^2$  convergence from the corresponding results in [12, 17] (instead, the results on pointwise and uniform convergence have no equivalence in those papers). We prefer not to follow such a way because the introduction of a new norm affects the constants which appears in the convergence estimates. Furthermore, in the case of perturbed data, the convergence rate is proved in [12, 17] under the assumption that the Fréchet derivative of the nonlinear operator is Lipschitz continuous. We don't need the introduction of this assumption, see Sec. 4. The reason is as follows: the results of [17], used in [12], are of an abstract nature, not relying on the structure of the

Volterra equation. In contrast with this, we make use of the specific structure of the Volterra equation from the outset, and we combine the technique of the *separation of the diagonal of the kernel* as in [1, 15] and the *monotonicity assumption 4* in order to prove our results. Moreover, the condition  $K(t, t) = I$  permit the use of certain stability properties. Hence, the linear terms turns out to have a permanent role. We note however that the purely monotonicity methods (and the additional regularity assumptions on the Fréchet derivatives of the nonlinear operator in the Volterra equation) allows the treatment of cases which we cannot approach at the present stage of our research: autoconvolution equation, as in [12] and the case that the matrix  $K$  is non regular, for example the case that  $K$  is an Abel kernel, as in [10].

An obvious case to which the result of this paper can be applied is the case of the Hammerstein equations, i.e. the case in which  $F(t, s, u) = F_0(t, s)\Phi(u)$ . This results in an Hammerstein equation of the type:

$$y(t) = \int_0^t K(t, s)u(s)ds + \int_0^t F_0(t, s)\Phi(u(s))ds.$$

Of course the assumptions 1–7 are in force, in particular  $K(t, t) = I$ . In this case the assumptions are satisfied if  $F_0(t, t) \geq 0$  and  $\Phi(u)$  is monotonic. Of course in this case we might use the linear theory in order to identify  $[u + \Phi(u)]$  from which  $u$  can be computed. However, the method in this paper gives directly an approximant  $v_\varepsilon$  of  $u$ . Explicit inversion of  $u \rightarrow [u + \Phi(u)]$  is not required.

Finally we note that assumption 5 can be achieved if  $K(t, t)$  is selfadjoint invertible, with bounded inverse.

Equality (3) is

$$\varepsilon v(t) + \int_0^t [K(t, s)v(s) + F(t, s, v(s))]ds = \int_0^t [K(t, s)u(s) + F(t, s, u(s))]ds \quad (5)$$

where  $0 \leq t \leq T$ ,  $\varepsilon \geq 0$ .

It is a fact that the solution  $v_\varepsilon(t)$  of (5) exists and is unique on  $[0, \infty)$ . See the appendix for a sketch of the proof.

From (5) we have:

$$\varepsilon e(t) = - \int_0^t K(t, s)e(s)ds - \int_0^t [F(t, s, v(s)) - F(t, s, u(s))]ds - \varepsilon u(t). \quad (6)$$

In order to separate the values of  $K$  and  $F$  on the diagonal  $(t, s)$ , we compute the derivative of both the sides. Using the notations in (4), we find

$$\begin{aligned} \varepsilon e'(t) = & - e(t) - \int_0^t K_t(t, s)e(s)ds - [\phi(t, v) - \phi(t, u)] + \\ & - \int_0^t [\psi(t, s, v) - \psi(t, s, u)]ds - \varepsilon u'(t). \end{aligned} \quad (7)$$

This equality is the starting point of our proofs.

**Remark 2.1** *The previous equality holds on each interval  $[t, t_i)$ . The function  $e$  has jump on the point  $t_i$  and we shall take into account this fact, see Sec. 3.2.*

### 3 The convergence of $v$ to $u$

In order to be as clear as possible we shall study first the case that the input  $u$  has no jump and we study separately  $L^2$  and uniform convergence. We find convenient to use the results concerning  $L^2$  convergence in the proof of uniform convergence and to keep separated the two proofs. Of course, we cannot have uniform convergence in the general case, see theorems 3.2 and 3.4 for precise statements. In the proofs we shall use  $M$  to denote a suitable constants which can change at every occurrence.

#### 3.1 The case $u \in W^{1,2}(0, T)$

We prove first  $L^2$  convergence of  $v = v_\varepsilon$  to  $u$ .

**Theorem 3.1** *Let  $v(t)$  be the solution of (5). If  $u$  is  $W^{1,2}(0, T)$  then there exists a number  $M$ , which does not depend on  $\varepsilon$ , such that*

$$\|v_\varepsilon - u\|_{L^2(0, T)} < M\sqrt{\varepsilon}. \quad (8)$$

**Proof.** We compute from (7)

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \varepsilon \|e(t)\|^2 = & -\|e(t)\|^2 - \langle e(t), \delta(t) \rangle - \langle e(t), \phi(t, v) - \phi(t, u) \rangle \\ & - \varepsilon \langle e(t), u'(t) \rangle - \langle e(t), \xi(t) \rangle. \end{aligned} \quad (9)$$

Integration on the interval  $[0, t]$ ,  $t \in [0, T]$  gives

$$\begin{aligned} \frac{1}{2} \varepsilon \|e(t)\|^2 - \frac{1}{2} \varepsilon \|u(0)\|^2 + \int_0^t \|e(s)\|^2 ds = & - \int_0^t \langle e(s), \delta(s) \rangle ds + \\ - \int_0^t \langle e(s), \phi(s, v) - \phi(s, u) \rangle ds - \varepsilon \int_0^t \langle e(s), u'(s) \rangle ds - \int_0^t \langle e(s), \xi(s) \rangle ds. \end{aligned} \quad (10)$$

Using the monotonicity of the function  $\phi$  (Assumption 4) and ignoring the positive term  $\varepsilon \|e(t)\|^2/2$  on the left side we get

$$\begin{aligned} R_\varepsilon(t) = R(t) = \int_0^t \|e(s)\|^2 ds \leq & \frac{1}{2} \varepsilon \|u_o\|^2 + \varepsilon \int_0^t \|e(s)\| \cdot \|u'(s)\| ds + \\ & + \int_0^t \|e(s)\| \cdot \|\xi(s)\| ds + \int_0^t \|e(s)\| \cdot \|\delta(s)\| ds. \end{aligned} \quad (11)$$

Our goal is to prove

$$\lim_{\varepsilon \rightarrow 0^+} R_\varepsilon(T) = 0.$$

We use  $ab \leq (a^2 + b^2)/2$  and Cauchy-Schwarz. We get:

$$R(t) = \int_0^t \|e(s)\|^2 ds \leq \frac{1}{2} \varepsilon \|u_o\|^2 + \frac{\varepsilon}{2} \left( \int_0^t \|e(s)\|^2 ds \right) + \frac{\varepsilon}{2} \|u'(s)\|_{L^2(0, T)}^2 +$$

$$+ \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \left[ \left( \int_0^t \|\xi(s)\|^2 ds \right)^{\frac{1}{2}} + \left( \int_0^t \|\delta(s)\|^2 ds \right)^{\frac{1}{2}} \right]. \quad (12)$$

We now use assumption 6 and we see that

$$\|\delta(t)\|^2 = \left\| \int_0^t K_t(t,s)e(s)ds \right\|^2 \leq B \left( \int_0^t \|e(s)\|^2 ds \right) = BR(t). \quad (13)$$

Using assumption 3 we see that

$$\|\xi(t)\|^2 \leq L \left( \int_0^t \|e(s)\|^2 ds \right) = LR(t). \quad (14)$$

We collect the previous inequalities and we get

$$R(t) \leq \frac{1}{2}\varepsilon \left( \|u_o\|^2 + \|u'\|_{L^2(0,T)}^2 \right) + \frac{1}{2}\varepsilon R(t) + [R(t)]^{\frac{1}{2}}(\sqrt{L} + \sqrt{B}) \left( \int_0^t R(s)ds \right)^{\frac{1}{2}}$$

and finally

$$\left(1 - \frac{1}{2} - \frac{\varepsilon}{2}\right)R(t) \leq \frac{1}{2}\varepsilon \left( \|u_o\|^2 + \|u'\|_{L^2(0,T)}^2 \right) + \frac{1}{2}(\sqrt{L} + \sqrt{B})^2 \int_0^t R(s)ds.$$

We apply Gronwall's inequality and we obtain the following estimate:

$$R(T) = \|e\|_{L^2(0,T)}^2 \leq \left[ \frac{1}{2}e^{3(\sqrt{L}+\sqrt{B})^2T} \left( \|u_o\|^2 + \|u'\|_{L^2(0,T)}^2 \right) \right] \varepsilon \quad (15)$$

as wanted.  $\diamond$

In particular, inequality (8) gives a convergence estimate which can be combined with (14) to see that for each  $t \in [0, T]$  we have

$$\|\xi(t)\| = \|\xi_\varepsilon(t)\| \leq \mathbf{M}\sqrt{\varepsilon}. \quad (16)$$

The number  $\mathbf{M}$  does not depend on  $\varepsilon$ .

**Remark 3.1** *We see from this proof that the monotonicity of the function  $\phi(t, u)$  was not really used. We only used that the following integral is non negative for every  $t$ :*

$$\int_0^t \langle u(s) - v(s), \phi(s, u) - \phi(s, v) \rangle ds$$

*i.e. we used that  $u(\cdot) \rightarrow \phi(\cdot, u(\cdot))$  is causal and monotonic as a transformation from  $L^2(0, T; \mathbb{R}^n)$  to itself. Similar remarks apply to the following proofs.*

We shall use the estimate proved above in order to study pointwise and uniform convergence. Surely in the case  $u(0) = u_o \neq 0$  we don't have convergence for  $t = 0$  because  $v(0) = 0$  for every  $\varepsilon > 0$ .

Using the monotonicity assumption and equality (9) we see that

$$\begin{aligned} \|e(t)\|^2 &\leq e^{-\frac{2}{\varepsilon}t} \|u_o\|^2 - \int_0^t 2e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), u'(s) \rangle ds + \\ &- \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \xi(s) \rangle ds - \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \delta(s) \rangle ds \end{aligned} \quad (17)$$

We use this in order to prove:

**Theorem 3.2** *Let  $u \in W^{1,2}(0, T)$ . We have  $v(t) \rightarrow u(t)$  uniformly on  $[0, T]$ , in the case that  $u(0) = 0$ , while when  $u(0) = u_o \neq 0$ , we have  $v(t) \rightarrow u(t)$  uniformly on  $[\sigma, T]$  for all strictly positive  $\sigma$ .*

**Proof.** We consider the second addendum of (17) and we use (8). We obtain for this the following estimate

$$\int_0^t 2e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), u'(s) \rangle ds \leq C \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \|u'\|_{L^2(0, T)} \leq \mathbf{M}_1 \sqrt{\varepsilon}.$$

Now we use inequality (16) in order to estimate the third addendum of (17). We obtain

$$\begin{aligned} \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \xi(s) \rangle ds &= \int_0^t \left\langle \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{\frac{1}{\varepsilon}(t-s)} e(s), \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{\frac{1}{\varepsilon}(t-s)} \xi(s) \right\rangle ds \leq \\ &\leq \frac{\sqrt{2}}{\sqrt{\varepsilon}} \left[ \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \|e(s)\|^2 ds \right]^{\frac{1}{2}} \cdot \left[ \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \|\xi(s)\|^2 ds \right]^{\frac{1}{2}}. \end{aligned}$$

Finally from (16) we obtain

$$\int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \xi(s) \rangle ds \leq \frac{\sqrt{2}}{\sqrt{\varepsilon}} M \sqrt{\varepsilon} \sup_{t \in [0, T]} \|\xi(t)\| \left[ \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} ds \right]^{\frac{1}{2}} \leq \mathbf{M}_2 \sqrt{\varepsilon}.$$

The last addendum in (17) is estimated analogously using (8), (13):

$$\begin{aligned} \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \delta(s) \rangle ds &\leq \frac{\sqrt{2}}{\sqrt{\varepsilon}} M \sqrt{\varepsilon} \sqrt{B} \left[ \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} R(s) ds \right]^{\frac{1}{2}} \\ &\leq M \left[ R(t) \cdot \left( \int_0^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} ds \right) \right]^{\frac{1}{2}} \leq \mathbf{M}_3 \sqrt{\varepsilon}. \end{aligned}$$

We combine the previous estimates and we obtain the expression

$$\|e(t)\|^2 \leq e^{-\frac{2}{\varepsilon}t} \|u_o\|^2 + (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \cdot \sqrt{\varepsilon}$$

We conclude that for  $\varepsilon \rightarrow 0^+$  we have  $e(t) \rightarrow 0$  and the limit is uniform on  $[0, T]$  if  $u(0) = 0$ , and uniform on  $[\sigma, T]$ ,  $\sigma > 0$ , if  $u(0) \neq 0$ .  $\diamond$

**Remark 3.2** *If  $u(0) \neq 0$  and in the unusual case that  $u(0)$  is known then we can adapt the method so to get uniform convergence on  $[0, T]$ .*

### 3.2 The case that $u$ is discontinuous

Now we consider the case that the input function  $u$  is only piecewise  $W^{1,2}(0, T)$ . Assumption 7 implies the existence of the directional limits at the points  $t_i$ . It is sufficient to show the proof of our results in the case in which we have just one point of discontinuity, let it be  $t = t_0$ . We can use the previous theorems on the interval  $[0, t_0]$  and we only need to study here the case  $t \in [t_0, T]$ .

As a preliminary observation we note that the function  $v(t) = v_\varepsilon(t)$  is continuous on  $[0, T]$  so that  $e(t)$  has the same jumps as  $u$  does (i.e. it has a jump at  $t_0$ .) We need the following piece of information:

**Lemma 3.1** *There exists a number  $C$  which does not depend on  $\varepsilon$  and such that*

$$\|e(t_0^+)\| \leq C.$$

In fact, we observe that the existence of the directional limits at  $t_0$  of the input function  $u(t)$  allows us to apply theorem 3.2 on the interval  $[0, t_0]$ . So we can write

$$\|e(t_0^+)\| = \|v(t_0^+) - u(t_0^+)\| \leq \|v(t_0^+)\| + \|u(t_0^+)\| = \|v(t_0^-)\| + \|u(t_0^+)\| \leq C$$

where  $C$  is a constant that not depends on  $\varepsilon$  because  $v(t_0^-) = v_\varepsilon(t_0^-) \rightarrow u(t_0^-)$  from Theorem 3.2.  $\diamond$

We proceed as in Sec. 3.1 and we prove  $L^2$  convergence first:

**Theorem 3.3** *Let  $v(t)$  be the solution of (5). If  $u$  is piecewise  $W^{1,2}(0, T)$  then*

$$\|v - u\|_{L^2(0, T)} \leq M\sqrt{\varepsilon}.$$

*The constant  $M$  does not depend on  $\varepsilon$ .*

**Proof.** We know from Theorem 3.1 that  $v(t) \rightarrow u(t)$  in  $L^2(0, t_0)$ . We integrate (9) on  $[t_0, T]$  and we obtain

$$\begin{aligned} \frac{1}{2}\varepsilon \|e(t)\|^2 - \frac{1}{2}\varepsilon \|e(t_0^+)\|^2 &= - \int_{t_0}^t \|e(s)\|^2 ds - \int_{t_0}^t \langle e(s), \delta(s) \rangle ds + \\ &- \int_{t_0}^t \langle e(s), \phi(s, v) - \phi(s, u) \rangle ds - \int_{t_0}^t \langle e(s), \xi(s) \rangle ds - \varepsilon \int_{t_0}^t \langle e(s), u'(s) \rangle ds. \end{aligned} \quad (18)$$

Using the monotonicity of the function  $\phi$  we have:

$$\frac{1}{2}\varepsilon \|e(t)\|^2 + \int_{t_0}^t \|e(s)\|^2 ds \leq \frac{1}{2}\varepsilon \|e(t_0^+)\|^2 + \frac{\varepsilon}{2} \int_{t_0}^t \|e(s)\|^2 ds + \frac{\varepsilon}{2} \|u'\|_{L^2(t_0, T)}^2 +$$

$$+ \int_{t_0}^t \langle e(s), \delta(s) \rangle ds + \int_{t_0}^t \langle e(s), \xi(s) \rangle ds. \quad (19)$$

According to the definitions of  $\xi$  and  $\delta$  in (4), for  $t > t_0$  we have

$$\begin{aligned} \delta(t) &= \int_0^t K_t(t, s)e(s)ds = \int_0^{t_0} K_t(t, s)e(s)ds + \int_{t_0}^t K_t(t, s)e(s)ds, \\ \xi(t) &= \int_0^{t_0} [\psi(t, s, v(s)) - \psi(t, s, u(s))]ds + \int_{t_0}^t [\psi(t, s, v(s)) - \psi(t, s, u(s))]ds. \end{aligned}$$

From (19) we obtain

$$\begin{aligned} \int_{t_0}^t \|e(s)\|^2 ds &\leq \frac{1}{2}\varepsilon(\|e(t_0^+)\|^2 + \|u'\|_{L^2(t_0, T)}^2) + \frac{\varepsilon}{2} \int_{t_0}^t \|e(s)\|^2 ds + \\ &+ \int_{t_0}^t \langle e(s), \int_0^{t_0} [\psi(s, r, v(r)) - \psi(s, r, u(r))]dr \rangle ds + \end{aligned} \quad (20a)$$

$$+ \int_{t_0}^t \langle e(s), \int_{t_0}^s [\psi(s, r, v(r)) - \psi(s, r, u(r))]dr \rangle ds + \quad (20b)$$

$$+ \int_{t_0}^t \langle e(s), \int_0^{t_0} K_s(s, r)e(r)dr \rangle ds + \int_{t_0}^t \langle e(s), \int_{t_0}^s K_s(s, r)e(r)dr \rangle ds \quad (20c)$$

We estimate (20a). We introduce a constant  $\gamma$  whose value will be chosen below. Inequality (8) and assumption 3 are used here:

$$\begin{aligned} &\int_{t_0}^t \langle e(s), \int_0^{t_0} [\psi(s, r, v(r)) - \psi(s, r, u(r))]dr \rangle ds \\ &\leq \left( \int_{t_0}^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \sqrt{L(T-t_0)} \cdot \left( \int_0^{t_0} \|e(s)\|^2 ds \right)^{\frac{1}{2}} \leq \frac{M^2}{2\gamma}\varepsilon + \frac{\gamma}{2} \int_{t_0}^t \|e(s)\|^2 ds. \end{aligned}$$

Now let us consider (20b). We compute as follows.

$$\begin{aligned} &\int_{t_0}^t \langle e(s), \int_{t_0}^s [\psi(s, r, v(r)) - \psi(s, r, u(r))]dr \rangle ds \\ &\leq \left( \int_{t_0}^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \sqrt{L} \cdot \left[ \int_{t_0}^t \int_{t_0}^s \|e(s)\|^2 ds \right]^{\frac{1}{2}} \leq \\ &\leq \frac{1}{2}\gamma \int_{t_0}^t \|e(s)\|^2 ds + \frac{M^2}{2\gamma} \int_{t_0}^t \left( \int_{t_0}^s \|e(r)\|^2 dr \right) ds. \end{aligned}$$

We consider the first term in (20c) and we estimate it as follows, again using inequality (8):

$$\int_{t_0}^t \langle e(s), \int_0^{t_0} K_s(s, r)e(r)dr \rangle ds \leq \frac{M^2}{2\gamma}\varepsilon + \frac{\gamma}{2} \int_{t_0}^t \|e(s)\|^2 ds.$$

The last term in (20c) is handled in a similar way:

$$\int_{t_0}^t \langle e(s), \int_{t_0}^s K_s(s,r)e(r)dr \rangle ds \leq \frac{1}{2}\gamma \int_{t_0}^t \|e(s)\|^2 ds + \frac{M^2}{2\gamma} \int_{t_0}^t \left( \int_{t_0}^s \|e(r)\|^2 dr \right) ds.$$

Let  $R_0(t) = \int_{t_0}^t \|e(s)\|^2 ds$ . We collect the previous inequalities and we obtain

$$(1 - \frac{\varepsilon}{2} - 2\gamma)R_0(t) \leq \frac{M^2}{\gamma}\varepsilon + \frac{1}{2}\varepsilon \left[ \|e(t_0^+)\|^2 + \|u'\|_{L^2(0,T)}^2 \right] + \frac{M^2}{\gamma} \int_{t_0}^t R_0(s) ds. \quad (21)$$

This inequality holds true for each  $\gamma$ . We can fix  $\varepsilon_0$  and  $\gamma$  such that the quantity  $(1 - \frac{\varepsilon}{2} - 2\gamma) > \frac{1}{2}$  holds for every  $\varepsilon \in (0, \varepsilon_0)$  and we can apply Gronwall's inequality to (21) so to obtain

$$R_0(t) \leq M\varepsilon. \quad (22)$$

The result follows from this and Theorem 3.1 applied to the interval  $[0, t_0]$ .  $\diamond$

Now we study pointwise and uniform convergence. Again we analyze the case there is just one point of discontinuity  $t = t_0$ .

Using (22) we see that there exists a number  $\mathbf{M}$ , independent of  $t$  and  $\varepsilon$ , such that for every  $t \in [0, T]$  we have:

$$\|\xi_0(t)\| = \left\| \int_{t_0}^t [\psi(t, s, v(s)) - \psi(t, s, u(s))] ds \right\| \leq \mathbf{M}\sqrt{\varepsilon}. \quad (23)$$

We prove:

**Lemma 3.2** *We have  $v(t) \rightarrow u(t)$  uniformly on  $[t_0 + \rho, T]$ ,  $\forall \rho > 0$ .*

**Proof.** Similar calculations to those in the proof of Theorem 3.2 show that, for  $t > t_0$ ,

$$\begin{aligned} \|e(t)\|^2 &\leq e^{-\frac{2}{\varepsilon}(t-t_0)} \|e(t_0^+)\|^2 - \int_{t_0}^t 2e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), u'(s) \rangle ds + \\ &- \int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \xi(s) \rangle ds - \int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \delta(s) \rangle ds. \end{aligned} \quad (24)$$

Using (22), we estimate the second addendum of (24). We obtain

$$\int_{t_0}^t 2e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), u'(s) \rangle ds \leq 2 \left( \int_{t_0}^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \|u'\|_{L^2(t_0, T)} \leq \widehat{M}_1 \sqrt{\varepsilon}.$$

Now we estimate the third addendum of (24). We use assumption 3 and the inequalities (8), (22), (23). We get

$$\begin{aligned}
& \int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \xi(s) \rangle ds = \int_{t_0}^t \left\langle \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} e(s), \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} \xi(s) \right\rangle ds \\
& = \int_{t_0}^t \left\langle \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} e(s), \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} \int_0^{t_0} [\psi(s, r, v(r)) - \psi(s, r, u(r))] dr \right\rangle ds + \\
& + \int_{t_0}^t \left\langle \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} e(s), \frac{\sqrt{2}}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}(t-s)} \xi_0(s) \right\rangle ds \leq \\
& \leq \frac{\sqrt{2}}{\sqrt{\varepsilon}} \left[ \int_{t_0}^t \|e(s)\|^2 ds \right]^{\frac{1}{2}} \cdot \left\{ \left[ \int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \left( \int_0^{t_0} \| [\psi(s, r, v(r)) - \psi(s, r, u(r))] \|^2 dr \right) ds \right]^{\frac{1}{2}} + \right. \\
& \left. + \left[ \int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \|\xi_0(s)\|^2 ds \right]^{\frac{1}{2}} \right\} \leq \widehat{M}_2 \sqrt{\varepsilon}.
\end{aligned}$$

A similar calculation as above gives the following estimate for the last addendum of (24):

$$\int_{t_0}^t \frac{2}{\varepsilon} e^{-\frac{2}{\varepsilon}(t-s)} \langle e(s), \delta(s) \rangle ds \leq \widehat{M}_3 \sqrt{\varepsilon}.$$

We combine the previous estimates and we obtain

$$\|e(t)\|^2 \leq e^{-\frac{2}{\varepsilon}(t-t_0)} \|e(t_0^+)\|^2 + (\widehat{M}_1 + \widehat{M}_2 + \widehat{M}_3) \sqrt{\varepsilon} \quad (25)$$

We know that the  $\|e(t_0^+)\| \leq C$  and  $C$  does not depend on  $\varepsilon$ . We conclude that  $e(t) \rightarrow 0$  for every  $t > 0$  when  $\varepsilon \rightarrow 0^+$ . The limit is uniform on  $[t_0 + \rho, T]$ .  $\diamond$

We can summarize the results of this paragraph:

**Theorem 3.4** *Let  $u$  be piecewise  $W^{1,2}(0, T)$ . We have  $v_\varepsilon(t) \rightarrow u(t)$  uniformly on  $([0, T] - \bigcup_{i=0}^n (t_i, t_i + \rho))$ , if  $u(0) = 0$ , while when  $u(0) \neq 0$ ,  $v_\varepsilon(t) \rightarrow u(t)$  uniformly on  $([\sigma, T] - \bigcup_{i=0}^n (t_i, t_i + \rho))$ ,  $\forall \sigma > 0$ .*

## 4 Noisy observation

The effect of the noise on the observation  $y$  will now be considered. We assume that the available signal is

$$\eta(t) = y(t) + \theta(t) \quad (26)$$

where  $y(t)$  is the unknown true value and  $\theta(t)$  is its perturbation. We assume that  $\theta(t)$  is a measurable function and that

$$\text{either } 1) \quad \|\theta(t)\|_\infty \leq h \quad \text{or } 2) \quad \|\theta(t)\|_{L^2(0, T)} \leq h \quad (27)$$

where  $h$  is a known tolerance.

We note that in many applications the signal is only read at a finite number of time instants  $\tau_k$  so that we must assume  $\eta(t) = y(\tau_k) + \theta_k$  for  $t \in [\tau_k, \tau_{k+1})$ .

If an *a priori* estimate for  $u$  is known, this effect can be modelled as a pointwise error of known tolerance.

We wish to apply also in this case the Lavrent'ev method conveniently modified. The use of Lavrent'ev method require some degree of regularity of the terms of the equation. For this reason, instead then imposing the equality (3) we impose

$$\varepsilon v(t) + \int_0^t K(t, s)v(s)ds + \int_0^t F(t, s, v(s))ds = \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\eta(s)ds. \quad (28)$$

This is suggested by the mollification approach to ill posed problems, see for example [18]. We note that there are different possibilities. For example if we are willing to introduce a further parameter  $\tau$  we can proceed as in [9] and we could replace the last addendum in (28) by the average

$$\frac{1}{\tau} \int_{t-\tau}^t \eta(s)ds.$$

In this section we confine ourselves to study  $L^2$  convergence and we prove:

**Theorem 4.1** *Let  $\varepsilon = \varepsilon(h)$  be a function such that*

$$\lim_{h \rightarrow 0^+} \varepsilon(h) = 0, \quad \lim_{h \rightarrow 0^+} \frac{h}{\varepsilon(h)} = 0 \quad (29)$$

*and let  $v_{\varepsilon(h)}(t)$  solve (28). If one of the conditions in (27) holds, we have*

$$v_{\varepsilon(h)}(t) \longrightarrow u(t)$$

*in  $L^2(0, T; \mathbb{R}^n)$ .*

**Proof.** In this proof we use the following known fact (see [2, 5]):

$$\lim_{\varepsilon \rightarrow 0^+} \left\| \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\phi(s)ds - \phi(t) \right\| = 0. \quad (30)$$

The norm is that of  $L^2(0, T)$  if  $\phi \in L^2(0, T)$ ; if  $\phi$  is continuous and  $\phi(0) = 0$  the norm is that of  $C(0, T)$ .

We study the case that  $u$  is continuous first. Eq. (28) is

$$\begin{aligned} \varepsilon v(t) + \int_0^t K(t, s)v(s)ds + \int_0^t F(t, s, v(s))ds &= \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\theta(s)ds + \\ + \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \left[ \int_0^s K(s, r)u(r)dr \right] ds &+ \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \left[ \int_0^s F(s, r, u(r))dr \right] ds \end{aligned} \quad (31)$$

Integration by parts gives

$$\frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \left[ \int_0^s K(s, r)u(r)dr \right] ds = \int_0^t K(t, s)u(s)ds + \quad (32)$$

$$- \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}u(s)ds - \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \int_0^s K_s(s, r)u(r)drds. \quad (33)$$

Analogous integration by parts on the last term in (31) shows that

$$\begin{aligned}
\varepsilon e(t) &= -\varepsilon u(t) - \int_0^t K(t,s)e(s)ds - \int_0^t [F(t,s,v(s)) - F(t,s,u(s))]ds + \\
&- \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}u(s)ds - \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \int_0^s K_s(s,r)u(r)drds - \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\phi(s,u(s))ds + \\
&- \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \int_0^s \psi(s,r,u(r))drds + \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\theta(s)ds. \tag{34}
\end{aligned}$$

We introduce the following term, which appears when we take the derivative of both the sides of (34).

$$\Upsilon_\varepsilon(t) = - \left[ u(t) - \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}u(s)ds \right] \tag{35a}$$

$$- \left\{ \int_0^t K_t(t,s)u(s)ds - \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \int_0^s K_s(s,r)u(r)drds \right\} \tag{35b}$$

$$- \left\{ \phi(t,u(t)) - \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\phi(s,u(s))ds \right\} \tag{35c}$$

$$- \left\{ \int_0^t \psi(t,s,u(s))ds - \frac{1}{\varepsilon} \int_0^t e^{-\frac{1}{\varepsilon}(t-s)} \int_0^s \psi(s,r,u(r))drds \right\}. \tag{35d}$$

With this notation we see that

$$\begin{aligned}
\varepsilon e'(t) &= -\varepsilon u'(t) - e(t) - \int_0^t K_t(t,s)e(s)ds - [\phi(t,v(t)) - \phi(t,u(t))] \\
&- \int_0^t [\psi(t,s,v(s)) - \psi(t,s,u(s))]ds + \Upsilon_\varepsilon(t) + \frac{1}{\varepsilon} \frac{d}{dt} \left( \int_0^t e^{-\frac{1}{\varepsilon}(t-s)}\theta(s)ds \right). \tag{36}
\end{aligned}$$

We multiply (36) by  $e(t)$  and we proceed as in Sec. 3.1. We obtain

$$\begin{aligned}
\frac{1}{2}\varepsilon \|e(t)\|^2 + \int_0^t \|e(s)\|^2 ds &\leq \frac{1}{2}\varepsilon \|u(0)\|^2 - \varepsilon \int_0^t \langle e(s), u'(s) \rangle ds - \int_0^t \langle e(s), \delta(s) \rangle ds + \\
&- \int_0^t \langle e(s), \xi(s) \rangle ds + \left\{ \int_0^t \langle e(s), \Upsilon_\varepsilon(s) \rangle ds + \int_0^t \langle e(s), \frac{1}{\varepsilon} \frac{d}{ds} \int_0^s e^{-\frac{1}{\varepsilon}(s-r)}\theta(r)dr \rangle ds \right\}. \tag{37}
\end{aligned}$$

Thanks to (30), we know that

$$\|\Upsilon_\varepsilon(t)\|_{L^2(0,T)} \leq \widetilde{M}\beta(\varepsilon), \quad \lim_{\varepsilon \rightarrow 0^+} \beta(\varepsilon) = 0. \tag{38}$$

(explicit estimates of  $\beta(\varepsilon)$  can be given under additional regularity of the functions  $K$  and  $F$ ). Now, we estimate the perturbed term of (37), i.e.

$$\int_0^t \langle e(s), \frac{1}{\varepsilon} \frac{d}{ds} \int_0^s e^{-\frac{1}{\varepsilon}(s-r)}\theta(r)dr \rangle ds,$$

in the cases 1) or 2) in (27).

In case 1) we compute the derivatives and we take the norms:

$$\begin{aligned} \left| \int_0^t \langle e(s), \frac{1}{\varepsilon} \frac{d}{ds} \int_0^s e^{-\frac{1}{\varepsilon}(s-r)} \theta(r) dr \rangle ds \right| &\leq \int_0^t \|e(s)\| \left\| \frac{1}{\varepsilon} \theta(s) - \frac{1}{\varepsilon} \int_0^s \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(s-r)} \theta(r) dr \right\| ds \leq \\ &\leq \frac{Mh}{\varepsilon} \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

In case 2), we proceed analogously

$$\begin{aligned} \left| \int_0^t \langle e(s), \frac{1}{\varepsilon} \frac{d}{ds} \int_0^s e^{-\frac{1}{\varepsilon}(s-r)} \theta(r) dr \rangle ds \right| &\leq \int_0^t \|e(s)\| \left\| \frac{1}{\varepsilon} \theta(s) - \frac{1}{\varepsilon} \int_0^s \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(s-r)} \theta(r) dr \right\| ds \leq \\ &\leq \frac{1}{\varepsilon} \int_0^t \|e(s)\| \cdot \|\theta(s)\| ds + \frac{1}{\varepsilon} \int_0^t \|e(s)\| \cdot \int_0^s \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(s-r)} \|\theta(r)\| dr ds \leq \\ &\leq \frac{1}{\varepsilon} \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t \|\theta(s)\|^2 ds \right)^{\frac{1}{2}} + \\ &+ \frac{1}{\varepsilon} \left\{ \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \left[ \int_0^t \left( \int_0^s \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(s-r)} \|\theta(r)\| dr \right)^2 ds \right]^{\frac{1}{2}} \right\}. \end{aligned}$$

Finally we use Jensen inequality and we find

$$\left| \int_0^t \langle e(s), \frac{1}{\varepsilon} \frac{d}{ds} \int_0^s e^{-\frac{1}{\varepsilon}(s-r)} \theta(r) dr \rangle ds \right| \leq \frac{2h}{\varepsilon} \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}}.$$

Hence, the norm of the brace is less then

$$M \left\{ \beta(\varepsilon) + \frac{h}{\varepsilon} \right\} = \zeta(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \zeta(\varepsilon) = 0.$$

The previous terms have already been encountered. As in Sec. 3.1, we find

$$\begin{aligned} R(t) &= \int_0^t \|e(s)\|^2 ds \leq \frac{1}{2} \varepsilon \left( \|u(0)\|^2 + \|u'\|_{L^2(0,T)}^2 \right) + \frac{\varepsilon}{2} R(t) + \frac{1}{2} R(t) + \\ &+ M \int_0^t R(s) ds + \left( \int_0^t \|e(s)\|^2 ds \right)^{\frac{1}{2}} \zeta(\varepsilon) \end{aligned} \quad (39a)$$

We use (38) and we see that

$$\left( \frac{1}{2} - \frac{\varepsilon}{2} - \gamma \right) R(t) \leq M(\varepsilon + \zeta^2(\varepsilon)) + M \int_0^t R(s) ds \quad (40)$$

This inequality holds true for each  $\gamma$ . We proceed as in Sec. 3.1 and we get the result.

We consider now the case that  $u$  has a jump at  $t_0$ . We note that inequality (38) still holds and that  $L^2$  convergence has been already proved on  $[0, t_0]$ . Hence, Lemma 3.1 still holds and similarly as in the proof of Lemma 3.1 we achieve the result.  $\diamond$

## 5 Few simulations

We present now two simulations in the simplest case of the Hammerstein equation

$$y(t) = \int_0^t u(s)ds + \int_0^t F_0(t-s)\Phi(u(s))ds.$$

The final time is  $T = 10$  and the signal  $u$  that we want to reconstruct is discontinuous, given by

$$u(t) = \begin{cases} 1 + \cos 2t & \text{if } \cos 2t > 0 \\ 0 & \text{if } \cos 2t = 0 \\ -1 + \cos 2t & \text{if } \cos 2t < 0. \end{cases}$$

We present two simulations which are extreme cases for different reasons. In one case  $\Phi$  mimic a saturation, given by  $\Phi(u) = \arctan 20u$  while in the second case  $\Phi(u) = u^3$ . The function  $F_0(t)$  is identically 1 in the second case  $\Phi(u) = u^3$  while it is  $F_0(t) = \sqrt{t}$  in the first case. The tolerance in the measures is in both the cases 1% of the measured output  $y$ . Additional errors are introduced by the discretization of the integrals.

Figure 1 presents the input  $u$  (continuous line) and, dotted, the function  $\Phi(u(t))$ .

Figure 2 presents the input signal, the same as in figure 1, with superimposed the reconstructed signal  $v$ . The value of  $\epsilon$  we used is the same in both the figures,  $\epsilon = 0.1$ .

## 6 Appendix

In this section we prove

**Theorem 6.1** *The solution  $v(t)$  of (5) exists and is unique on  $[0, +\infty)$ .*

**Proof.** It is not restrictive to work with  $\varepsilon = 1$ . Unicity and local existence is a standard fact. We prove that the solution can be extended to  $[0, +\infty)$ . Let the solution to

$$v(t) + \int_0^t [K(t,s)v(s) + F(t,s,v(s))]ds = y(t) \quad (41)$$

exists on  $[0, T_0)$ . We know (see [11, p. 343]) that if a constant  $M$  can be found such that

$$\|v(t)\| \leq M \quad \forall t \in [0, T_0) \quad (42)$$

then  $v(t)$  is continuable beyond  $T_0$ .

Now we prove (42). Using (41) we find

$$v'(t) = -v(t) - F(t,t,v(t)) - \int_0^t K_t(t,s)v(s)ds - \int_0^t \psi(t,s,v(s))ds + y'(t) \quad (43)$$

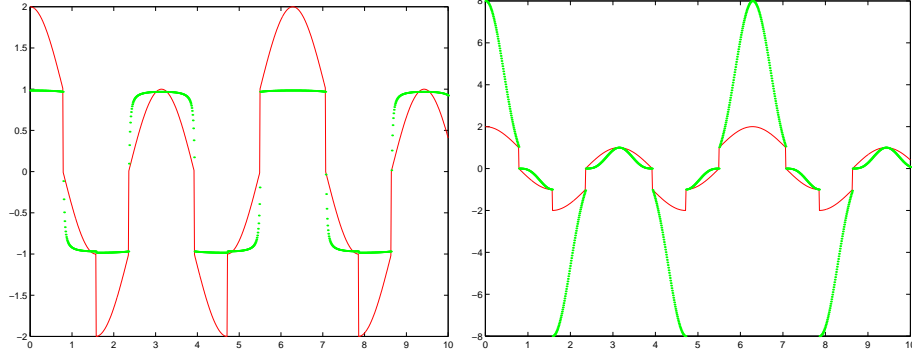


Figure 1: The functions  $u(t)$  and  $\Phi(u(t))$ .

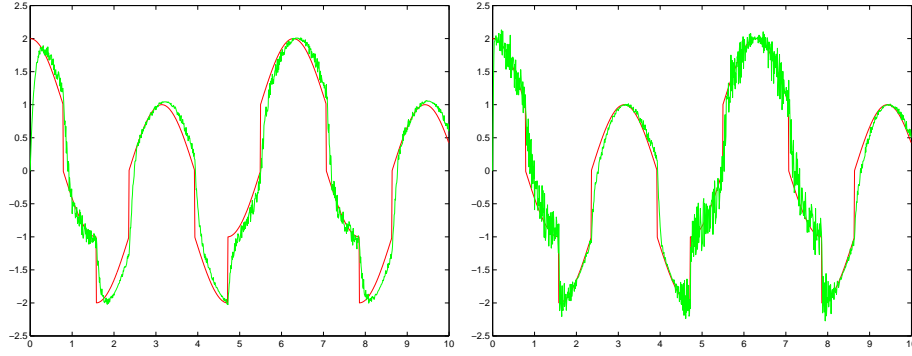


Figure 2: The functions  $u(t)$  and  $v_\epsilon(t)$ .

and

$$\begin{aligned}
\frac{d}{dt} \frac{1}{2} \|v(t)\|^2 &= -\|v(t)\|^2 - \langle v(t) - 0, \phi(t, v(t)) - \phi(t, 0) \rangle - \langle v(t), \delta(t) \rangle + \\
&\quad - \langle v(t), \int_0^t \psi(t, s, v(s)) ds \rangle + \langle v(t), y'(t) + \phi(t, 0) \rangle \\
&\leq -\|v(t)\|^2 - \langle v(t), \delta(t) \rangle - \langle v(t), \int_0^t \psi(t, s, v(s)) ds \rangle + \langle v(t), y'(t) + \phi(t, 0) \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2} \|v(t)\|^2 + \int_0^t \|v(s)\|^2 ds \leq \\
&\leq \left( \int_0^t \|v(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t \|\delta(s)\|^2 ds \right)^{\frac{1}{2}} + \left( \int_0^t \|v(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t \left\| \int_0^s \psi(s, r, v(r)) dr \right\|^2 ds \right)^{\frac{1}{2}} +
\end{aligned}$$

$$\begin{aligned}
& + \left( \int_0^t \|v(s)\|^2 ds \right)^{\frac{1}{2}} \cdot (\|y'\|_{L^2(0,T_0)} + \|\phi(t,0)\|_{L^2(0,T_0)}) \leq \\
& \leq \frac{1}{2} \int_0^t \|v(s)\|^2 ds + \frac{1}{2} \int_0^t \|v(s)\|^2 ds + \\
& + \frac{1}{2} C \int_0^t \int_0^s \|v(r)\|^2 dr ds + \frac{1}{2} \int_0^t \|v(s)\|^2 ds + \frac{1}{2} \|y'(t)\|_{L^2(0,T_0)}^2 + \frac{1}{2} \|\phi(t,0)\|_{L^2(0,T_0)}^2.
\end{aligned}$$

From this we obtain

$$\frac{1}{2} \|v(t)\|^2 \leq C(T_0) \int_0^t \|v(s)\|^2 ds + \frac{1}{2} \|y'(t)\|_{L^2(0,T_0)}^2 + \frac{1}{2} \|\phi(t,0)\|_{L^2(0,T_0)}^2.$$

Gronwall inequality implies the result.  $\diamond$

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