The controllability of the Gurtin-Pipkin equation A cosine operator approach

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Abstract

In this paper we give a semigroup-based definition of the solution of the Gurtin-Pipkin equation with Dirichlet boundary conditions. It turns out that the dominant terms is the control to displacement operator of the wave equation. Hence it is surjective if the time interval is long enough. We use this observation in order to prove exact controllability in finite time of the Gurtin-Pipkin equation.

1 Introduction

The Gurtin-Pipkin equation, proposed in [15], models the temperature evolution in a thermal system with memory:

$$\theta_t(t,x) = \int_0^t b(t-s)\Delta\theta(s,x)\,\mathrm{d}s + f(t,x) \qquad x \in \Omega\,,\ t \geq 0\,. \tag{1}$$

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We shall study this equation in a bounded region Ω with "regular" boundary Γ (this assumption is discussed below).

We associate an initial condition and a boundary condition of Dirichlet type to (1):

$$\theta(0,x)=\theta_0(x)\in L^2(\Omega)\,,\qquad \theta(t,s)=u(s)\in L^2_{\rm loc}(0,+\infty;L^2(\Gamma))\,. \quad (2)$$

Our first goal will be to define the solutions to the problem (1), (2). After that we study some regularity properties of the solutions (see Sect. 3) and, in Sect. 4, we prove exact controllability. As suggested in [4], we shall use a cosine operator approach. The bonus of this approach is that controllability is a direct consequence of the (known) corresponding property of the wave equation:

$$\begin{array}{l} w_{tt}(t,x)=\Delta w(t,x) \qquad \text{in } \Omega \\ w(0,x)=w_0(x)\,,\quad w_t(0,x)=w_1(x) \qquad \text{in } \Omega \\ w(t,s)=u(s) \text{ on } \Gamma. \end{array} \quad (3)$$

The controllability of the Gurtin-Pipkin equation, and of the coupling of this equation and wave-type equations, has been studied by many authors, see the references below. However, these previous papers had to study the equation itself, without relying on previously known controllability results.

Our approach to the controllability of Eq. (1) is suggested by the well known fact that Eq. (1) displays an hyperbolic behavior (see [4, 9, 15]).

Now we list and discuss the standing assumptions of this paper:

Assumption 1 The assumption that we make on the kernel b is:

- $b(t)$ is twice continuously differentiable and $b(0)=\mu>0$.
- Controllability will be proved in section 4 under the additional assumption:
  - stronger regularity: $b\in C^3$;
  - stronger positivity condition: $b(t)$ is integrable on $[0,+\infty)$ and
$$\hat b(0)=\int_0^{+\infty} b(t)\,{\rm d}t>0\,.$$

The assumption that we make on the region Ω is:

- We assume that the region Ω is simply connected, with boundary of class C^2.

Remark 2 The condition $b(0) > 0$ is crucial in order to have a hyperbolic type behavior. In order to simplify the notations we shall put $b(0) = 1$. This amount to the choice of a suitable time scale and it is not restrictive. The assumption on Ω is stronger then needed: for most of the results in Sect.s 2, 3 we only need that the Dirichlet map D:

$$\theta = Du \iff \begin{cases} \Delta\theta = 0 \text{ in } \Omega \\ \theta = u \text{ on } \Gamma \end{cases}$$

transform $L^2(\Gamma)$ to $H^{1/2}(\Omega)$. Instead Theorem 11 and the controllability results in Sect. 4 requires the existence of a suitable regular vector field on Ω, related to the outer normal to Γ. This is discussed in details in [17, 23] and, in the wider generality, in [20]. This allows also regions which are not simply connected and the control acting on a suitable part of the boundary. Under this conditions, Eq. (3) is controllable in finite time. We don't need to go into these technicalities since the point of view of our paper is to prove controllability of Eq. (1) as a consequence of the known controllability properties of Eq. (3). ∎

1.1 Comments on the literature

As we said, Eq. (1) was proposed in [15] in order to get a model for heat transfer with finite signal speed. That Eq. (1) might display hyperbolic type properties is suggested by the special case $b \equiv 1$ and $u = 0$. In this case Laplace transform gives

$$\lambda\hat{\theta}(\lambda) - \theta_0 = \frac{1}{\lambda}\Delta\hat{\theta}(\lambda) + \hat{f}(\lambda)$$

which displays some "hyperbolic look". This idea has been pursued in [8, 9] to prove hyperbolicity when the kernel b has a strictly proper rational Laplace transform.

The obvious approach to the solution of Eq. (1) at least when $u = 0$ is to reduce it, by integration, to a Volterra integral equation with a kernel which takes values unbounded operators, see for example [13, 25, 14].

The fact that the Laplacian with homogeneous Dirichlet boundary condition generates a holomorphic semigroup suggests the arguments for examples in [6, 7, 13] while other papers explicitly used the hyperbolicity of the problem, see for example [24, 4].

Subsequently, a great number of researchers investigated Eq. (1) and suitable generalizations. We need not cite them explicitly here. We note however that it goes along the two lines outlined above: the use of ideas from holomorphic semigroups, as in [10, 11] or ideas from hyperbolic equations, as in [1, 28].

Several version of the controllability problem for Eq. (1) have been studied in recent times.

The paper [22] presents a precise study of exact controllability for Eq. (1) in one space dimension. The control acts on one side of the boundary. Remark 1 in [22] notes that the regularity properties of the solutions to Eq. (1) resembles those derived in [18], a paper which makes explicit use of the cosine operator theory.

The paper [3] consider the approximate controllability in the "parabolic" case (i.e. when the memory integral is added as a perturbation to a parabolic equation) in m space dimensions and an exact controllability problem for Eq. (1), in one space dimension. The control now acts as a distributed control on a part of the region, and the results are then used to recover boundary controllability with the control acting on the entire boundary of the domain. We note that in the presentation of the problem we also assumed the control to act on the entire boundary, but this is not really an assumption in our paper, see Remark 2, since we only use that the wave equation is controllable.

Controllability of the interconnection of Eq. (1) and a wave equation is studied in [26].

Further papers on controllability of materials with memory are, for example, [5, 16, 30].

2 Cosine operators and the solutions of the Gurtin-Pipkin equation

Our first step is the definition of the solutions of the Gurtin-Pipkin equation (1). We shall use the cosine operator theory, as presented in [12, 29] and used in [4, 18]. We shall use the same notations as in [4].

Let A be the generator of an exponentially stable holomorphic semigroup on a Hilbert space X. In our application, it will be

$$X = L^2(\Omega)\,, \qquad \operatorname{dom} A = H^2(\Omega) \cap H_0^1(\Omega)\,, \qquad A\theta = \Delta\theta\,.$$

Let moreover $\mathcal{A} = i(-A)^{1/2}$. It is known (see [4]) that $e^{\mathcal{A}t}$ is a C_0-*group* of operator on X. The strongly continuous operator valued function

$$R_+(t) = \frac{1}{2}\left[e^{\mathcal{A}t} + e^{-\mathcal{A}t}\right] \qquad t \in \mathbb{R}$$

is called *cosine operator* (the cosine operator generated by A.) It is usually denoted $C(t)$. It is convenient to introduce the operators

$$R_-(t) = \frac{1}{2}\left[e^{\mathcal{A}t} - e^{-\mathcal{A}t}\right], \qquad S(t) = \mathcal{A}^{-1}R_-(t), \qquad t \in \mathbb{R}.$$

The operator $S(t)$ is called the *sine operator* (generated by A.) The following properties are known, or easily proved (see [12, 18, 29]):

- $S(t)z = \int_0^t R_+(r)z \,\mathrm{d}r \qquad \forall z \in X$
- $S(t)$ takes values in $\mathrm{dom}\,\mathcal{A}$
- $R_+(t)z = z + A\int_0^t rR_+(t-r)z \,\mathrm{d}r$
- for every $z \in \mathrm{dom}\,\mathcal{A}$ we have $\frac{\mathrm{d}}{\mathrm{d}t}R_+(t)z = \mathcal{A}R_-(t)z = AS(t)z$, $\frac{\mathrm{d}}{\mathrm{d}t}R_-(t)z = \mathcal{A}R_+(t)z$.

Consequently:

Lemma 3 *Let $\xi(s) \in C^1(0,T;X)$. We have*

$$\int_0^t R_+(t-s)\dot{\xi}(s)\,\mathrm{d}s = \xi(t) - R_+(t)\xi(0) + \mathcal{A}\int_0^t R_-(t-s)\xi(s)\,\mathrm{d}s \quad (4)$$

$$\int_0^t R_-(t-s)\dot{\xi}(s)\,\mathrm{d}s = -R_-(t)\xi(0) + \mathcal{A}\int_0^t R_+(t-s)\xi(s)\,\mathrm{d}s \quad (5)$$

Proof. We know that

$$\int_0^t e^{\mathcal{A}(t-s)}\dot{\xi}(s)\,\mathrm{d}s = \xi(t) - e^{\mathcal{A}t}\xi(0) + \mathcal{A}\int_0^t e^{\mathcal{A}(t-s)}\xi(s)\,\mathrm{d}s$$

(see [27, p. 107]). We use this formula and the definitions of $R_+(t)$ and $R_-(t)$ (note that $R_-(0) = 0$.) ∎

Now we make some formal computations. Linearity of the equation shows that the effect of the distributed input f will be the same as computed in [4]:

$$\int_0^t R_+(t-s)f(s)\,\mathrm{d}s\,. \quad (6)$$

So, in the following computation we assume f to be zero, and the effect of f will then be added to the final formulas.

Let D be the Dirichlet map:

$$\theta = Du \iff \begin{cases} \Delta\theta = 0 \\ \theta_{|\Gamma} = u\,. \end{cases}$$

We write Eq. (1) as

$$\theta_t = \int_0^t b(t-s)A\left\{\theta(s) - Du(s)\right\}\,\mathrm{d}s\,.$$

This is clearly legitimate if $\theta(t,x)$ is a classical solution of (1). Now, as in [18], we introduce $\xi(t) = \theta(t) - Du(t)$ and, if u is regular and θ is a classical solution, we see that

$$\xi_t = \int_0^t b(t-s)A\xi(s)\,\mathrm{d}s - Du'(t) \tag{7}$$

(dependence on the space variable x will not be indicated unless needed for clarity and the derivative with respect to time is denoted either with an index or with an apex).

The previous formula is not yet justified, since we don't even know the existence of the solutions. It will suggest a formula which will then be used to *define* the solutions to Eq. (1).

We apply the operator $R_+(t-s)$ to both the sides of the equality (7) and we integrate from 0 to t. A formula for the left hand side is suggested by (4). Instead, if u is smooth enough, we have the legitimate equality

$$\int_0^t R_+(t-s)Du'(s)\,\mathrm{d}s = Du(t) - R_+(t)Du(0) + \mathcal{A}\int_0^t R_-(t-s)Du(s)\,\mathrm{d}s\,.$$

We use (5) in order to formally compute

$$\begin{aligned}
&\int_0^t R_+(t-s)\int_0^s b(s-r)A\xi(r)\,\mathrm{d}r\;\mathrm{d}s \\
&= \mathcal{A}\int_0^t R_+(t-s)\int_0^s b(s-r)\mathcal{A}\xi(r)\,\mathrm{d}r\;\mathrm{d}s \\
&= \int_0^t R_-(t-s)\left[\mathcal{A}\xi(s) + \int_0^s b'(s-r)\mathcal{A}\xi(r)\,\mathrm{d}r\right]\,\mathrm{d}s\,.
\end{aligned}$$

We compare the three terms and we find that ξ "solves" the following integral equation:

$$\begin{aligned}
&\xi(t) = R_+(t)\xi(0) + \int_0^t R_-(t-s)\int_0^s b'(s-r)\mathcal{A}\xi(r)\,\mathrm{d}r\;\mathrm{d}s \\
&-Du(t) + R_+(t)Du(0) - \mathcal{A}\int_0^t R_-(t-s)Du(s)\,\mathrm{d}s\,.
\end{aligned}$$

It is now convenient to introduce a further transformation which, although not essential for the following, simplifies certain formulas. It will be freely used when needed. We note that we don't yet have a definition of θ, hence of ξ, so that the computations are still at a formal level.

We replace ξ with $\theta - Du$ and we find

$$\theta(t) = R_+(t)\theta_0 - \mathcal{A}\int_0^t R_-(t-s)D\left[u(s) + \int_0^s b'(s-r)u(r)\,\mathrm{d}r\right]\,\mathrm{d}s$$
$$+\mathcal{A}\int_0^t R_-(t-s)\int_0^s b'(s-r)\theta(r)\,\mathrm{d}r\ \mathrm{d}s\,. \tag{8}$$

The Volterra equation

$$v(s) = u(s) + \int_0^s b'(s-r)u(r)\,\mathrm{d}r \tag{9}$$

admits precisely one solution $u \in L^2(0,T;L^2(\Gamma))$ for every $v \in L^2(0,T;L^2(\Gamma))$ and the transformation from v to u is linear, continuous and with continuous inverse, for every $T > 0$. So, we can work with one of the two equivalent integral equations for θ: Eq. (8) or

$$\theta(t) = R_+(t)\theta_0 - \mathcal{A}\int_0^t R_-(t-s)Dv(s)\,\mathrm{d}s$$
$$+\mathcal{A}\int_0^t R_-(t-s)\int_0^s b'(s-r)\theta(r)\,\mathrm{d}r\ \mathrm{d}s\,. \tag{10}$$

Remark 4 Expressions (8), equivalently (10) are not yet in the final form we want to reach, since the last integral contains unbounded operators. Also the first integral contains an unbounded operator but it is proved in [19] that

$$\mathcal{A}\int_0^t R_-(t-s)Du(s)\,\mathrm{d}s = A\int_0^t S(t-s)Du(s)\,\mathrm{d}s$$

defines a linear and continuous transformation from $L^2(0,T;L^2(\Gamma))$ to $C(0,T;L^2(\Omega))$, for every $T > 0$. ∎

Now we elaborate further: we use the assumption that the kernel b is of class C^2 in order to write

$$\mathcal{A}\int_0^t R_-(t-s)\int_0^s b'(s-r)\theta(r)\,\mathrm{d}r\ \mathrm{d}s =$$
$$\int_0^t R_+(t-s)\left[b'(0)\theta(s) + \int_0^s b''(s-r)\theta(r)\,\mathrm{d}r\right]\,\mathrm{d}s \tag{11}$$
$$-\int_0^t b'(t-r)\theta(r)\,\mathrm{d}r\,. \tag{12}$$

Clearly, this is a formal integration by parts, suggested by (4), since the existence and properties of θ are not yet known.

We sum up: we formally obtained the following two Volterra integral equation for θ, which are equivalent (here we insert the effect of the distributed input f, see (6)):

$$\begin{aligned}\theta(t) &= \Big\{R_+(t)\theta_0 - \mathcal{A}\int_0^t R_-(t-s)Dv(s)\,\mathrm{d}s \\ &+ \int_0^t R_+(t-s)f(s)\,\mathrm{d}s\Big\} + \int_0^t L(t-s)\theta(s)\,\mathrm{d}s\,, \qquad (13)\\ \theta(t) &= \Big\{R_+(t)\theta_0 - \mathcal{A}\int_0^t R_-(t-s)Du(s)\,\mathrm{d}s \\ &- \int_0^t L(t-s)Du(s)\,\mathrm{d}s + \int_0^t R_+(t-s)f(s)\,\mathrm{d}s\Big\} \\ &+ \int_0^t L(t-s)\theta(s)\,\mathrm{d}s\,. \qquad (14)\end{aligned}$$

Here v and u are related by (9) and

$$L(t)\theta = R_+(t)b'(0)\theta - b'(t)\theta + \int_0^t R_+(t-\nu)b''(\nu)\theta\,\mathrm{d}\nu\,. \qquad (15)$$

As noted in Remark 4, the braces belong to $C(0,+\infty;L^2(\Omega))$ and depends continuously on $\theta_0 \in L^2(\Omega)$ and $v \in L^2(0,T;L^2(\Gamma))$ (equivalently, on $u \in L^2(0,T;L^2(\Gamma))$.) This is a Volterra integral equation with bounded operator valued kernel $L(t)$ and, as noted, both the brace and L are continuous X-valued function. Consequently, there exists a unique continuous function $\theta(t)$ which solves the Volterra equation (14), equivalently (13), for $t \geq 0$. Moreover, for each $T > 0$, the transformation from $\theta_0 \in L^2(\Omega)$, $u \in L^2(0,T;L^2(\Gamma))$, $f \in L^2(0,T;L^2(\Omega))$ to $\theta \in C(0,T;L^2(\Omega))$ is continuous.

Definition 5 The solution to Eq. (1) is the solution $\theta(t)$ to the Volterra integral equation (14), equivalently (13), where u and v are related as in (9). ∎

Proofs of the global existence of the solutions of a Volterra integral equations with bounded operator valued kernel are well known. However, we need to recall the main points of this proof for later use, in Sect. 4: We fix $T_0 > 0$ and we prove existence of a unique solution on $[0, T_0]$. Let $\chi > 0$ be a number whose value will be specified later on.

Let $\Phi(t)$ denote the brace in (14). We multiply both sides of Eq. (14) by $e^{-\chi t}$. It is then sufficient to prove the existence of the solution $e^{-\chi t}\theta(t) \in C(0,T;L^2(\Omega))$ to the new Volterra equation

$$\left[e^{-\chi t}\theta(t)\right] = e^{-\chi t}\Phi(t) + \int_0^t L_\chi(t-s)\left[e^{-\chi s}\theta(s)\right]\,\mathrm{d}s\,.$$

Here,

$$L_\chi(t) = e^{-\chi t}L(t)\,.$$

The operator

$$\theta(t) \longrightarrow \int_0^t L_\chi(t-s)\theta(s)\,\mathrm{d}s$$

from $C([0,T_0];X)$ to itself has norm less then 1 when χ is so large that

$$\int_0^{T_0} e^{-\chi t}||L(t)||\,\mathrm{d}t < 1\,.$$

We fix χ with this property. Existence of solutions now follows from Banach fixed point theorem.

Remark 6 The key points to note for later use are:

- the number χ will be nonnegative. Il θ solves Eq. (1) then $\eta(t) = e^{-\chi t}\theta(t)$ solves

$$\eta_t = -\chi\eta + \int_0^t e^{-\chi(t-s)}b(t-s)\Delta\eta(s)\,\mathrm{d}s\,, \qquad \eta_{|\Gamma}(t) = e^{-\chi t}u(t)\,;$$

- for $t = T$ fixed, the subspaces spanned by $\theta(T)$ and $\eta(T) = e^{-\chi T}\theta(T)$ (when u varies in $L^2(0,T;L^2(\Gamma))$) coincide;
- let $\theta_0 = 0$, as in Sect. 4 and let us define

$$\tilde{\mathcal{R}}_T = \left\{\mathcal{A}\int_0^T R_-(T-s)Du(s)\,\mathrm{d}s\,, \qquad u \in L^2(0,T;L^2(\Gamma))\right\}.$$

  This is a subspace of $L^2(\Omega)$ which is not changed by the transformation used above (which, for fixed $t = T$, is just multiplication by $e^{-\chi T}$). ∎

3 Regularity results

In order to study the regularity properties of the solution $\theta(t)$ to Eq. (1), it is convenient to use the more direct formula (14).

We already noted time continuity of θ. Now we prove further regularity properties, which extend to the case of boundary inputs results which are known in the case of distributed inputs.

The solution of the Volterra integral equation (14) will be denoted $\theta(\cdot;\theta_0,f,u)$ or, simply, $\theta(t)$. Moreover, we use also the shorter notations $L^2(Q)=L^2(0,T;L^2(\Omega))$, $L^2(G)=L^2(0,T;L^2(\Gamma))$.

We already noted:

Theorem 7 *The transformation $(\theta_0,f,u)\to\theta(\cdot;\theta_0,f,u)$ is linear and continuous from $L^2(\Omega)\times L^2(Q)\times L^2(G)$ to $C(0,T;L^2(\Omega))$.*

We recall that $\operatorname{dom} A = H^2(\Omega)\cap H^1_0(\Omega)$ so that $\operatorname{dom}\mathcal{A}=H^1_0(\Omega)$.

We have the following result which completely justifies the definition we chose for the solution:

Theorem 8 *1) If $f\in C^1(0,T;L^2(\Omega))$, $u\in C^2(0,T;L^2(\Gamma))$ and if $\theta_0-Du(0)\in\operatorname{dom}\mathcal{A}$ then $\xi(t)=\theta(t)-Du(t)$ belongs to $C^1(0,T;L^2(\Omega))\cap C(0,T;\operatorname{dom}\mathcal{A})$.*
2) if furthermore $\theta_0-Du(0)\in\operatorname{dom}A$, $f(0)-Du'(0)=0$, $f\in C^2(0,T;L^2(\Omega))$ and $u\in C^3(0,T;L^2(\Gamma))$ then: a) if $f''(t)$ and $u'''(t)$ are exponentially bounded, the function $\xi(t)$, $\xi'(t)$, $A\xi(t)$ are exponentially bounded; b) $\theta(t)$ solves Eq. (1) in the sense that $\xi(t)$ is continuously differentiable, takes values in $\operatorname{dom}A$ and for every $t\geq 0$ we have

$$\xi_t=\int_0^t b(t-s)A\xi(s)\,\mathrm{d}s-Du'(t)+f(t)\,.$$

Proof. We replace $\xi=\theta-Du$ in (14). We use (4) (and $u\in C^1$) to represent

$$\xi(t)=R_+(t)\xi(0)+\mathcal{A}\int_0^t R_+(t-s)\mathcal{A}^{-1}\left[f(s)-Du'(s)\right]\,\mathrm{d}s+\int_0^t L(t-s)\xi(s)\,\mathrm{d}s\,. \tag{16}$$

We use now (5) (and $f\in C^1$, $u\in C^2$) to represent

$$\begin{aligned}\xi(t)=&\Big\{R_+(t)\xi(0)+\int_0^t R_-(t-s)\mathcal{A}^{-1}\left[f'(s)-Du''(s)\right]\,\mathrm{d}s\\&+R_-(t)\mathcal{A}^{-1}[f(0)-Du'(0)]\Big\}+\int_0^t L(t-s)\xi(s)\,\mathrm{d}s\,.\end{aligned} \tag{17}$$

Our assumptions imply that the brace belong to $\operatorname{dom}\mathcal{A}$, which is invariant under $L(t)$. Hence, the previous Volterra integral equation can be solved not only in $L^2(0,T;L^2(\Omega))$ but also in $L^2(0,T;\operatorname{dom}\mathcal{A})$. Moreover, the brace belongs to $C(0,T;\operatorname{dom}\mathcal{A})$, so that the solution $\xi(t)$ is a continuous, $\operatorname{dom}\mathcal{A}$ valued function. the brace belongs to $C^1(0,T;L^2(\Omega))$ and the last integral can now be written

$$\int_0^t L(t-s)\mathcal{A}^{-1}[\mathcal{A}\xi(s)]\,\mathrm{d}s\,.$$

Hence it belongs to $C^1(0,T;L^2(\Omega))$ so that $\xi(t)$ is differentiable too.

Now we consider the more stringent set of assumptions 2). We use (4) to represent

$$\begin{array}{l}\int_0^t R_-(t-s)\mathcal{A}^{-1}\left[f'(s)-Du''(s)\right]\,\mathrm{d}s=\\ \int_0^t R_+(t-s)A^{-1}\left[f''(s)-Du'''(s)\right]\,\mathrm{d}s-A^{-1}[f'(t)-Du''(t)]\\ +R_+(t)A^{-1}[f'(0)-Du''(0)]\in\operatorname{dom}A\,.\end{array}$$

So we can solve the Volterra equation in $\operatorname{dom}A$, i.e. $\xi(t)$ is a continuous $\operatorname{dom}A$ valued function.

The exponential bound on $\xi(t)$ follows directly from (16) and Gronwall inequality. The exponential bound on $\xi'(t)$ is obtained because we already know the existence of $\xi'(t)$ so that differentiation of both sides of (16) gives an integral equation for $\xi'(t)$.

In order to obtain the exponential bound of $A\xi(t)$ we proceed as follows: we apply $A=\mathcal{A}^2$ to both the sides of (16) and we use the regularity of u and f in order to represent

$$\begin{array}{l}A\int_0^t R_+(t-s)[f(s)-Du'(s)])\,\mathrm{d}s=R_+(t)[f'(0)-Du''(0)]\\ -[f'(t)-u''(t)]+\int_0^t R_+(t-s)[f''(s)-Du'''(s)]\,\mathrm{d}s\,.\end{array}$$

Moreover we already know that $\xi(t)\in\operatorname{dom}A$ so that we can exchange A and $L(t)$. We obtain a Volterra integral equation for $A\xi(t)$, from which the exponential bound follows.

Now we go back to the formula (16). The regularity of $\xi(t)$ already noted shows that

$$\int_0^t L(t-s)\xi(s)\,\mathrm{d}s=\int_0^t R_+(t-s)\int_0^s b(s-r)A\xi(r)\,\mathrm{d}r\,\,\mathrm{d}s-\int_0^t R_-(t-s)\mathcal{A}\xi(s)\,\mathrm{d}s\,.$$

Hence, $\xi(t)$ solves

$$\xi(t) - R_+(t)\xi(0) + \mathcal{A}\int_0^t R_-(t-s)\xi(s)\ \mathrm{d}s = \int_0^t R_+(t-s)f(s)\ \mathrm{d}s$$
$$+\int_0^t R_+(t-s)\int_0^s b(s-r)A\xi(r)\ \mathrm{d}r\ \ \mathrm{d}s - \int_0^t R_+(t-s)Du'(s)\ \mathrm{d}s\,.$$

We use (4) and we see that

$$\int_0^t R_+(t-s)\left\{\xi'(s) + Du'(s) - f(s) - \int_0^s b(s-r)A\xi(r)\ \mathrm{d}r\right\}\ \mathrm{d}s = 0$$

For every $t \in [0,T]$; Now, T is arbitrary so that equality to zero holds on $[0,+\infty)$. Fix any value of T and change the definition of $f(t)$ and $u(t)$ for $t > 0$ so to have regular functions with bounded support. This does not change $\xi(t)$ on $[0,T]$. Now we can take the Laplace transform and we see that the brace is identically zero on $[0,T]$ for every $T > 0$. This completes the proof. ∎

Lemma 9 *If* $f = 0$, $u = 0$ *and* $\theta_0 \in \operatorname{dom} A^k$ *then* $\theta(t) \in C(0,T;\operatorname{dom} A^k)$.

Proof. If $k = 0$ this is a special case of Theorem 8. In general, it is deduced from formula (14), i.e. (16), with $f = 0$ and $u = 0$, and unicity of the solution, because

$$A^K L(t)\theta = L(t)A^k\theta \qquad \forall\theta \in \operatorname{dom} A^k\,. \quad ∎$$

The results that most interest us concern normal derivatives. Let us introduce the trace operator

$$\gamma_1\theta = \frac{\partial}{\partial n}\theta_{|\Gamma}$$

We now use the notation $b \star \theta$ to denote the convolution,

$$(b\star\theta)(t) = \int_0^t b(t-s)\theta(s)\ \mathrm{d}s\,.$$

Lemma 10 *Let* $u = 0$, $f = 0$. *We have:*

$$\gamma_1\,(b\star\theta) \in H^{-1}(0,T;L^2(\Gamma))$$

and the transformation from $\theta_0 \in L^2(\Omega)$ *to* $\gamma_1\,(b\star\theta) \in H^{-1}(0,T;L^2(\Gamma))$ *is continuous.*

Proof. The trace is well defined if θ_0 is "regular", i.e. $\theta_0 \in \mathrm{dom}\, A^k$ with k large. We prove continuity (in the stated sense) of the transformation, which is then extended by continuity to every $\theta_0 \in L^2(\Omega)$.

It is known from [20] that

$$\gamma_1 \theta = -D^* A\theta\,, \qquad \theta \in \mathrm{dom}\, A\,.$$

For "regular" data and zero boundary condition and affine term we have

$$\theta'(t;\theta_0) = b \star (A\theta(\cdot;\theta_0)) = A(b \star \theta(\cdot;\theta_0))\,.$$

Hence,

$$b \star \theta(\cdot;\theta_0) = A^{-1}(\theta'(t;\theta_0))\,, \qquad \gamma_1(b \star \theta(\cdot;\theta_0)) = D^*\theta'(t;\theta_0)\,.$$

Let now $\{\theta_n\}$ be a sequence of "regular" initial data, $\theta_n \to \theta_0$ in $L^2(\Omega)$. We proved that the solutions converge in $C(0,T;L^2(\Omega))$ so that

$$\theta'(t;\theta_n) \to \theta'(t;\theta_0) \qquad \text{in} \quad H^{-1}(0,T;L^2(\Omega))\,.$$

Hence, it is sufficient to prove that D^* has a continuous extension as an operator from $H^{-1}(0,T;L^2(\Omega))$ to $H^{-1}(0,T;L^2(\Gamma))$. Let for this

$$\tilde{D}\ : \quad (\tilde{D}u(\cdot)\,)(t) = Du(t)\,.$$

Let $\phi \in H^1_0(0,T;L^2(\Gamma))$, $\psi \in H^{-1}(0,T;L^2(\Omega))$. The operator $\tilde{D}^*$ is defined by

$$\langle\!\langle \tilde{D}\phi, \psi \rangle\!\rangle = \langle\!\langle \phi, \tilde{D}^*\psi \rangle\!\rangle$$

(where $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ denotes the proper pairing). This operator $\tilde{D}^*$ is the required extension by continuity of D^* since, for $\psi = \psi(t)$ a smooth function, equality of the pairings is equivalent to

$$\int_0^T \langle \tilde{D}\phi(t,x), \psi(t,x)\rangle = \int_0^T \langle D\phi(t,x), \psi(t,x)\rangle = \int_0^T \langle \phi(t,x), D^*\psi(t,x)\rangle$$

where $\langle \cdot,\cdot \rangle$ is the inner product in $L^2(\Omega)$. ∎

Let us consider now the wave equation

$$w_{tt} = \Delta w\,, \quad w(0) = w_0\,, \quad w'(0) = 0 \text{ in } \Omega\,, \qquad w_\Gamma = 0\,. \tag{18}$$

It is known that

$$w(t) = R_+(t)w_0\,, \qquad t \in \mathbb{R}$$

and $\gamma_1 w(t)$ exists as an element of $L^2(-T,T;L^2(\Gamma))$. Moreover,

$$w_0 \longrightarrow \gamma_1 w(t) = \gamma_1[R_+(t)w_0] \tag{19}$$

is continuous from $w_0 \in H_0^1(\Omega)$ to $L^2(-T,T;L^2(\Gamma))$ for every $T>0$, see [20, 23, 17]. Here the full strength of the assumption made on Ω is needed.

The sense in which the trace exists is as follows: it exists for the "regular" vectors w_0. It is proved the stated continuous dependence, so that the trace operator is extended by continuity to every initial condition in $H_0^1(\Omega)$.

We prove an analogous property of $\theta(t)$ but, instead then proving it directly, with a computation that mimics, for example, that in [23], we deduce it from the property of the solution of the wave equation just recalled.

Theorem 11 *Let f and u be zero. The transformation*

$$\theta_0 \to \gamma_1\theta$$

is continuous from $H_0^1(\Omega)$ to $L^2(0,T;L^2(\Gamma))$ for every $T>0$.

<u>Proof</u>. We proved (Lemma (9)) that if $\theta_0 \in \mathrm{dom}\, A^k$ then $\theta(t) \in \mathrm{dom} A^k$ for every t so that the trace $\gamma_1\theta(t)$ exists in the usual sense, provided that k is large enough. We prove continuity from $H_0^1(\Omega)$ to $L^2(G)$ so that the transformation $\theta_0 \to \gamma_1\theta(t)$ can be extended to every initial condition $\theta_0 \in H_0^1(\Omega)$.

Let $w(t) = R_+(t)\theta_0$ be the solution of problem (18) (with now $w_0 = \theta_0$). Then, $\theta(t)$ is the solution of the Volterra integral equation

$$\theta(t) = w(t) + \int_0^t \left[R_+(t-s)b'(0)\theta(s) + \int_0^{t-s} R_+(t-s-\nu)b''(\nu)[\theta(s)]\,\mathrm{d}\nu\right]\,\mathrm{d}s$$
$$-\int_0^t b'(t-s)\theta(s)\,\mathrm{d}s\,. \tag{20}$$

Let $T>0$ be fixed. The properties of the wave equation recalled above show that for every fixed s, the function

$$t \longrightarrow \gamma_1\left[R_+(t-s)\theta(s)\right]$$

is square integrable, hence it is integrable. We now proceed in two steps.

Step 1) we prove that $s \to \gamma_1 \left[R_+(t-s)\theta(s)\right]$ exists and belongs to $L^2(G)$ for a.e. t.

We shall prove below that the function of s

$$s \to \int_0^T \gamma_1 \left[R_+(t-s)\theta(s)\right] \,\mathrm{d}t \tag{21}$$

is continuous, hence square integrable. Granted this we can integrate,

$$\begin{aligned}
&\int_0^T \left|\left|\int_0^T \gamma_1 \left[R_+(t-s)\theta(s)\right] \,\mathrm{d}t\right|\right|^2 \mathrm{d}s \\
&\le \int_0^T \left\{T \cdot \int_0^T \left|\left|\gamma_1 \left[R_+(t-s)\theta(s)\right]\right|\right|^2 \mathrm{d}t\right\} \mathrm{d}s < +\infty\,.
\end{aligned} \tag{22}$$

Now, Fubini theorem proves that the function

$$s \longrightarrow \gamma_1 \left[R_+(t-s)\theta(s)\right]$$

exists a.e. and belongs to $L^2(G)$. Moreover,

$$\begin{aligned}
&\int_0^T \int_0^T ||\gamma_1 R_+(t-s)\theta(s)||^2 \,\mathrm{d}t \;\mathrm{d}s = \int_0^T \int_{-s}^{T-s} ||\gamma_1 R_+(r)\theta(s)||^2 \,\mathrm{d}r \;\mathrm{d}s \\
&\le M||\theta||^2_{L^2(0,T;H_0^1(\Omega))} \le \tilde{M}||\theta_0||^2_{H_0^1(\Omega)}\,.
\end{aligned} \tag{23}$$

Inequalities (22) and (23) show that the function

$$t \longrightarrow \int_0^t \gamma_1[R_+(t-s)\theta(s)] \,\mathrm{d}s\,,$$

as an element of $L^2(Q)$, depends continuously on $\theta_0 \in H_0^1(\Omega)$.

In order to complete this argument we prove continuity of the function in (21). We represent

$$\begin{aligned}
&\int_0^T \left[\gamma_1 R_+(t-s)\theta(s) - \gamma_1 R_+(t-s')\theta(s')\right] \,\mathrm{d}t \\
&= \int_0^T \left[\gamma_1 R_+(t-s)\theta(s) - \gamma_1 R_+(t-s)\theta(s')\right] \,\mathrm{d}t \qquad (24)\\
&+ \int_0^T \left[\gamma_1 R_+(t-s)\theta(s') - \gamma_1 R_+(t-s')\theta(s')\right] \,\mathrm{d}t \qquad (25)
\end{aligned}$$

We know from Theorem 8 point 1) that if $\theta_0 \in H_0^1(\Omega) = \mathrm{dom}\,\mathcal{A}$ then the solution $\theta(s)$ is continuous from s to $H_0^1(\Omega)$. Hence,

$$\lim_{s' \to s} ||\theta(s) - \theta(s')||_{H_0^1(\Omega)} = 0\,.$$

This shows that the integral in (24) tends to zero, thanks to the regularity of the trace operator of the *wave* equation.

The integral in (25) is represented as

$$\int_0^T \left[\gamma_1 R_+(t-s'+(s'-s))\theta(s') - \gamma_1 R_+(t-s')\theta(s')\right]\, \mathrm{d}t\,.$$

This tends to zero for $s-s'\to 0$, thanks to the Lebesgue theorem on the continuity of the shift.

We recapitulate: we have now proved that $\gamma_1 R_+(t-s)\theta(s)$ is well defined, as an element of L^2, both as a function of t and as a function of s. In order to complete the proof:

Step 2) we prove that the trace $\gamma_1\theta(s)$ exists in $L^2(G)$ and depends continuously on $\theta_0\in H_0^1(\Omega)$. We go back to the equation (20) that we now represent as

$$\theta(t) = F(t) + \int_0^t b'(t-s)\theta(s)\,\mathrm{d}s\,.$$

Here

$$F(t) = y(t) + \int_0^t \left[R_+(t-s)b'(0)\theta(s) + \int_0^{t-s} R_+(t-s-\nu)b''(\nu)[\theta(s)]\,\mathrm{d}\nu\right]\,\mathrm{d}s$$

and we proved that $\gamma_1 F(t)\in L^2(G)$ is a continuous function of $\theta_0\in H_0^1(\Omega)$.

The function $b'(t)$ is scalar, so that for $\theta_0\in \mathrm{dom}\,A^k$, k large enough, we have

$$\gamma_1\theta(t) = \gamma_1 F(t) + \int_0^t k(t-s)\gamma_1 F(s)\,\mathrm{d}s$$

where $k(t)$ is the resolvent kernel of $b'(t)$. The required continuity property of $\gamma_1\theta(t)$ now follows because the right hand side of this equality is a continuous function of $\gamma_1 F(t)\in L^2(G)$. ∎

4 Controllability

In this section we prove the following controllability result:

Theorem 12 *Let Assumption 1 hold and let $f=0$, $\theta_0=0$. There exists $T>0$ such that for every $\theta_1\in L^2(\Omega)$ there exists $u\in L^2(G)$ such that the corresponding solution $\theta(t;u)$ of Eq. (1) (with $f=0$) satisfies*

$$\theta(T;u)=\theta_1\,.$$

In order to prove this theorem we use the known results on the controllability of the wave equation and a compactness argument, and we prove first that the reachable space is closed with finite codimension for T large. We then characterize the orthogonal of the reachable space and we prove that it is 0 (compare [30] for a similar argument.)

We need some preliminaries: we shall equivalently work with both the representations (14) and (13) of the solution of Eq. (1) (and $\theta_0 = 0$, $f = 0$). For definiteness (and with an abuse of language), in the first case the solution is denoted $\theta(t; u)$; in the second case $\theta(t; v)$. For every $T > 0$, we denote

$$\mathcal{R}_T = \Big\{\theta(T; v), \quad v \in L^2(G)\Big\} = \Big\{\theta(T; u), \quad u \in L^2(G)\Big\}$$

(implicitly, u and v are related by (9)). Instead, $\tilde{\mathcal{R}}_T$ denotes the ($L^2(\Omega)$ component of the) reachable set of the wave equation:

$$\begin{array}{ll} w_{tt} = \Delta\eta\,, \qquad t > 0\,,\ x \in \Omega & \\ w(0, x) = 0\,,\ w_t(0, x) = 0 \quad x \in \Omega\,, \qquad w(t, x) = v(t, x) \quad \text{on } \Gamma\,. & \end{array} \tag{26}$$

I.e.

$$\tilde{\mathcal{R}}_T = \Big\{w(T; v)\,, \quad v \in L^2(G)\Big\}\,.$$

It is known that for T large we have $\tilde{\mathcal{R}}_T = L^2(\Omega)$, see [23].

It is proved in [18] that the solution to (26) is

$$\eta(t; v) = \mathcal{A} \int_0^t R_-(t - s) D[-v(s)]\, \mathrm{d}s$$

while we recall

$$\theta(t; v) = \mathcal{A} \int_0^t R_-(t - s) D[-v(s)]\, \mathrm{d}s + \int_0^t L(t - s)\theta(s)\, \mathrm{d}s$$

We shall replace v with $-v$ for convenience of notations. The reachable spaces are not changed. We noted that the operator

$$v(\cdot) \longrightarrow \mathcal{A} \int_0^t R_-(t - s) D v(s)\, \mathrm{d}s = \Big(\mathcal{S}v\Big)(t)$$

is continuous from $L^2(G)$ to $C(0, T; L^2(\Omega))$.

We recall from Remark 6 that we can multiply by $e^{-\chi t}$ and we obtain that the operator L_χ: $C(0, T, L^2(\Omega))$ to itself has norm less then a fixed

$q < 1$. In fact we do more: it is convenient to represent $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ where

$$\Big(\mathcal{L}_1\theta\Big)(t) = \int_0^t \Big[b'(t-s)\theta(s) - \int_0^{t-s} R_+(t-s-\nu)b''(\nu)[\theta(s)]\,\mathrm{d}\nu\Big]\,\mathrm{d}s$$

$$\Big(\mathcal{L}_2\theta\Big)(t) = b'(0)\int_0^t R_+(t-s)\theta(s)\,\mathrm{d}s\,.$$

Multiplication by $e^{-\chi t}$ replaces these operators with the operators

$$\Big(\mathcal{L}_{\chi,1}\theta\Big)(t) = \int_0^t e^{-\chi(t-s)}\Big[b'(t-s)\theta(s) - \int_0^{t-s} R_+(t-s-\nu)b''(\nu)[\theta(s)]\,\mathrm{d}\nu\Big]\,\mathrm{d}s$$

$$= \int_0^t H_\chi(t-s)\theta(s)\,\mathrm{d}s$$

$$\Big(\mathcal{L}_{\chi,2}\theta\Big)(t) = b'(0)\int_0^t e^{-\chi(t-s)}R_+(t-s)\theta(s)\,\mathrm{d}s\,.$$

Of course,

$$\mathcal{L}_\chi = \mathcal{L}_{\chi,1} + \mathcal{L}_{\chi,2}\,.$$

The operator $\mathcal{S}$ is transformed to $\mathcal{S}_\chi$,

$$\left(\mathcal{S}_\chi v\right)(t) = \mathcal{A}\int_0^t e^{-\chi(t-s)}R_-(t-s)Dv(s)\,\mathrm{d}s$$

(here of course we wrote $v(t)$ instead then $e^{-\chi t}v(t)$). We choose χ so to have

$$||\mathcal{L}_1|| < q < \frac{1}{2M}\,, \qquad ||\mathcal{L}_2|| < q < \frac{1}{2M} \qquad M = \max\{||\mathcal{S}||\,,\,1\}\,. \quad (27)$$

The crucial lemma that we now prove is:

Lemma 13 *For every $T > 0$ there exists a bounded boundedly invertible operator J_T in $L^2(G)$ and a compact operator K_T from $L^2(G)$ to $L^2(\Omega)$ such that*

$$\mathcal{R}_T = \mathrm{im}\Big\{\mathcal{S}_T J_T + \mathcal{K}_T\Big\}\,. \quad (28)$$

In the proof of this lemma, χ has been fixed so to have (27). In order to prove this result, T is fixed so that the index "T" is omitted. Conditions (27) shows that $||\mathcal{L}_\chi|| < q < 1$ so that

$$\theta(T;v) = \sum_{n=0}^{+\infty}\Big(\mathcal{L}_\chi^n\mathcal{S}v\Big)(T)\,. \quad (29)$$

We examine the individual terms of the series. We note first that for T given

$$\left|\left|\mathcal{L}_\chi^n \mathcal{S}\right|\right| \leq M q^n \sum_{i=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{1}{M^{n-1}} \tag{30}$$

and we can reorder the terms in the series which is obtained after replacing $\mathcal{L}_\chi = \mathcal{L}_{\chi,1} + \mathcal{L}_{\chi,2}$ in (29).

Now we prove:

Lemma 14 *Let $b \in C^3(0,T)$. For every $T > 0$ the transformation*

$$v(\cdot) \longrightarrow \left(\mathcal{L}_{\chi,1}\mathcal{S}_\chi v\right)(T)$$

from $L^2(G)$ to $L^2(\Omega)$ is compact.

Proof. In fact,

$$\begin{aligned} \left(\mathcal{L}_{\chi,1}\mathcal{S}_\chi v\right)(T) &= \int_0^T H_\chi(T-s)\mathcal{A}\int_0^s R_-(s-r)Dv(r)\,\mathrm{d}r\ \mathrm{d}s \\ &= \int_0^T \mathcal{A}\int_r^T R_-(s-r)H_\chi(T-s)[Dv(r)]\,\mathrm{d}s\ \mathrm{d}r\,. \end{aligned} \tag{31}$$

The function $s \to H_\chi(T-s)[Dv]$ is continuously differentiable so that we can use (4) to integrate by parts the inner integral in (31). We find

$$\begin{aligned} &\int_0^{T-r} R_+(T-r-\nu)H'_\chi(\nu)[Dv(r)]\,\mathrm{d}\nu - H_\chi(T-r)[Dv(r)] \\ &+R_+(T-r)H_\chi(0)[Dv(r)]\,. \end{aligned}$$

Now we observe that $\operatorname{im} D \subseteq H^{1/2}(\Omega) = \operatorname{dom}\mathcal{A}^{1/2}$ is invariant under $R_\pm(t)$ so that $\left(\mathcal{L}_{\chi,1}\mathcal{S}_\chi v\right)(T)$ takes values in $H^{1/2}(\Omega)$, compactly embedded in $L^2(\Omega)$. ∎

We recapitulate: $\left(\mathcal{L}_{\chi,1}^n\mathcal{S}_\chi\,\cdot\right)(T)$ is a compact operator for every $k > 0$. Hence, in a special case we soon get the results we are looking for (with $J = I$):

Corollary 15 *If $b'(0) = 0$ then the representation (28) holds.*

We consider now the case $b'(0) \neq 0$. In order to study the possibly non compact operators appearing in (29) we must study the operators $\mathcal{L}_{\chi,2}^n\mathcal{S}_\chi$. The first observation is

$$\left|\left|\mathcal{L}_{\chi,2}^n\mathcal{S}_\chi\right|\right| < ||\mathcal{S}||\, q^n\,, \qquad q < 1/(2M)\,, \qquad M = \max\{||\mathcal{S}||\,,\ 1\} \tag{32}$$

(note that $||\mathcal{S}_\chi|| \le ||\mathcal{S}||$ because $\chi \le 0$.) We shall use the following equality:

$$R_+(t-s)R_-(s-r) = \frac{1}{2}\Big[R_-(t-r) - R_-(t-2s+r)\Big] \qquad (33)$$

so that for $0 \le t \le T$ we have

$$\Big(\mathcal{L}_{\chi,2}\mathcal{S}_\chi v\Big)(t) = \frac{1}{2}b'(0)\mathcal{A}\int_0^t\int_0^s e^{-\chi(t-r)}R_-(t-r)Dv(r)\;\mathrm{d}r\;\;\mathrm{d}s$$
$$-\frac{1}{2}b'(0)\mathcal{A}\int_0^t\int_0^s e^{-\chi(t-r)}R_-(t-2s+r)Dv(r)\;\mathrm{d}r\;\;\mathrm{d}s\,.$$

The last double integral can be written as

$$\int_0^t e^{-\chi(t-r)}\int_{r-t}^{t-r} R_-(\nu)[Dv(r)]\;\mathrm{d}\nu\;\;\mathrm{d}r\,.$$

It is zero because $R_-(\nu) = -R_-(-\nu)$. So,

$$\Big(\mathcal{L}_{\chi,2}\mathcal{S}_\chi v\Big)(t) = \frac{1}{2}b'(0)\mathcal{A}\int_0^t e^{-\chi(t-r)}R_-(t-r)\,[(t-r)Dv(r)]\;\mathrm{d}r\,.$$

We elaborate analogously and we find

$$\Big(\mathcal{L}_{\chi,2}^2\mathcal{S}_\chi v\Big)(t)$$
$$= \frac{1}{2^2}b'(0)^2\mathcal{A}\int_0^t R_-(t-r)\Big[e^{-\chi(t-r)}\frac{(t-r)^2}{2}Dv(r)\Big]\;\mathrm{d}r \qquad (34)$$
$$+\frac{b'(0)^2}{2\cdot 2^2}\int_0^t e^{-\chi(t-r)}\Big\{\int_0^{t-r} R_+(t-r-2\nu)[Dv(r)]\;\mathrm{d}\nu$$
$$-R_+(r-t)(t-r)Dv(r)\Big\}\;\mathrm{d}r \qquad (35)$$

(The integration by parts needed in this computations are justified by using the definition of $R_-(t)$ and [27, p. 107]. This is possible because $e^{\mathcal{A}t}$ is a C_0–group of operators.)

The operator (35) for $t = T$ takes values in $H^{1/2}(\Omega)$. Hence it is compact.

The norm of the operator obtained from (34) when $t = T$ as a transformation from $L^2(0,T;L^2(\Gamma))$ to $L^2(\Omega)$ is less then

$$||\mathcal{S}||\left[\frac{b'(0)}{2}\right]^2\frac{T^2}{2!}\,.$$

Hence, the norm of the compact operator (35) is less then

$$||\mathcal{S}|| \left\{ \left[\frac{b'(0)}{2} \right]^2 \frac{T^2}{2!} + q^2 \right\}$$

see (30).

Now we iterate the previous computation in order to elaborate $\left(\mathcal{L}^n_{\chi,2}\mathcal{S}_\chi v\right)(t)$. It turns out that this is the sum of two operators. When evaluated at T one of them is a compact operator, say $\mathcal{K}_n$ and the second one is

$$\mathcal{A}\int_0^T R_-(T-r)\left[e^{-\chi(T-r)}\left(\frac{b'(0)}{2}\right)^n \frac{(T-r)^n}{n!} Dv(r)\right] \,\mathrm{d}r\,. \qquad (36)$$

The norm of this noncompact operator is less then

$$||\mathcal{S}|| \cdot \left[\frac{|b'(0)|}{2} T\right]^n \frac{1}{n!}$$

and so the norm of $\mathcal{K}_n$ is less then

$$||\mathcal{S}|| \cdot \left\{ \left[\frac{|b'(0)|}{2} T\right]^n \frac{1}{n!} + q^n \right\},$$

see (32). This shows uniform convergence of the series, in particular of the series of the compact operators, and we find the representation (28) with J_T the multiplication by $\exp\{-(\chi - b'(0)/2)(T-r)\}$.

What we know on the controllability of the wave equation now implies:

Theorem 16 *There exists $T > 0$ such that the reachable set $\mathcal{R}_T$ is closed with finite codimension.*

We noted that the reachable set does not depend on χ. Hence, once this result has been proved, we can proceed in the computation without making use of the multiplication by $e^{-\chi t}$.

Now we observe:

Lemma 17 *The reachable set $\mathcal{R}(T)$ increases with time.*

Proof. The proof is easy and expected, but it must be explicitly checked because $L^2(\Omega)$ is not the state space of Eq. (1). We introduce the resolvent operator $K_L(t)$ of the kernel $L(t)$ and we see that

$$\theta(t;v) = \mathcal{A}\int_0^t R_-(t-s)Dv(s)\,\mathrm{d}s + \int_0^t \mathcal{A}\int_r^t K_L(t-s)R_-(s-r)[Dv(r)]\,\mathrm{d}s\,\mathrm{d}r\,.$$

It is now easily checked that if θ_1 is reached at time T using the input $v(\cdot)$ then it is also reached at time $T+\tau$ using the input which is zero for $t \le \tau$, $v(t-\tau)$ on $(\tau, T+\tau)$. ∎

Consequently,

$$\mathcal{R}_\infty = \bigcup_{T>0} \mathcal{R}_T = \bigcup_n \mathcal{R}_n$$

and Baire Theorem implies:

Lemma 18 *There exists T such that $\mathcal{R}_T = L^2(\Omega)$ if and only if $\mathcal{R}_\infty = L^2(\Omega)$.*

Now we use Lemma 10 in order to characterize the elements of $[\mathcal{R}_\infty]^\perp$ (a finite dimensional subspace of $L^2(\Omega)$) by using the adjoint equation

$$\xi_t = -\int_t^T b(s-t)\Delta\xi(s)\,\mathrm{d}s\,, \qquad \xi_{|\Gamma} = 0\,, \qquad \xi(T) = \xi_0\,. \tag{37}$$

We observe that the substitution

$$\xi(t) = y(T-t)$$

shows that $y(t)$ solves

$$y_t = \int_0^t b(t-s)\Delta y(s)\,\mathrm{d}s\,, \qquad y_{|\Gamma} = 0\,, \qquad y(0) = \xi_0\,. \tag{38}$$

So, it is equivalent to work with (37) or (38).

In order to prove the next crucial theorem, it is more convenient to work with the original input u and to indicate explicitly dependence on the space variable x. So, θ is now $\theta(t,x;u)$ simply denoted $\theta(t,x)$.

Theorem 19 *The vector $\xi_0 \in L^2(\Omega)$ belongs to $[\mathcal{R}_T]^\perp$ if and only if the solution $\xi(t)$ of equation (38) satisfies*

$$\gamma_1 \int_t^T b(r-t)\xi(r,x)\,\mathrm{d}r = 0 \qquad 0 \le t \le T\,. \tag{39}$$

equivalently if the solution $y(t)$ to Eq. (37) satisfies

$$\gamma_1 \int_0^t b(t-s)y(s,x)\,\mathrm{d}s = 0 \qquad 0 \le t \le T\,. \tag{40}$$

Proof. We observe that in order to characterize $[\mathcal{R}_T]^\perp$ we can assume that u as a smooth control both in space and time variables. If it happens that $\xi_0 \in \operatorname{dom} A^k$, k large enough, then the next computation is justified:

$$\begin{aligned}
&\int_\Omega \xi_0(x)\theta(T,x)\,\mathrm{d}x = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \xi(t,x)\theta(t,x)\,\mathrm{d}x\ \mathrm{d}t\\
&= -\int_0^T\int_\Omega\int_t^T b(r-t)\Delta\xi(r,x)\,\mathrm{d}r\,\theta(t,x)\,\mathrm{d}x\ \mathrm{d}t\\
&+\int_0^T\int_\Omega \xi(t,x)\int_0^t b(t-s)\Delta\theta(s,x)\,\mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t\\
&= -\int_0^T\int_\Gamma \gamma_1\Big[\int_t^T b(r-t)\xi(r,x)\,\mathrm{d}r\Big]u(t,x)\,\mathrm{d}x\ \mathrm{d}t\,.
\end{aligned}$$

The previous computation is only justified if ξ_0 is regular, which needs not be. Otherwise we approximate ξ_0 with a sequence $\xi_n \in \operatorname{dom} A^k$. We intend the exterior double integral as the pairing of $H^{-1}(0,T;L^2(\Omega))$ and $H^1_0(0,T;L^2(\Omega))$ and pass to the limit. We obtain the equality

$$\int_\Omega \xi_0(x)\theta(T,x)\,\mathrm{d}x = -\int_0^T\int_\Gamma \gamma_1\Big[\int_t^T b(r-t)\xi(r,x)\,\mathrm{d}r\Big]u(t,x)\,\mathrm{d}x\ \mathrm{d}t$$

thanks to Lemma 10.

Arbitrary varying $u(\cdot)$ within the smooth elements of $H^1_0(0,T;L^2(\Omega))$ we see that $\xi_0 \in [\mathcal{R}_T]^\perp$ if and only if

$$\gamma_1\int_t^T b(r-t)\xi(r,x)\,\mathrm{d}r = 0 \qquad \text{i.e.} \qquad \gamma_1\int_0^{T-t} b(T-t-s)y(s,x)\,\mathrm{d}s = 0\,. \tag{41}$$

∎

We now prove that $[\mathcal{R}_\infty]^\perp$ is invariant under the action of the equation (38):

Lemma 20 *Let $\xi_0 \in [\mathcal{R}_\infty]^\perp$ and let $t_0 > 0$ be fixed. Let y solve (38). Then, $y(t_0) \in [\mathcal{R}_\infty]^\perp$.*

Proof. It is sufficient to prove that $y(t_0) \perp \mathcal{R}_N$ for every N. We note that $y(t_0) = \xi(T-t_0)$ (and ξ solves (37)) and T can be arbitrarily fixed. We fix T so large that $T - t_0 > N$.

We repeat the same computation as in the proof of Theorem 19 but we integrate on $[T-t_0, T]$ instead then on $[0,T]$. The conditions

$$\gamma_1\int_t^T b(r-t)\xi(r,x)\,\mathrm{d}r = 0 \qquad \text{and} \qquad \int_\Omega \xi_0(x)\theta(T,x)\,\mathrm{d}x = 0$$

now hold by assumption because $\xi_0 \in [\mathcal{R}_\infty]^\perp$ and we get

$$\int_\Omega \xi(T-t_0,x)\theta(T-t_0,x;u)\,\mathrm{d}x = 0$$

for every $u \in H^1_0(0,T;L^2(\Omega))$. Hence, $\xi(T-t_0) = y(t_0) \in [\mathcal{R}_N]^\perp$ for every N, as wanted. ∎

We shall now use the Laplace transform of $y(t)$ in order to prove the following theorem. We find

$$\hat{y}(\lambda) = [\lambda I - \hat{b}(\lambda)A]^{-1}\xi_0 \tag{42}$$

The fact that $b(t) \in L^1(0,T)$ and

$$\hat{b}(0) = \int_0^{+\infty} b(s)\,\mathrm{d}s > 0$$

show that $\hat{b}(\lambda)$ is well defined and positive in a right neighborhood of $\lambda = 0$, on the real axis. Hence, the inverse operator in (42) exists and depends continuously on λ. This justifies the previous computation when ξ_0 is regular so that $y(t)$ is a differentiable solution of (38). Equality is then extended by continuity to every $\xi_0 \in L^2(\Omega)$. Alternatively, we can obtain this formula from the Volterra integral equation (14) (with $f = 0$ and $u = 0$.)

Lemma 21 *The subspace $[\mathcal{R}_\infty]^\perp$ is invariant under A^{-1}.*

Proof. Let $\xi_0 \in [\mathcal{R}_\infty]^\perp$ so that $y(t) \in [\mathcal{R}_\infty]^\perp$ for every t. We compute the Laplace transform of $y(t)$. This takes values in the finite dimensional subspace $[\mathcal{R}_\infty]^\perp$ i.e.

$$\left[\lambda I - \hat{b}(\lambda)A\right]^{-1}\xi_0 \in [\mathcal{R}_\infty]^\perp .$$

Our assumption is that $\hat{b}(0) > 0$. We compute with $\lambda = 0$ and we see that $A^{-1}\xi_0 \in [\mathcal{R}_\infty]^\perp$. ∎

The fact that $[\mathcal{R}_\infty]^\perp$ is a *finite dimensional* invariant subspace of A^{-1} shows the existence of an eigenvector of A^{-1} in $[\mathcal{R}_\infty]^\perp$:

$$A^{-1}\xi_0 = \mu\xi_0\,, \quad \mu \neq 0\,, \qquad \text{i.e.} \quad A\xi_0 = \lambda\xi_0\,.$$

And, $\xi_0 \in [\mathcal{R}_\infty]^\perp$. Moreover,

$$\xi_0 \in \operatorname{dom} A \qquad \text{i.e.} \qquad \xi_0 \in H^2(\Omega) \cap H^1_0(\Omega)\,.$$

Let now ϕ solve the scalar integrodifferential equation

$$\phi'(t) = \lambda \int_0^t b(t-s)\phi(s)\,\mathrm{d}s\,, \qquad \phi(0) = 1\,.$$

Note that $\phi(t)$ is continuous not identically zero. Then, $y(t) = \phi(t)\xi_0$ solves (38) with an initial condition which belongs to $[\mathcal{R}_\infty]^\perp$. Hence,

$$0 = \gamma_1\left(\left[\int_0^t b(t-s)\phi(s)\,\mathrm{d}s\right]\xi_0\right) = \left(\int_0^t b(t-s)\phi(s)\,\mathrm{d}s\right)\gamma_1\xi_0\,.$$

Using that ϕ is not identically zero we see that its convolution with b is not identically zero so that $\xi_0 = \xi_0(x)$ solves

$$\Delta\xi_0 = \lambda\xi_0\,, \qquad \xi_{0|_\Gamma} = 0\,, \qquad \frac{\partial}{\partial n}\xi_{0|_\Gamma} = 0\,.$$

This implies that $\xi_0 = 0$, see [23, Cap. I, Cor. 5.1]; i.e., $[\mathcal{R}_\infty]^\perp = 0$ and $\mathcal{R}_\infty = L^2(\Omega)$, as wanted.

Remark 22 We note:

- In the previous results we assumed $\theta_0 = 0$. In fact, the addition of a nonzero initial condition results in an additive constant added to $\theta(T)$. Hence it is also true that the reachable set from every $\theta_0 \in L^2(\Omega)$ is $L^2(\Omega)$. Analogous observation if $f \neq 0$. The case $f \neq 0$ is important since f contains information on the evolution of the temperature θ before the initial time $t = 0$.

- We noted that the positivity condition $b(0) > 0$ is needed in order to have a hyperbolic type behavior. Instead, the positivity condition $\hat{b}(0) > 0$ is more subtle. Once we have reachability at a given T, we can arbitrarily change $b(t)$ for $t > T$ and this does not affect the reachable set. Hence, we could change b so to "destroy" the condition $\hat{b}(0) > 0$. However, this can be done once the reachable time T is known. It seems that in order to have controllability in a still unspecified time T a global positivity condition must be imposed, see for example [3]. ∎

5 Conclusion

In this paper we gave an alternative definition of the solution of the Gurtin-Pipkin equation with $L^2(0,T;L^2(\Gamma))$ Dirichlet boundary condition. Of course, previous paper already defined the solutions, see for example [26]. In contrast with the previous papers, the definitions presented here follows the ideas in [4] and it shows a formula for the solution which immediately suggest that the controllability of Eq. (1) could be a consequence of the known controllability properties of the wave equation. Using this observation, controllability was proved for the Gurtin-Pipkin equation on multidimensional domains: controllability holds on those domains over which the wave equation is controllable.

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