

TRACKING CONTROL OF PARABOLIC SYSTEMS

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Abstract We consider the tracking problem for parabolic systems with boundary control. Assuming that the reference signal is bounded and measurable, we prove various regularity results as well representation formulas for the optimal control and the optimal trajectory.

Keywords: Regulator problem, distributed systems, boundary control

1. Introduction and preliminaries

The quadratic regulator problem for distributed parameter systems was analyzed in the monographs [1], [2] and [6]; in particular the (time and space) regularity properties of the Riccati equations, arising in boundary control of PDEs, is deeply studied in [6]. However variants of the quadratic regulator problem like the tracking and cheap control did not receive much attention in the boundary control case.

The aim of this paper is to partially fill this gap, by investigating the tracking problem. This consists in finding a control v to force the output z of a given system to follow a desired reference signal y ; we refer to [4, 9] for an introduction to this problem in finite dimensions. We obtain regularity results for the optimal control v as well as useful representation formulas involving v and the optimal trajectory w , see in

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particular Theorems 8, 10 and 13. These are extensions of known finite dimensional formulas (see [4]) to the present boundary control case.

Since the presence of the reference signal y entails a lack of regularity in the solution and a different form of the optimal control, our theorems are not contained in the monograph [6]. Moreover the results which are presented here will be applied to the study of the cheap control problem in a forthcoming paper.

The system that we consider is described by

$$\dot{w}(t) = Aw(t) + Bv(t), \quad t \in (0, T), \quad w(0) = w_0, \quad (1)$$

where A generates an exponentially stable holomorphic semigroup on a Hilbert space X and $v \in L^2(0, T; X)$. Exponential stability is assumed only in order to simplify the notations. The operator B takes values in $(\text{dom}A^*)'$. Here A^* denotes the adjoint of A and $(\text{dom}A^*)'$ stands for the topological dual of $\text{dom}A^*$ (the space $\text{dom}A^*$ is endowed with the graph norm); see [6] for more details as well as for several applications of (1). We assume that for some $\gamma \in [0, 1)$,

$$D = (-A)^{-\gamma}B \in \mathcal{L}(U, X), \quad (2)$$

i.e. D is a bounded linear operator from U to X , where U is a second Hilbert space (if A is not stable then the notation $(-A)^\gamma$ is to be replaced by $(-A - rI)^\gamma$ with r large enough). Note that (2) is equivalent to **requiring** that $B \in \mathcal{L}(U, [\text{dom}(-A^*)^\gamma]')$ and implies that, for some σ and $M > 0$,

$$\|B^*e^{A^*t}\| \leq \frac{Me^{-\sigma t}}{t^\gamma}, \quad t > 0. \quad (3)$$

The tracking problem is the following:

$$\begin{aligned} \min_v J_\alpha(w_0; v), \quad J_\alpha(w_0; v) &= \int_0^T \left\{ \|z(t) - y(t)\|^2 + \alpha \|v(t)\|^2 \right\} dt, \\ z(t) &= z(t; w_0, v) = Cw(t; w_0, v), \end{aligned} \quad (4)$$

where $\alpha > 0$ is fixed, y is a prescribed reference signal and $w(t; w_0, v)$ denotes the solution to (1); further C is a linear and bounded operator from X to a third Hilbert space Y . The cheap control problem consisting in studying the limit for $\alpha \rightarrow 0^+$ will be studied in the **sequel** (it is for this future use that we keep the penalization α ; **however since in this paper α is fixed, we could take $\alpha = 1$ without restriction**). The standing assumption on y is that it is measurable and bounded, i.e.

$$y \in L^\infty(0, T; Y).$$

We note that we can reformulate the problem in an equivalent way: we add the term $Ce^{At}w_0$ to y and then consider system (1) with null initial condition. We prefer not to follow this way, so to stress those results which require stronger regularity assumptions of y .

Let us introduce the following operators

$$Lv(t) = \int_0^t e^{A(t-s)}Bv(s)ds, \quad \Lambda = CL$$

$$L^*w(t) = B^* \int_t^T e^{A^*(s-t)}w(s)ds, \quad \Lambda^* = L^*C^*; \quad \Gamma w_0(t) = Ce^{At}w_0,$$

$w_0 \in X$. The properties of these operators have been precisely studied in [6]. For further use we recall from [6, p. 13 and p. 23]:

Theorem 1 *Let $\gamma \in [0, 1)$ as in (2). We have:*

$$L \in \mathcal{L}(L^2(0, T; U), L^2(0, T; \text{dom}(-A)^{1-\gamma}))$$

and

$$L \in \mathcal{L}(L^\infty(0, T; U), C([0, T]; \text{dom}(-A)^\theta)), \quad \text{for every } 0 \leq \theta < 1 - \gamma.$$

We observe that we are in the second smoothing case studied in [6] so that we can freely use all the regularity results in that book, which concern the solutions of the Riccati equation and the operators L and Λ . Using a result in [7], we can improve Lemma 1 as follows (as usual, C^σ denotes the space of Hölder continuous functions):

Theorem 2 *For any $\theta \in [0, 1 - \gamma)$, we have:*

$$L \in \mathcal{L}(L^\infty(0, T; U), C^{1-\gamma-\theta}([0, T]; \text{dom}(-A)^\theta)).$$

In particular $L \in \mathcal{L}(L^\infty(0, T; U), C^{1-\gamma}([0, T]; X))$.

Proof We write:

$$Lv(t) = \int_0^t e^{A(t-s)}Bv(s)ds = (-A)^\gamma \int_0^t e^{A(t-s)}Dv(s)ds,$$

see (2). Now recall that in Proposition 4.2.2 in [7], see also Sec. 2.2.2 in [7], it is proved that the operator R ,

$$Rf(t) = \int_0^t e^{A(t-s)}f(s)ds$$

belongs to $\mathcal{L}(L^\infty(0, T; X), C^{1-\alpha}([0, T]; \text{dom}(-A)^\alpha))$, for any $\alpha \in (0, 1)$. Remark that this implies also that $R \in \mathcal{L}(L^\infty(0, T; X), C^{1-\alpha}([0, T]; X))$, for any $\alpha \in (0, 1)$. Applying this result to L we get the assertion.

2. The tracking problem

Let us consider (4). The existence of the optimal control v_α is clear,

$$v_\alpha = (\alpha + \Lambda^* \Lambda)^{-1} \Lambda^* (y - \Gamma w_0). \quad (5)$$

Let w_α be the state produced by v_α , i.e. the solution of Eq. (1) when $v = v_\alpha$. Let moreover $z_\alpha = Cw_\alpha = \Lambda v_\alpha + \Gamma w_0$ be the corresponding output. We easily obtain from (5) a second representation formula for the optimal control:

$$v_\alpha = \frac{1}{\alpha} \Lambda^* [y - z_\alpha]. \quad (6)$$

Using Young inequalities, we see that $\Lambda^*(y - \Gamma w_0)$ is continuous on $[0, T]$ and $\|\Lambda^*(y - \Gamma w_0)(t)\| \leq M(T - t)^{1-\gamma}$. Moreover from [6, Theorem 1.4.4.4] $(\alpha + \Lambda^* \Lambda)^{-1}$ is boundedly invertible on $C_\gamma(0, T; U)$, where

$$C_\gamma(0, T; U) = \left\{ f : [0, T] \rightarrow U \text{ continuous on } [0, T] \text{ such that} \right. \\ \left. \sup_{t \in (0, T)} (T - t)^\gamma |f(t)| < +\infty \right\}.$$

Hence we have that v_α is continuous on $[0, T]$. In addition the usual bootstrap argument shows:

Theorem 3 *The optimal control v_α is continuous on $[0, T]$. Hence, also $w_\alpha(t)$ and $z_\alpha(t)$ are continuous too. Moreover, $\|v_\alpha(t)\| = \mathcal{O}(T - t)^{1-\gamma}$.*

The proof, based on Young inequalities, is sketched in the appendix.

Combining Lemmas 2 and 3 we obtain:

Theorem 4 *The function w_α is Hölder continuous on every compact interval contained in $(0, T]$ with values in X . The Hölder exponent is $1 - \gamma$.*

Proof From

$$w_\alpha(t) = e^{At} w_0 + \int_0^t e^{A(t-s)} B v_\alpha(s) ds,$$

we need only to prove Hölder continuity of the integral (since the first addendum is continuously differentiable for $t > 0$, because e^{At} is a holomorphic semigroup). To this end it is enough to apply Lemma 2.

In the next result, we give two representations of the minimum value of the cost.

Theorem 5 *We have:*

$$J_\alpha(w_0; v_\alpha) = \langle y - \Gamma w_0, y - z_\alpha \rangle = \langle y - \Gamma w_0, \alpha(\alpha + \Lambda \Lambda^*)^{-1} (y - \Gamma w_0) \rangle.$$

Proof In the following computation, $\tilde{y} = y - \Gamma w_0$ and norm and inner product are in L^2 . We note that

$$\begin{aligned} \|\Lambda v_\alpha - \tilde{y}\|^2 + \alpha \|v_\alpha\|^2 &= \langle \Lambda v_\alpha, \Lambda v_\alpha \rangle - 2\langle \Lambda v_\alpha, \tilde{y} \rangle + \|\tilde{y}\|^2 + \alpha \|v_\alpha\|^2 \\ &= \langle v_\alpha, \Lambda^* \Lambda (\alpha + \Lambda^* \Lambda)^{-1} \Lambda^* \tilde{y} \rangle - 2\langle v_\alpha, \Lambda^* \tilde{y} \rangle + \|\tilde{y}\|^2 + \alpha \|v_\alpha\|^2 \\ &= \langle v_\alpha, \Lambda^* \tilde{y} - \alpha (\alpha + \Lambda^* \Lambda)^{-1} \Lambda^* \tilde{y} \rangle - 2\langle v_\alpha, \Lambda^* \tilde{y} \rangle + \|\tilde{y}\|^2 + \alpha \|v_\alpha\|^2 \\ &= -\langle v_\alpha, \Lambda^* \tilde{y} \rangle + \|\tilde{y}\|^2 = \langle \tilde{y}, \tilde{y} - \Lambda v_\alpha \rangle \end{aligned}$$

since

$$\tilde{y} - \Lambda v_\alpha = y - z_\alpha.$$

This is the first representation. The second representation is obtained from here, since

$$\Lambda(\alpha + \Lambda^* \Lambda)^{-1} \Lambda^* = (\alpha + \Lambda \Lambda^*)^{-1} \Lambda \Lambda^*.$$

The proof is complete.

We introduce now the explicit form of (6):

$$v_\alpha(t) = \frac{1}{\alpha} \int_t^T B^* e^{A^*(s-t)} C^* [y(s) - z_\alpha(s)] ds = -\frac{1}{\alpha} B^* p_\alpha(t) \quad (7)$$

where

$$p_\alpha(t) = - \int_t^T e^{A^*(s-t)} C^* [y(s) - z_\alpha(s)] ds. \quad (8)$$

We note that $p_\alpha \in C([0, T]; X)$, since C is a bounded operator and

$$\|p(t)\| \leq M(T-t)^{(1-\gamma)}.$$

For $t = 0$ we have the equality

$$p_\alpha(0) = - \int_0^T e^{A^*s} C^* [y(s) - z_\alpha(s)] ds = -\Gamma^*(y - z_\alpha)$$

so that we find a third representation for the optimal cost,

$$J_\alpha(w_0; v_\alpha) = \langle y, y - z_\alpha \rangle_{L^2(0, T; Y)} + \langle w_0, p_\alpha(0) \rangle_X.$$

The function p_α is the weak solution of

$$\dot{p} = -A^* p - C^* [C w_\alpha - y], \quad p(T) = 0. \quad (9)$$

We have the condition $p(T) = 0$ since the final value of w is not penalized. In this way we arrive at the usual hamiltonian system

$$\begin{cases} \dot{w} &= Aw - \frac{1}{\alpha} BB^* p & w(0) &= w_0, \\ \dot{p} &= -C^* C w - A^* p + C^* y & p(T) &= 0. \end{cases} \quad (10)$$

The functions p_α and w_α solve (10) in a weak sense. We improve the regularity of p_α in the next Lemma.

Theorem 6 *We have $p_\alpha \in C^{1-\theta}([0, T]; \text{dom}(-A^*)^\theta)$, for every $0 < \theta < 1$, and $p_\alpha \in L^2(0, T; \text{dom } A^*)$.*

Proof The first assertion follows from (8), taking into account that the function $C^*[Cw_\alpha - y]$ is bounded and applying Proposition 4.2.2 in [7]. The second statement can be proved as in [5], see also [6, p. 4].

Let us consider now the special but important case when y is Hölder continuous. Using [7, Theorem 4.3.4] and [8, Theorem 3.5.], we get:

Theorem 7 *If there exists $\eta \in (0, 1)$, $\eta \leq 1 - \gamma$, such that $y \in C^\eta([\epsilon, T]; Y)$ for every $\epsilon > 0$, then p_α is continuously differentiable, and the derivative is Hölder continuous too, i.e., $p_\alpha \in C^{1+\eta}([\epsilon, T], X)$, for every $\epsilon > 0$.*

In the next result we consider the regularity of the map $B^*p_\alpha = -\alpha v_\alpha$.

Theorem 8 *Let $\eta \in (0, 1)$ and $y \in C^\eta([\epsilon, T]; Y)$, for any $\epsilon > 0$. The functions B^*p_α (and so also v_α) is Hölder continuous on compact intervals contained in $(0, T]$, of exponent $\min\{\eta, 1 - \gamma\}$.*

Proof Let $f = -C^*[Cw_\alpha - y]$. The function f is bounded. Now fix $\epsilon > 0$ and take $t'' > t' \geq \epsilon$. We obtain

$$[p_\alpha(t'') - p_\alpha(t')] = \int_0^{T-t''} e^{A^*s} f(s+t'') ds - \int_0^{T-t'} e^{A^*s} f(s+t') ds.$$

Hence,

$$\begin{aligned} [B^*p(t'') - B^*p(t')] &= B^* \int_{T-t''}^{T-t'} e^{A^*s} f(s+t') ds \\ &+ B^* \int_0^{T-t''} e^{A^*s} [f(s+t'') - f(s+t')] ds. \end{aligned}$$

The function f is bounded so that the first addendum is less than

$$\int_{T-t''}^{T-t'} \frac{M_0}{s^\gamma} ds \leq M_1 \left\{ (T-t')^{1-\gamma} - (T-t'')^{1-\gamma} \right\} \leq M(t'' - t')^{(1-\gamma)}.$$

The second integral is the sum of the following two terms:

$$B^* \int_0^{T-t''} e^{A^*s} C^* [y(s+t') - y(s+t'')] ds, \quad (11)$$

$$B^* \int_0^{T-t''} e^{A^*s} C^* C [w_\alpha(s+t'') - w_\alpha(s+t')] ds. \quad (12)$$

Hölder continuity of y and condition (3) imply that the norm of (11) is less than

$$M_\epsilon [t'' - t']^\eta.$$

The second integral is treated analogously, and we get a similar estimate, with exponent $1 - \gamma$ on every interval $[\epsilon, T]$, $\epsilon > 0$, see Lemma 4.

A further regularity result that is needed below is as follows:

Theorem 9 *Let $x_0 \in \text{dom } A^*$. The function:*

$$t \longrightarrow \langle x_0, \int_0^t e^{A(t-s)} B v_\alpha(s) ds \rangle = \langle (-A^*)^\gamma x_0, \int_0^t e^{A(t-s)} D v_\alpha(s) ds \rangle,$$

is differentiable on $[0, T]$ and moreover

$$\begin{aligned} & \frac{d}{dt} \langle x_0, \int_0^t e^{A(t-s)} B v_\alpha(s) ds \rangle \\ &= \langle (-A^*)^\gamma x_0, D v_\alpha(t) \rangle + \langle A^* x_0, \int_0^t e^{A(t-r)} B v_\alpha(r) dr \rangle. \end{aligned}$$

Proof First recall that the function $t \rightarrow D v_\alpha(t)$ is continuous on $[0, T]$. Then let $\delta > 0$; because the semigroup is holomorphic, we have

$$\begin{aligned} & \frac{d}{dt} \langle x_0, \int_0^{t-\delta} e^{A(t-s)} B v_\alpha(s) ds \rangle = \frac{d}{dt} \langle (-A^*)^\gamma x_0, \int_0^{t-\delta} e^{A(t-s)} D v_\alpha(s) ds \rangle \\ &= \langle (-A^*)^\gamma x_0, e^{A\delta} D v_\alpha(t - \delta) \rangle + \langle (-A^*)^\gamma x_0, A \int_0^{t-\delta} e^{A(t-s)} D v_\alpha(s) ds \rangle \\ &= \langle (-A^*)^\gamma x_0, e^{A\delta} D v_\alpha(t - \delta) \rangle + \langle A^* x_0, (-A)^\gamma \int_0^{t-\delta} e^{A(t-s)} D v_\alpha(s) ds \rangle. \end{aligned}$$

We see from here that

$$\left\| \frac{d}{dt} \langle x_0, \int_0^{t-\delta} e^{A(t-s)} B v_\alpha(s) ds \rangle \right\| \leq M \|v_\alpha\|_\infty \{ \|(-A^*)^\gamma x_0\| + \|A^* x_0\| \}.$$

Thanks to this estimate, we can use dominated convergence theorem and we can pass to the limit for $\delta \rightarrow 0^+$ in the following equality:

$$\begin{aligned} & \langle (-A^*)^\gamma x_0, \int_0^{t-\delta} e^{A(t-s)} D v_\alpha(s) ds \rangle = \int_\delta^{t-\delta} \left\{ \langle (-A^*)^\gamma x_0, e^{A\delta} D v_\alpha(s - \delta) \rangle \right. \\ & \left. + \langle A^* x_0, \int_0^{s-\delta} e^{A(s-r)} B v_\alpha(r) dr \rangle \right\} ds. \end{aligned}$$

We differentiate both sides of the resulting equality and we get:

$$\begin{aligned} & \frac{d}{dt} \langle x_0, \int_0^t e^{A(t-s)} B v_\alpha(s) ds \rangle \\ &= \langle (-A^*)^\gamma x_0, Dv_\alpha(t) \rangle + \langle A^* x_0, \int_0^t e^{A(t-r)} B v_\alpha(r) dr \rangle. \end{aligned}$$

The proof is complete.

If $y = 0$ it is well known that the optimal control can be put in feedback form. This is not possible if $y \neq 0$ since at a given time t the future values of y , which affect the optimal control, are unknown. But, if $\dim X < +\infty$, a special representation formula for p_α is known, which precisely shows the effect of noncausality. We extend this result:

Theorem 10 *We have*

$$p_\alpha(t) = P_\alpha(t)w_\alpha(t) + d_\alpha(t)$$

where P_α solves the Riccati differential equation

$$\begin{aligned} \langle \frac{d}{dt} P_\alpha(t)x, y \rangle &= -\langle P_\alpha(t)Ax, y \rangle - \langle P_\alpha(t)x, Ay \rangle \\ &- \langle Cx, Cy \rangle + \frac{1}{\alpha} \langle B^* P_\alpha(t)x, B^* P_\alpha(t)y \rangle, \quad P_\alpha(T) = 0. \end{aligned} \quad (13)$$

Here x and y are arbitrary elements in $\text{dom } A$. The function d_α is continuous on $[0, T]$ and is zero for $t = T$. It depends on y but not on w_0 and it is given by

$$\begin{aligned} d_\alpha(t) &= - \int_t^T e^{A^*(s-t)} C^* y(s) ds \\ &+ \int_t^T e^{A^*(s-t)} C^* C \int_t^s e^{A(s-r)} B [(\alpha + \Lambda_t \Lambda_t^*)^{-1} \Lambda_t^* y](r) dr ds, \\ &\text{where } (\Lambda_t u)(r) = \int_t^r C e^{A(r-s)} B u(s) ds. \end{aligned}$$

Proof The operator valued function $P_\alpha(t)$ is the solution of the usual Riccati equation. For each $x \in X$, $P_\alpha(\cdot)x$ is continuous on $[0, T]$ and it is zero for $t = T$. Moreover, it is differentiable on $(0, T)$, with continuous and bounded derivative in closed subintervals, see Theorem [6, p. 19-20]. In order to prove the theorem, it is sufficient to show that the continuous function

$$d_\alpha(t) = p_\alpha(t) - P_\alpha(t)w_\alpha(t)$$

only depends on the tracking signal y . In particular, we want to show that it does not depend on w_0 . We introduce the functions $v^+(s; t, x_0)$

and $w^+(s; t, x_0)$, $x_0 \in X$, the solutions of the optimization problem under study, in the case that $y = 0$ and with initial condition x_0 at time t instead then 0. Hence,

$$v^+ = -(\alpha + \Lambda_t^* \Lambda_t)^{-1} \Lambda_t^* \Gamma_t x_0, \quad \text{where } \Gamma_t x_0(s) = C e^{(s-t)A} x_0,$$

(note that v^+ depends on α). Moreover, we use the following representation formula for $P_\alpha(t)$:

$$P_\alpha(t)x = \int_t^T e^{A^*(r-t)} C^* C w^+(r; t, x) dr.$$

Hence,

$$\begin{aligned} p_\alpha(t) - P_\alpha(t)w_\alpha(t) &= - \int_t^T e^{A^*(s-t)} C^* y(s) ds \\ &+ \int_t^T e^{A^*(s-t)} C^* C [w_\alpha(s; 0, w_0) - w^+(s; t, w_\alpha(t; 0, w_0))] ds. \end{aligned}$$

For clarity, in this formula we indicated explicitly the initial time and initial value of w_α . Now we use dynamic programming: the optimal control on $[t, T]$, with initial condition $w_\alpha(t; 0, w_0)$ is the restriction to $[t, T]$ of v_α . This holds for every given reference signal y . Hence,

$$\begin{aligned} &w_\alpha(s; 0, w_0) - w^+(s; t, w_\alpha(t; 0, w_0)) \\ &= w_\alpha(s; t, w_\alpha(t; 0, w_0)) - w^+(s; t, w_\alpha(t; 0, w_0)) \\ &= \left(e^{A(s-t)} [w_\alpha(t; 0, w_0) - w_\alpha(t; 0, w_0)] \right) + \int_t^s e^{A(s-r)} B [v_\alpha(r) - v^+(r)] dr \\ &= \int_t^s e^{A(s-r)} B \left\{ (\alpha + \Lambda_t^* \Lambda_t)^{-1} [\Lambda_t^* (y - \Gamma_t w_0) + \Lambda_t^* (\Gamma_t w_0)] \right\} (r) dr \\ &= \int_t^s e^{A(s-r)} B \left[(\alpha + \Lambda_t^* \Lambda_t)^{-1} \Lambda_t^* y \right] (r) dr, \end{aligned}$$

as wanted.

In fact, in finite dimensions d_α solves

$$\dot{d}_\alpha(t) = - \left(A^* - \frac{1}{\alpha} P_\alpha(t) B B^* \right) d_\alpha(t) + C^* y(t), \quad d_\alpha(T) = 0. \quad (14)$$

We will show that an analogous result holds in general. We prove first:

Theorem 11 *We have, for every $x \in \text{dom} A$,*

$$\begin{aligned} &\frac{d}{dt} \langle x, d_\alpha(t) \rangle \\ &= \langle -Ax, d_\alpha(t) \rangle + \frac{1}{\alpha} \langle B^* P_\alpha(t)x, B^* d_\alpha(t) \rangle + \langle x, C^* y(t) \rangle, \quad d_\alpha(T) = 0. \end{aligned} \quad (15)$$

Proof We already know the continuity of d_α and that $d_\alpha(T) = 0$. Moreover, from [6, p. 21, formula 1.2.2.19], $B^*P_\alpha \in \mathcal{L}(X; C([0, T]; U))$.

Recall that $d_\alpha(t) = p_\alpha(t) - P_\alpha(t)w_\alpha(t)$. First let us treat p_α . Using the Hamilton equation (10), we see that

$$\frac{d}{dt}\langle x, p_\alpha(t) \rangle = -\langle C^*Cx, w_\alpha(t) \rangle - \langle Ax, p_\alpha(t) \rangle + \langle Cx, y(t) \rangle. \quad (16)$$

Now we consider differentiability of $P_\alpha(t)$. We use formula [6, (1.2.2.14)] and Lemma 9 in order to compute

$$\frac{d}{dt}\langle x, P_\alpha(t)w_\alpha(t) \rangle = \frac{d}{dr}\langle x, P_\alpha(r)w_\alpha(t) \rangle|_{r=t} + \frac{d}{dr}\langle x, P_\alpha(t)w_\alpha(r) \rangle|_{r=t}.$$

The first addendum is

$$\langle -A^*P_\alpha(t)x - P_\alpha(t)Ax - C^*Cx + \frac{1}{\alpha}P_\alpha(t)BB^*P_\alpha(t)x, w_\alpha(t) \rangle. \quad (17)$$

The second addendum is computed from Lemma 9 (recall that $P_\alpha(t)x \in \text{dom } A^*$ since $x \in \text{dom } A$, see [6, property vii], p. 20]). We get

$$\begin{aligned} & \langle (-A^*)^\gamma P_\alpha(t)x, Dv_\alpha(t) \rangle + \langle A^*P_\alpha(t)x, w_\alpha(t) \rangle \\ &= -\frac{1}{\alpha}\langle B^*P_\alpha(t)x, B^*p_\alpha(t) \rangle + \langle A^*P_\alpha(t)x, w_\alpha(t) \rangle. \end{aligned} \quad (18)$$

Now we subtract (17) and (18) from (16). The result follows.

Now we improve our information on the regularity of $d_\alpha(t)$:

Theorem 12 *We have $d_\alpha \in C^{1-\theta}([0, T]; \text{dom}(-A^*)^\theta)$, for every $0 < \theta < 1$, and $d_\alpha \in L^2(0, T; \text{dom } A^*)$.*

Proof We derive an integral representation formula for d_α , which displays the desired regularity properties. For any $s \in [0, T]$, set $[P_\alpha(s)B] = [B^*P_\alpha(s)]^*$. From [6, p. 21] it follows that $P_\alpha B$ is linear and continuous from U to $C([0, T]; X)$. Moreover,

$$B^*d_\alpha(s) = B^*p_\alpha(s) - B^*P_\alpha(s)w_\alpha(s)$$

is well defined and continuous, the first addendum from Lemma 3 and the second one from the continuity of $w_\alpha(s)$ and of $B^*P_\alpha(s)$. For the same reason, $s \rightarrow [P_\alpha(s)B]B^*d_\alpha(s)$ is continuous so that, from (15), the following representation formula holds:

$$\begin{aligned} d_\alpha(t) &= -\int_t^T e^{A^*(s-t)} \left\{ \frac{1}{\alpha}[P_\alpha(s)B]B^*d_\alpha(s) + C^*y(s) \right\} ds \\ &= \int_t^T e^{A^*(s-t)} f(s) ds, \end{aligned}$$

where $f(s)$ is bounded on $[0, T]$ with values in X . Now we conclude as in Lemma 6.

We are going to prove a variation of constants formula for $d_\alpha(t)$. Namely, we want to prove

Theorem 13 *The function $d_\alpha(t)$ is given by*

$$d_\alpha(t) = - \int_t^T U(T-t, T-s) C^* y(s) ds$$

where $U(t, s)$ is an evolution operator which is exponentially bounded, strongly continuous and which transforms X into $\text{dom} B^* = \text{dom}(-A^*)^\gamma$, for a.e. $t > s$.

In order to reduce the notation to a more usual form, it is convenient to replace $\xi(t) = d_\alpha(T-t)$. A simple transformation shows that $\xi(t)$ solves

$$\xi(t) = \int_0^t e^{A^*(t-r)} \{ [\tilde{P}(r)B] B^* \xi(r) - C^* \tilde{y}(r) \} dr$$

where $\tilde{P}(r) = -\frac{1}{\alpha} P_\alpha(T-r)$, $\tilde{y}(r) = y(T-r)$. To prove the previous theorem we need the next result.

Theorem 14 *There exists a unique strongly continuous and exponentially bounded evolution family $U(t, s)$ which is defined by*

$$U(t, s)x = e^{A^*(t-s)}x + \int_s^t U(t, r) [\tilde{P}(r)B] B^* e^{A^*(r-s)} x dr, \quad t \geq s, s \in \mathbf{R},$$

for every $x \in X$. Moreover, for a.e. $t > s$, we have $U(t, s)X \subseteq \text{dom} B^*$ and $[\tilde{P}(\cdot)B] B^* U(\cdot, s)x$ is locally integrable on $[s, +\infty)$, for every $x \in X$. The evolution family $U(t, s)$ verifies

$$U(t, s)x = e^{A^*(t-s)}x + \int_s^t e^{A^*(t-r)} [\tilde{P}(r)B] B^* U(r, s)x dr, \quad (19)$$

for $t \geq s$ and $x \in X$.

Proof We use Theorem 9.19, p. 487, in [3]. Let $U_0(t, s) = e^{A^*(t-s)}$ and let $\mathcal{B}(t) = [\tilde{P}(t)B] B^*$. Then, $\text{dom} \mathcal{B}(t) = \text{dom} B^*$ is constant and we have $\mathcal{B}(\cdot)U_0(\cdot, s)$ strongly continuous for $t > s$, with

$$\|\mathcal{B}(t)U_0(t-s)\| = \|[\tilde{P}(t)B] B^* e^{A^*(t-s)}\| \leq \frac{M}{(t-s)^\gamma},$$

locally integrable, from (3). The conclusion follows from this.

Proof of Theorem 13. We introduce

$$\tilde{\xi}(t) = - \int_0^t U(t, s) C^* \tilde{y}(s) ds.$$

We are going to prove that $\xi(t) = \tilde{\xi}(t)$. We see from (19) that $s \rightarrow B^*U(t, s)x$ is integrable on $[0, t]$, for any $x \in X$. Since B^* is closed, it is straightforward to check, by using suitable Riemann sums, that

$$B^* \tilde{\xi}(t) = - \int_0^t B^* U(t, s) C^* \tilde{y}(s) ds.$$

Moreover, we see from (19)

$$\tilde{\xi}(t) = - \int_0^t e^{A^*(t-s)} C^* \tilde{y}(s) ds + \int_0^t e^{A^*(t-s)} [\tilde{P}(s) B] B^* \tilde{\xi}(r) dr.$$

Hence, $\xi(t)$ and $\tilde{\xi}(t)$ solve the same Volterra integral equation, and

$$[\xi(t) - \tilde{\xi}(t)] = \int_0^t e^{A^*(t-r)} [\tilde{P}(r) B] B^* [\xi(r) - \tilde{\xi}(r)] dr \quad (20)$$

so that also

$$(B^*[\xi(t) - \tilde{\xi}(t)]) = \int_0^t B^* e^{A^*(t-r)} [\tilde{P}(r) B] (B^*[\xi(r) - \tilde{\xi}(r)]) dr.$$

Thanks to the inequality (3), Young inequalities and continuity of $[\tilde{P}(r) B]$, the operator on $L^2(0, T)$ defined by

$$\phi(\cdot) \longrightarrow \int_0^t B^* e^{A^*(t-r)} [\tilde{P}(r) B] \phi(r) dr$$

has norm less than MT^γ for a suitable number M ; so it is a contraction on $L^2(0, T_1)$, $T_1 < (1/M)^{1/\gamma} = T_1$. Hence $B^*[\xi(t) - \tilde{\xi}(t)]$ is zero on $[0, T_1]$ and, for $t > T_1$,

$$(B^*[\xi(t) - \tilde{\xi}(t)]) = \int_{T_1}^t B^* e^{A^*(t-r)} [\tilde{P}(r) B] (B^*[\xi(r) - \tilde{\xi}(r)]) dr.$$

The same argument shows that $(B^*[\xi(t) - \tilde{\xi}(t)])$ is zero on $[T_1, 2T_1]$ too. In fact, it is easily seen that the norm of the operator

$$\phi(\cdot) \longrightarrow \int_{T_1}^t e^{A^*(t-r)} [\tilde{P}(r) B] \phi(r) dr$$

from $L^2(T_1, 2T_1)$ in itself, is less than MT_1^γ , with the same coefficient M as above. After a finite number of steps we see that $B^*[\xi(t) - \tilde{\xi}(t)]$ is zero on $[0, T]$ so that, from (20) we have $\xi(t) = \tilde{\xi}(t)$ on $[0, T]$. This finishes the proof.

Appendix: Proof of Lemma 3

We sketch now the proof that w_α and v_α are continuous functions on $[0, T]$. Recall that

$$w_\alpha(t) = e^{At}w_0 + \int_0^t e^{A(t-s)}Bv_\alpha(s)ds, \quad t \in [0, T],$$

where v_α is the optimal control. We use an usual bootstrap argument, based on Young inequalities. It is sufficient to prove that

$$\int_0^t e^{A(t-s)}Bv_\alpha(s)ds \quad (21)$$

is continuous since then $w_\alpha(t)$ is continuous for $t \in [0, T]$ and

$$v_\alpha(t) = \frac{1}{\alpha} \int_t^T B^* e^{A^*(s-t)} C^* [y(s) - Cw_\alpha(s)] ds$$

is continuous, and even

$$\|v_\alpha(t)\| = O(T-t)^{1-\gamma}. \quad (22)$$

We note that (21) is a convolution with a kernel which is singular at $t = 0$, of the order $1/t^\gamma$, $0 \leq \gamma < 1$, hence of class $L^p(\mathbf{R})$, $p = (1/\gamma) - \epsilon$ for every $\epsilon > 0$. In particular, $p > 1$. Now we recall the Young inequalities, which show that

$$H * k$$

belongs to L^r if $H \in L^p$, $k \in L^q$ and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

is nonnegative. If r so defined turns out to be negative, and $p > 1$, then the convolution is continuous.

We know that $v_\alpha \in L^2$ so that from Young inequalities,

$$w_\alpha \in L^r, \quad r > \frac{2}{2\gamma - 1} - \delta,$$

for every positive δ . Note that if $2\gamma - 1 > 0$ then $r > 2$ (since $\gamma \in [0, 1)$). Otherwise w_α is continuous. If $2\gamma - 1 > 0$ we use again Young inequality to deduce that

$$v_\alpha \in L^\sigma, \quad \sigma = \frac{2}{4\gamma - 3} - \delta$$

for every $\delta > 0$. We iterate, once that we know $v_\alpha \in L^\sigma$ we can deduce $w_\alpha \in L^r$, $v_\alpha \in L^{\sigma'}$ where

$$r > \frac{\sigma}{1 - \sigma(1 - \gamma)} - \delta, \quad \sigma' = \frac{\sigma}{1 - 2\sigma(1 - \gamma)} - \delta,$$

for every positive δ . We recall $\sigma > 2$ and we see that, as long as $1 - 2\sigma(1 - \gamma) > 0$,

$$\sigma' - \sigma = \frac{\sigma}{1 - 2\sigma(1 - \gamma)} - \sigma - \delta > 8(1 - \gamma).$$

Hence, at the k -th iteration we see that $v \in L^{\sigma_k}$ with

$$\sigma_k - \sigma > 8k \tag{23}$$

After that we see that the exponent r for w_α is

$$r_k > \frac{\sigma_k}{1 - \sigma_k(1 - \gamma)} - \delta.$$

Thanks to (23) this exponent becomes negative for k large. Hence w_α is continuous and v_α is continuous too, and satisfies inequality (22).

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