A quadratic regulator problem related to identification problems and singular systems*

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Abstract

In this paper we study a new form of the quadratic regulator problem which is suggested by recent applications to singular systems and to identification problems. The new feature of the quadratic regulator problem under study is the penalization of the values taken by the control at individual instants of time.

1 Introduction

The quadratic regulator problem has a key role in control theory and it has been studied from many different point of view. We study here a new version which is encountered in the study of singular systems and of an inverse problem. The system that we study is described by

$$\dot{x} \;=\; Ax + Bu\,, \qquad y(t) = Gu(t)\,. \tag{1}$$

We note that $y(t)$ depends on $u(t)$.

The operators A, B and G in (1) satisfy:

- the operator A generates a C_0–semigroup on a Hilbert space X;
- the operators B and G are linear and continuous, $B \in \mathcal{L}(U,X)$, $G \in \mathcal{L}(U,Y)$ where U and Y are Hilbert spaces.
- we assume $\ker G^* = 0$. This is not restrictive.

The quadratic cost we consider is non standard. It has the following form:

$$J(x_0;u) = \int_0^T F(x(t)-\xi(t),u(t))\,\mathrm{d}t + \langle \begin{bmatrix} x(\tau_0)-\xi_0 \\ y(\tau_0) \end{bmatrix}, \tilde{M} \begin{bmatrix} x(\tau_0)-\xi_0 \\ y(\tau_0) \end{bmatrix} \rangle \tag{2}$$

$$+\langle \begin{bmatrix} x(T)-\xi_T \\ y(T) \end{bmatrix}, M \begin{bmatrix} x(T)-\xi_T \\ y(T) \end{bmatrix} \rangle\,. \tag{3}$$

Here $0 \le \tau_0 \le T$ and

$$F(x,u) = \langle x, Qx \rangle + ||u||^2 \tag{4}$$

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while ξ, ξ_0, ξ_T are given reference signals. We *do not* assume that ξ_0 and ξ_T are the values of the function ξ at the corresponding points.

The novelty of this problem stems from the intermediate and final penalization of the control. Due to these terms, the quadratic functional is neither continuous nor closed on $L^2(0,T;U)$.

In order to simplify the notations we shall put

$$\begin{aligned}
&\Phi_{\tau_0}(x,u) = \langle \begin{bmatrix} x-\xi_0 \\ Gu \end{bmatrix}, \tilde{M} \begin{bmatrix} x-\xi_0 \\ Gu \end{bmatrix} \rangle, \qquad \Phi_T(x,u) = \langle \begin{bmatrix} x-\xi_T \\ Gu \end{bmatrix}, M \begin{bmatrix} x-\xi_T \\ Gu \end{bmatrix} \rangle \\
&J_{\rm int}(x_0;u) = \int_0^T F(x(t)-\xi(t), u(t))\,{\rm d}t \\
&J_{\rm fin}(x_0;u) = \Phi_{\tau_0}(x,u) + \Phi_T(x,u)
\end{aligned} \tag{5}$$

Standing assumptions: Q and M and $\tilde{M}$ are symmetric nonnegative continuous linear operators. We assume furthermore that

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} \qquad M_{22} > cI > 0\,, \qquad \tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^* & \tilde{M}_{22} \end{bmatrix} \qquad \tilde{M}_{22} > cI > 0\,.$$

This setup is encountered in particular in the study of important classes of singular control systems, as documented in [6]. Moreover, recently in [1, 8] similar quadratic control problems have been studied for the solution of identification (inverse) problems. See sect. 2 for details.

Remark 1 As we said, the novelty of this problem consists in the presence of the penalization of the final and intermediate values of the control. We have been stimulated to study this problem by identification problems and by the theory of singular control systems. These applications require that we study the problem in an infinite dimensional Hilbert space. This kind of problem has rarely been studied even for finite dimensional systems. See [4] for this. The problem with $\tau_0 = 0$ and null matrices $\tilde{M}_{12}$, $\tilde{M}_{22}$ is studied in [10], in the contest of Kalman filtering.

It may seem that $y = Hx + Gu$ gives a more general problem than (1) and (2). However, it is easily seen that this case can be treated as below. This is left to the reader.

Finally, let us consider the general case

$$F(x,u) = F_0(x,u) + \langle u,u \rangle\,, \qquad F_0(x,u) = \langle x, Qx\rangle + 2\Re e\,\langle x, Q_{12}Gu\rangle + \langle Gu, Q_{22}Gu\rangle \ge 0$$

(see sect. 2.2 concerning the application to singular systems). This can be reduced to the form (4) as follows: first we absorb $\langle y, Q_{22}y\rangle = \langle u, G^*Q_{22}Gu\rangle$ into the last term which takes the form $\langle u, (I + G^*Q_{22}G)u\rangle$. A coordinate transformation in U reduce this to be $\langle u,u\rangle$ again. We then apply a feedback $u = -Q_{12}x + v$ in order to absorb the mixed term. This changes the operator A to $A - BQ_{12}$. We assume that these transformations have been already performed. ∎

We are going to study this quadratic regulator problem with $u \in L^2(0,T;U)$. Clearly, the quadratic cost is not even defined for every $u \in L^2(0,T;U)$. So, we introduce a suitable domain over which the quadratic cost makes sense.

Let

$$\mathrm{ess}\lim_{t\to\tau_0} u(t) = l$$

when for every $\epsilon > 0$ there exists $\delta > 0$ such that the following set has zero Lebesgue measure:

$$\{t\,,\ \text{such that } ||u(t)-l|| > \epsilon\,,\ |t-\tau_0| < \delta\}\,.$$

Analogous definition for the left limit at T.

If two functions u and u' belong to the same equivalent class $[u] \in L^2(0,T)$ then the ess lim exists for the first if it exists for the second one, and the limit itself is the same.

We introduce the linear space $\mathcal{U}_{\rm ess}$ of those equivalent classes in $L^2(0,T;U)$ identified by a representative u such that the essential limits for $t \to T-$ and $t \to \tau_0$ exist and we define for $u \in \mathcal{U}_{\rm ess}$

$$u(\tau_0) = \text{ess}\lim_{t\to\tau_0} u(t)\,, \qquad u(T) = \text{ess}\lim_{t\to T-} u(t)\,.$$

The subspace over which we study our quadratic regulator problem is

$$\mathcal{U} = \mathcal{U}_{\rm ess} \bigcap \mathcal{U}_0$$

where $\mathcal{U}_0 \subseteq L^2(0,T;U)$ is

$$\mathcal{U}_0 = \left\{ u \in L^2(0,T;U)\,,\ u(t) = 0 \text{ for } t > T_0 \right\}\,, \qquad T_0 > 0\,.$$

Here T_0 is fixed.

The introduction of the space $\mathcal{U}_0$ is suggested by the identification problem described in Sect. 2 while the definition of the subspace $\mathcal{U}_{\rm ess}$ is suggested in [9, Ch. 1].

If $T_0 > T$ then we intend $\mathcal{U} = \mathcal{U}_{\rm ess}$. If $\tau = T$ then we simply ignore the term Φ_{τ_0}.

In general, the quadratic cost we are studying does not admit an optimal control. The optimal control might exist for special initial conditions x_0. An initial condition which admits an optimal control will be called "optimizable" and $\mathcal{O}$ is the set of all the optimizable initial conditions.

It is convenient to introduce the following additional notations:

- when it exists, the optimal control on $[s,T]$ which corresponds to the initial condition x_0 (at the initial time s) is denoted $u_s^+(\cdot;x_0)$. The corresponding optimal trajectory is $x_s^+(\cdot;x_0)$. The index s is omitted when $s=0$. We shall see uniqueness of the optimal control, so that the notation is unambiguous.

- an *optimal pair* is the pair of an optimal control and the corresponding trajectory, for a given initial condition x_0 at the "initial time" s.

The organization of the paper is as follows: In the next section 2 we present two applications of the problem we are studying and we derive a preliminary characterization of the optimal control. We shall see that the optimal control will only exist for special initial conditions. But, the infimum of the cost is always finite (and nonnegative). The properties of the infimum are studied in sect. 3 where it is proved that the infimum is always a continuous quadratic form. The infimum of the cost as a function of the initial time $s \geq 0$ is studied in sect. 3.2. If in particular $T_0 \geq T = \tau$, it is proved that the infimum of the cost is a continuous function of the initial time. With the application to singular systems in mind, this should be contrasted with the result in [3] where it is proved that when the cost is singular but the system is regular then the infimum of the cost is an upper semicontinuous function of the initial time. In [3] an example is given, which shows that in general it is not continuous.

In sect. 4 we concentrate on the problem which is most important for the applications to singular systems, the problem with $\tau_0 = T < T_0$ and reference signals equal to zero. Also in this case the optimal control exists only for suitable initial conditions. The set of the optimizable initial conditions and the corresponding optimal controls are characterized in sect. 4.1 where we also clarify the relation of the optimal control with the Riccati equation. A noteworthy result is that the set of the optimizable initial conditions is a closed subspace of X which is characterized as the kernel of a certain linear operator.

The arguments of sect. 4 are easily extended to the case $\tau_0 \in [0,T)$.

2 Examples, applications and the optimal control

We present two examples in this section, which justify our study. We then derive the two point characterization of the optimal control which shows that in general the optimal control does not exists. This justifies our analysis of the properties of the infimum of the cost.

2.1 An identification problem

This example is taken from [1]. In applications, delay systems are often used as simple models of more complex distributed parameter systems. Let a signal $\tilde{y}$ be measured and let us assume that we guessed a delay system

$$y' = Ay + By(t-h) \tag{6}$$

which could be used as an approximation of the device which produces $\tilde{y}$. The solution y depends on the initial condition $\phi(\cdot)$, $y(t) = \phi(t)$, $t \in [-h, 0]$ and the approximation is "tuned" by choosing ϕ in the "best possible" way. In [1] the index to be minimized for the choice of ϕ is a quadratic index,

$$\frac{\alpha}{2}\int_{-h}^{0} ||\phi(t) - \tilde{\phi}(t)||^2 \, \mathrm{d}t + \frac{1}{2}\int_0^T ||y(t) - \tilde{y}(t)||^2 \, \mathrm{d}t$$
$$+\frac{\beta}{2}||\phi(0) - \tilde{\phi}(0)||^2 + \frac{\gamma}{2}||y(0) - \tilde{y}(0)||^2 \, .$$

It is known (see [2]) that system (6) can be represented as the semigroup system in (1) in the space $M^2 = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$. We can consider $\phi(\cdot)$ as a "control" by assuming zero initial condition and $u(t) = \phi(t-h)$ for $t \in [0, h]$, $u(t) = 0$ for $t > T_0 = h$. In this way we get the control system

$$Y' = \mathcal{A}Y + \mathcal{B}u$$

with a quadratic cost as in (2) and the admissible controls must be zero for $t > T_0 = h$.

2.2 Application to singular systems

Let $m(x)$ be a continuous nonnegative function defined on a Jordan region Ω bounded by a smooth curve, for example of class C^2 (as we are giving an example, we don't need to use the most general assumptions). The operator A is the laplacian on Ω, with Dirichlet homogeneous conditions. We consider the system

$$\frac{\partial}{\partial t}[m(x)\eta(t,x)] = A\eta + m(x)A^{-1}u \tag{7}$$

and initial conditions

$$\lim_{t\to 0+} m(x)\eta(t,x) = m(x)\eta_0(x) \, , \qquad \eta_0 \in H_0^1(\Omega) \, .$$

The cost functional naturally associated to the problem under study is the cost (2) with $\tau_0 = T$ and reference signals put equal to zero, i.e. a functional of the form

$$\int_0^T \left[||Q\eta(t)||^2 + ||u(t)||^2 \right] \, \mathrm{d}t + ||\eta(T)||^2$$

(see below for the norms used).

Let M be the multiplication operator by $m(x)$. It is proved in [6] that

$$||M(\lambda I - L)^{-1}||_{H^{-1}(\Omega)} \leq \frac{C}{1+|\lambda|}\,, \qquad \Re e\,\lambda \geq 0\,. \tag{8}$$

It is known that $A^{-1} \in \mathcal{L}(H^{-1}(\Omega), H^1_0(\Omega))$. This fact and the previous estimate suggest the choice $U = X = H^{-1}(\Omega)$.

Let now $T = MA^{-1}$ and let us split

$$X = [\ker T] \oplus [\overline{(\mathrm{im} T)}]\,.$$

It is a fact that, thanks to condition (8), the restriction $\tilde{T}$ of T to $[\overline{(\mathrm{im} T)}]$ has an inverse, let it be denoted $\tilde{T}^{-1}$, which generates a holomorphic semigroup. If we denote P the orthogonal projection of X on $\ker T$, it turns out that the system can be split as

$$\begin{cases} x' = \tilde{T}^{-1}x + (I-P)u\,, \\ y(t) = Pu \end{cases}$$

(here $x = (I-P)\eta$, $y = P\eta$). This is a system of the form (1). We introduce the variables x and y in the quadratic cost and we see that the final value $u(T)$ is now penalized (and we see also that mixed terms in x and u appear. These terms can be removed from the integral, see Remark 1).

2.3 The optimal control

An optimality condition is easily obtained, provided that the optimal control exists. We present this condition mainly in order to show that in general the optimal control *does not exist* and this justifies our interest in the properties of the infimum of the cost.

Let the optimal control $u^+(\cdot;x_0)$ exist, for a fixed initial condition x_0. As x_0 is now fixed, we shall simply denote it u^+ and x^+ is the corresponding trajectory. Moreover, we introduce the error $e^+(t) = x^+(t) - \xi(t)$. If $t = \tau_0$ or $t = T$ we introduce $e^+_{\tau_0} = x^+(\tau_0) - \xi_0$, $e^+_T = x^+(T) - \xi_T$. In general, these vectors are not the values of $e^+(t)$ for $t = \tau_0$ or $t = T$.

We compute the Gateaux derivative of the cost at u^+. Standard computations give that the following conditions must hold:

$$\int_0^T \langle u^+(t) + B^*p_0(t) + B^*e^{A(T-s)}[M_{11}e^+_T + M_{12}Gu^+(T)], v(s)\rangle\,\mathrm{d}s$$
$$+ \int_0^{\tau_0} \langle B^*e^{A^*(\tau_0-s)}[\tilde{M}_{11}e^+_{\tau_0} + \tilde{M}_{12}Gu^+(\tau_0)]\,, v(s)\rangle\,\mathrm{d}s = 0 \tag{9}$$

and (10)

$$\begin{cases} \langle G^*\tilde{M}_{22}Gu^+(\tau_0) + G^*\tilde{M}_{12}e^+_{\tau_0}, v(T)\rangle = 0\,, \\ \langle G^*M_{22}Gu^+(T) + G^*M_{12}e^+_T, v(T)\rangle = 0 \end{cases} \tag{11}$$

where

$$p_0(t) = \int_t^T e^{A^*(s-t)}Qe^+(s)\,\mathrm{d}s\,.$$

Here v is *every admissible* input, i.e. every $v \in \mathcal{U}$. From (11) we obtain that the following conditions must hold (here we use $\ker G^* = 0$):

$$\begin{cases} \tilde{M}_{22}Gu^+(\tau_0) = -\tilde{M}^*_{12}e^+_{\tau_0} & \text{if } T_0 > \tau_0 \\ M_{22}Gu^+(T) = -M^*_{12}e^+_T & \text{if } T_0 > T. \end{cases} \tag{12}$$

Condition (9) shows that the optimal control is in general discontinuous at τ_0 and the following relations must hold:

$$u(t) = -B^* p(t) \quad 0 \le t \le T_0\,, \qquad u(t) = 0 \quad t > T_0 \tag{13}$$

where

$$\begin{aligned} & p(t) = p_0(t) + \mathbf{1}(\tau_0 - t) e^{A^*(\tau_0 - t)} \left\{ \tilde{M}_{11} e^+_{\tau_0} + \tilde{M}_{12} G u^+(\tau_0) \right\} \\ & + e^{A^*(T-t)} \left\{ M_{11} e^+_T + M_{12} G u^+(T) \right\} . \end{aligned} \tag{14}$$

The function $\mathbf{1}(t)$ is the Heaviside function, equal to 1 for $t \ge 0$, equal to 0 otherwise (used formally to say that the corresponding term does not appear if the argument of $\mathbf{1}$ is negative. Strictly speaking, in this case this term might be meaningless).

We can use the compatibility conditions (12) in order to replace $Gu^+(\tau_0)$ and $Gu^+(T)$ in this last expression and we find

$$\begin{aligned} & p(t) = p_0(t) + \mathbf{1}(\tau_0 - s) e^{A^*(\tau_0 - t)} \left\{ \tilde{M}_{11} - \tilde{M}_{12} \tilde{M}_{22}^{-1} \tilde{M}_{12}^* \right\} e^+_{\tau_0} \\ & + e^{A^*(T-t)} \left\{ M_{11} - M_{12} M_{22}^{-1} M_{12}^* \right\} e^+_T . \end{aligned}$$

It is easily seen that when $G = 0$ and $T_0 > \tau_0 = T$ these condition are the usual two-point problem for the proposed cost.

It is now possible to see that the optimal control does not exists, even in the simplest cases. Let $X = \mathbb{R}$, $A = 0$, $B = 1$, $T_0 = \tau = T = 1$. Let $M_{22} = 1$, $M_{11} = 1$ $M_{12} = 0$, $Q = 0$. Let the reference signals be zero. The optimality system takes the form

$$\dot{x} = -p\,, \qquad x(0) = 1\,, \qquad \dot{p} = -p\,, \qquad p(1) = x(1)$$

togheter with the "compatibility condition" $p(1) = 0$, which cannot be met since if $p(1) = 0$ then $p(t) \equiv 0$ and $x(t) \equiv 1$, in contrast with the compatibility condition.

The previous relations have been derived as conditions which an optimal control must satisfy. The converse holds:

Theorem 2 *If x_0 is an optimizable initial condition then the optimal control is unique.*

Let the functions x and p solve (1) and (14) when u is given by (13) and let furthermore conditions (12) be satisfied. Then, x_0 is an optimizable initial condition and the control u in (13) is the optimal control.

Proof. We prove uniqueness first. Let u_1 and u_2 be different optimal controls for the same initial condition x_0 and let $v = [u_1 + u_2]/2$. Then, $J_{\rm int}(x_0; v) < J_{\rm int}(x_0; u_i)$ while $J_{\rm fin}(x_0; v) \le J_{\rm fin}(x_0; u_i)$ so that v gives a smaller value of the cost if $u_1 \ne u_2$. This is not possible, so that $u_1 = u_2$.

The second part follows because if we replace u with $u + v$ and the functions x, p and u satisfy the stated condition, we get a quadratic form in v, which has a minimum when $v = 0$. ∎

As we said already, this uniqueness result justifies the notations chosen to denote the unique optimal control.

The infimum of the cost is always non negative and a minimizing sequence can be constructed as follows: We define

$$\tilde{u} = \arg\min \left\{ J_{\rm int}(x_0; u) + \langle M_{11}[x(T) - \xi_T], x(T) - \xi_T \rangle \right\} \tag{15}$$

($\tilde{x}$ is the corresponding solution). Let moreover

$$\hat{u}_{\tau_0} = \arg\min \Phi_{\tau_0}(\tilde{x}(\tau_0), u)\,, \qquad \hat{u}_T = \arg\min \left\{ \Phi_T(\tilde{x}(T), u) - \langle M_{11}[\tilde{x}(T) - \xi_T], \tilde{x}(T) - \xi_T \rangle \right\} . \tag{16}$$

We define now

$$\hat{u}_n(t) = \begin{cases} \tilde{u}(t) & t \notin [\tau_0 - 1/n, \tau_0 + 1/n] \cup [T - 1/n, T] \\ \hat{u}_{\tau_0} & t \in (\tau_0 - 1/n, \tau_0 + 1/n) \\ \hat{u}_T & t \in [T - 1/n, T]\,. \end{cases} \tag{17}$$

It is easy to check that $\{\hat{u}_n\}$ is a minimizing sequence.

The limit of $\{\hat{u}_n\}$ exists in $L^2(0,T;U)$ but in general it is not the optimal control.

A robustness property of this minimizing sequence will be given in sect. 3.1.

3 The infimum of the quadratic cost

We noted that in general the optimal control does not exist and in fact the cost functional as a function of $u(\cdot)$ is well defined on $\mathcal{U}$ but it is neither continuous nor closed in $L^2(0,T)$. We define

$$I(x_0; \xi_0, \xi_T, \xi) = \inf_{u \in \mathcal{U}} J(x_0, u)\,.$$

The most important case in the applications to singular control systems is when the reference signals are put equal to zero. In this case we use the simpler notation $I(x_0)$.

We easily see that the following inequalities hold:

$$\begin{aligned} & J(x_0; u) \geq ||u||_2^2 \\ & \inf_{u \in \mathcal{U}} J(x_0; u) \leq J(x_0; 0) \leq K \left\{ ||x_0||^2 + ||\xi_0||^2 + ||\xi_T||^2 + ||\xi(\cdot)||_2^2 \right\}\,. \end{aligned} \tag{18}$$

Here and below $||\cdot||$ denotes the norm in the spaces X, U or Y while $||\cdot||_2$ will be used to denote either the norm in $L^2(0,T;X)$ (as above) or in $L^2(0,T;U)$. Moreover

Lemma 3 *Let x_0 be fixed and let $\{u_n\} \in \mathcal{U}$ be a minimizing sequence. The sequences $\{Gu_n(\tau_0)\}$ and $\{Gu_n(T)\}$ are bounded.*

Proof. The first inequality (18) shows that $\{u_n\}$ is bounded in $L^2(0,T;U)$ so that the sequences $\{x(\cdot; x_0, u_n)\}$, $\{x(T; x_0, u_n)\}$ are bounded respectively in $C(0,T;X)$ and in X.

We prove boundedness of the second sequence $\{Gu_n(T)\}$. The first sequence is treated analogously.

Let by contradiction $\lim ||Gu_n(T)|| = +\infty$. In this case

$$2\Re e \,\langle x_n(T) - \xi_T, M_{12} G u_n(T) \rangle + \langle G u_n(T), M_{22} G u_n(T) \rangle$$

is unbounded since $\{x_n(T)\}$ is bounded and M_{22} is coercive. Hence, $\{u_n\}$ cannot be a minimizing sequence, a contradiction. ∎

So, every minimizing sequences $\{u_n(\cdot)\}$ is bounded in $L^2(0,T;U)$ and the sequences $\{Gu_n(\tau_0)\}$, $\{Gu_n(T)\}$ are bounded in U. Hence both are weakly compact. However, $\mathcal{U}$ is not closed and we cannot use this observation in order to deduce the existence of the minimum. In spite of this we have the following result which shows the dependence of the infimum on $(x_0, \xi_\tau, \xi_T) \in X^3$ and on $\xi \in L^2(0,T;X)$.

Theorem 4 *The nonnegative functional $I(x_0)$ is continuous and quadratic, i.e. there exists $\mathbf{P} = \mathbf{P}^* \geq 0$, $\mathbf{P} \in \mathcal{L}(X^3 \times L^2(0,T;X))$ such that*

$$I(x_0) = \langle (x_0, \xi_0, \xi_T, \xi), \mathbf{P}(x_0, \xi_0, \xi_T, \xi) \rangle\,.$$

Proof. For simplicity of notation, we give the proof in the most important case $\xi_0 = \xi_T = 0$ and $\xi = 0$. The proof can easily be repeated in general.

In order to prove the theorem we must show that

1. the transformation $x \to I(x)$ is continuous;

2. the parallelogram identity holds for $I(x)$,

see [7, sect. 9.2].

We prove continuity first. By contradiction, let there exists a sequence $\{x_n\}$ such that

$$\lim x_n = x_0 , \qquad \lim I(x_n) \neq I(x_0) .$$

This means that there exists $\epsilon > 0$ such that

$$\text{either} \quad \mathbf{1a)} \quad \limsup I(x_n) > I(x_0) + \epsilon \qquad \text{or} \quad \mathbf{1b)} \quad \liminf I(x_n) < I(x_0) - \epsilon .$$

We consider separately **1a)** and **1b)**.

1a) Let $\tilde{u}$ be such that

$$|I(x_0) - J(x_0; \tilde{u})| < \epsilon$$

so that

$$\limsup I(x_n) > J(x_0; \tilde{u}) + \epsilon/2 .$$

Hence, for a suitable subsequence still denoted $\{x_n\}$, we have for every n

$$I(x_n) > J(x_0; \tilde{u}) + \epsilon/3$$

and for every $u \in \mathcal{U}$ we have

$$J(x_n; u) > J(x_0; \tilde{u}) + \epsilon/3 .$$

In particular when $u = \tilde{u}$,

$$J(x_n; \tilde{u}) > J(x_0; \tilde{u}) + \epsilon/3 .$$

This cannot be since, with $\tilde{u}$ fixed,

$$\lim J(x_n; \tilde{u}) = J(x_0; \tilde{u}) .$$

Hence, case **1a)** is impossible.

If **1b)** holds then we can find $\{u_n\}$ such that

$$J(x_n; u_n) < I(x_0) - \epsilon/2 < J(x_0; u_n) - \epsilon/2 .$$

As already noted, the sequences $\{u_n\}$, $\{Gu_n(\tau_0)\}$ and $\{Gu_n(T)\}$ are bounded. Hence it is not restrictive to assume

$$u_n \rightharpoonup \tilde{u} , \qquad Gu_n(\tau_0) \rightharpoonup \tilde{\eta} , \qquad Gu_n(T) \rightharpoonup \eta .$$

Now we use the representation

$$J(x_0; u) = J_{\rm int}(x_0; u) + J_{\rm fin}(x_0; u) . \tag{19}$$

as in (5). We have

$$J(x_n; u_n) - J(x_0; u_n) < -\epsilon/2$$

so that (with $x_n(t)=x(t;x_0,u_n)$)

$$-\frac{\epsilon}{2} > \Big\{ J_{\rm int}(x_n;u_n) - J_{\rm int}(x_0;u_n) \Big\} \tag{20}$$

$$+\Big\{ \Big[\langle e^{A\tau_0}x_n, \tilde M_{11} e^{A\tau_0} x_n\rangle - \langle e^{A\tau_0}x_0, \tilde M_{11} e^{A\tau_0}x_0\rangle \Big] + 2\langle e^{A\tau_0}(x_n-x_0), \tilde M_{11}\Lambda_{\tau_0}u_n\rangle$$

$$+\, 2\langle e^{A\tau_0}(x_n-x_0), \tilde M_{12} G u_n(\tau_0)\rangle \Big\} \tag{21}$$

$$+\Big\{ \Big[\langle e^{AT}x_n, M_{11} e^{AT} x_n\rangle - \langle e^{AT}x_0, M_{11} e^{AT}x_0\rangle \Big] + 2\langle e^{AT}(x_n-x_0), M_{11}\Lambda_{T}u_n\rangle$$

$$+\, 2\langle e^{AT}(x_n-x_0), M_{12} G u_n(T)\rangle \Big\} \tag{22}$$

where

$$\Lambda_t u = \int_0^t e^{A(t-s)} B u(s)\, {\rm d}s\,.$$

The brace in (22) converges to zero because the sequences $\{u_n\}$ and $\{Gu_n(T)\}$ are bounded and

$$\lim x_n = x_0\,.$$

Analogously it is seen that the brace in (21) converges to zero.

The terms which contain solely u_n in the brace (20) cancel out and, as we noted, $x_n \to x_0$, $u_n(\cdot) \rightharpoonup u_0(\cdot)$ so that the limit of the brace in (20) is zero. This is a contradiction and proves continuity of $I(x)$.

We prove now the parallelogram identity

$$I(x+y) + I(x-y) = 2\left[I(x) + I(y) \right]\,.$$

It is known that parallelogram identity holds for J:

$$J(x+y;u+v) + J(x-y;u-v) = 2\left[J(x;u) + J(y;v) \right]$$

for every x, y in X and u, v in $\mathcal{U}$.

We fix x and y and $\epsilon>0$ and we choose u_x and u_y such that

$$J(x;u_x) < I(x) + \epsilon/2\,, \qquad J(y;u_y) < I(y) + \epsilon/2$$

so that

$$J(x+y;u_x+u_y) + J(x-y;u_x-u_y) = 2J(x;u_x) + 2J(y;u_y) < 2\left[I(x)+I(y) \right] + 2\epsilon\,.$$

This proves the inequality

$$I(x+y) + I(x-y) \le 2\left[I(x) + I(y) \right]\,.$$

We prove that the inequality cannot be strict. We prove that if ϵ satisfy

$$I(x+y) + I(x-y) \le 2\left[I(x) + I(y) \right] - \epsilon \tag{23}$$

then $\epsilon = 0$.

If (23) holds then we can find $\tilde u$ and $\tilde v$ such that

$$J(x+y;\tilde u) + J(x-y;\tilde v) \le 2\left[I(x)+I(y) \right] - \epsilon/2\,.$$

We define

$$u_0 = \frac{\tilde u + \tilde v}{2}\,, \qquad v_0 = \frac{\tilde u - \tilde v}{2}$$

so that

$$2\left[J(x;u_0)+J(y;v_0)\right]=J(x+y;u_0+v_0)+J(x-y;u_0-v_0)\le 2\left[I(x)+I(y)\right]-\epsilon/2\,.$$

However, we also have

$$I(x)+I(y)\le J(x;u_0)+J(y;v_0)\le\left[I(x)+I(y)\right]-\epsilon/2\,.$$

This shows $\epsilon=0$ so that parallelogram identity holds.

The operator $\mathbf{P}$ is now constructed by polarization,

$$4\langle x,\mathbf{P}y\rangle=I(x+y)-I(x-y)\,. \tag{24}$$

The same arguments can be repeated in the general case. ∎

3.1 Robustness of the minimizing sequence

The nonexistence of the optimal control forces us in general to relay on minimizing sequences. In this section we prove that the minimizing sequence constructed in (17) is robust under perturbations, in the sense that we explain below. Let $\epsilon\ge 0$ be a parameter whose nominal value should be 0 and let $\xi^\epsilon_{\tau_0}$, ξ^ϵ_T, ξ^ϵ, τ^ϵ depend continuously on ϵ (respectively in X, $L^2(0,T;X)$ and $[0,T]$), with the nominal values taken for $\epsilon=0$. Let $J^\epsilon(u^\epsilon_n)$, $J^\epsilon_{\text{int}}(u^\epsilon_n)$, $\Phi^\epsilon_{\tau^\epsilon_0}$, Φ^ϵ_T be the functionals which correspond to the value ϵ.

Let $\{u^\epsilon_n\}$ be the minimizing sequence constructed in (17) with $\epsilon\ge 0$ fixed. We prove:

Theorem 5 *Let $\delta>0$. There exists $\eta_\delta>0$ such that if $0\le\epsilon<\eta_\delta$ then we have*

$$||J(\hat u_n)-J^\epsilon(u^\epsilon_n)||<\delta\,.$$

Proof. As a preliminary observation we note that if y^ϵ depends continuously on ϵ and M is fixed and coercive, the minimum point v^ϵ of

$$\langle Mv,v\rangle+\langle y^\epsilon,v\rangle\,,\qquad \text{i.e.}\quad v^\epsilon=-M^{-1}y^\epsilon$$

is a continuous function of ϵ. Also the minimum value is a continuous function of ϵ.

The construction of the minimizing sequence $\{u^\epsilon_n\}$ in (17) is in two steps. In the first step we use (15) and we construct $\tilde u^\epsilon$. This does not depend on n, it is a continuous function of ϵ and also the corresponding solution $x^\epsilon(\cdot)$ is a $C(0,T;X)$ continuous function of ϵ thanks to the observation above. Moreover,

$$\{J^\epsilon_{\text{int}}+\langle M_{11}[x^\epsilon(T)-\xi^\epsilon_T],M_{11}[x^\epsilon(T)-\xi^\epsilon_T]\rangle\}$$

is a continuous function of ϵ. Hence, in order to prove the theorem it is sufficient to consider the finite terms. We consider

$$\hat u_{\tau^\epsilon_0}=\arg\,\min\Phi_{\tau^\epsilon_0}(\tilde x^\epsilon(\tau^\epsilon_0),u)\,.$$

The functionals $\Phi_{\tau^\epsilon_0}(\tilde x^\epsilon(\tau^\epsilon_0),\cdot)$ depends continuously on ϵ so that the minimum point and the minimum value do depend continuously on ϵ. When $\epsilon\to 0$ these tend respectively to $\hat u_{\tau_0}$ and $\Phi_{\tau_0}(\tilde x(\tau_0),\hat u_{\tau_0})$. Hence, given $\sigma>0$, when ϵ is small enough we have

$$|\Phi_{\tau_0}(\tilde x(\tau_0),\hat u_{\tau_0})-\Phi_{\tau^\epsilon_0}(\tilde x^\epsilon(\tau^\epsilon_0),\hat u^\epsilon_{\tau^\epsilon_0})|<\sigma\,.$$

The penalization at T can be treated analogously, and we get the result. ∎

3.2 The function $s \to I_s(x_0;\xi_0,\xi_T,\xi)$

It is clear that the previous arguments can be repeated on every interval $[s,T]$ and this leads to the construction of functions $s \to I_s(x_0;\xi_0,\xi_T,\xi)$ for every $s \in [0,T]$. Here x_0 is the "initial condition" at the "initial time" s.

It is convenient to introduce the following notations: $x(t;s,x_0,u)$ is the solution of eq. (1) when the initial time is s and the initial value at s is x_0; J_s and I_s are defined analogously to J and I. We are going to study the regularity properties of the function $s \to I_s(x_0;\xi_0,\xi_T,\xi)$.

We need a Lemma, which is analogous to Lemma 3:

Lemma 6 *Let $s_n \to \hat\tau$ and let $\{u_n\}$ be a sequence in $L^2(s_n,T)$ such that $J_{s_n}(x_0,u_n) < \beta$ (β is a fixed number). Then the sequences $\{Gu_n(\tau_0)\}$ and $\{Gu_n(T)\}$ are bounded (of course the sequence $\{Gu_n(\tau_0)\}$ is considered only if $\hat\tau \le \tau_0$).*

Proof. We see from (18) that $\{u_n\}$ is a bounded sequence in $L^2(0,T;U)$ so that the sequence $\{x(T;s_n,x_0,u_n)\}$ is bounded in X. The result now follows as in Lemma (3). ∎

Theorem 7 *The function $s \to I_s(x_0;\xi_0,\xi_T,\xi)$ is continuous for every x_0 and $s \ne \tau_0$. For $s = \tau_0$, it is left continuous and upper semicontinuous from the right.*

Proof. As above, we prove the result for $I_s(x_0)$. The general case is analogous. We prove separately upper and lower semicontinuity at every $\tau \in [0,T]$, $\tau \ne \tau_0$. The arguments we present holds also at τ_0 but only from the left while only the argument concerning upper semicontinuity holds at τ_0 from the right.

We prove first

$$\limsup_{s\to\hat\tau} I_s(x_0) \le I_{\hat\tau}(x_0)\,.$$

Let us fix $\alpha > I_{\hat\tau}(x_0)$. We prove

$$\limsup_{s\to\hat\tau} I_s(x_0) \le \alpha\,.$$

Let u satisfy

$$I_{\hat\tau}(x_0) < J_{\hat\tau}(x_0;u) < \alpha\,.$$

Now we distinguish the two limits, for $s \to \hat\tau+$ and $s \to \hat\tau-$.

If $s \to \hat\tau+$ we use

$$x(t;s,x_0,u) = \left\{e^{A(t-s)}x_0 + \int_s^t e^{A(t-r)}Bu(r)\,\mathrm{d}r\right\} \longrightarrow x(t;\hat\tau,x_0,u)$$

uniformly on every $[r,T]$, $r > \hat\tau$; and it remains bounded. Hence,

$$\lim_{s\to\hat\tau+} J_s(x_0;u_{|[s,T]}) = \begin{cases} J_{\hat\tau}(x_0;u) & \hat\tau \ne \tau_0 \\ J_{\tau_0}(x_0;u) - \Phi_{\tau_0}(x(\tau_0),u(\tau_0)) & \hat\tau = \tau_0\,. \end{cases}$$

In both the cases, the expression on the right side is less or equal to $J_{\hat\tau}(x_0;u) \le \alpha$.

For every s we have

$$I_s(x_0) \le J_s(x_0;u_{|[s,T]})\,.$$

Hence we have

$$\limsup_{s\to\hat\tau+} I_s(x_0) \le \alpha\,.$$

Analogously, let $s \to \hat\tau-$. The input u is as above and for every s we introduce u_s:

$$u_s(t) = \begin{cases} u(t) & \text{if } t > \hat\tau \\ 0 & \text{if } s < t < \hat\tau \end{cases}$$

so that we still have

$$\lim_{s\to\hat{\tau}-} x(t;s,x_0,u) = x(t;\tau_0,x_0,u)\,.$$

The limit is uniform on $[\tau_0, T]$. The result follows as above (and we don't need to single out the case $\hat{\tau} = \tau_0$).

We prove now lower semicontinuity,

$$\liminf_{s\to\hat{\tau}} I_s(x_0) \geq I_{\hat{\tau}}(x_0)\,.$$

We prove this for $\tau \neq \tau_0$ and when $\tau = \tau_0$ we prove the result only for the left limit. We choose any β and β' such that $\beta < \beta' < I_{\hat{\tau}}(x_0)$ and we prove

$$\liminf_{s\to\hat{\tau}} I_s(x_0) \geq \beta\,.$$

By contradiction, let $\liminf_{s\to\hat{\tau}} I_s(x_0) < \beta$ and let $s_n \to \hat{\tau}$, $\{u_n\}$ be sequences such that

$$J_{s_n}(x_0;u_n) < \beta'\,.$$

If $\hat{\tau} = \tau_0$ we assume $s_n < \tau_0$. Lemma 6 shows that $\{u_n(\cdot)\}$ is bounded in $L^2(0,T;U)$ and $\{Gu_n(\tau_0)\}$ $\{Gu_n(T)\}$ are bounded . We first consider the case $s \to \hat{\tau}-$. We shall prove that

$$|J_{\hat{\tau}}(x_0;u_n) - J_{s_n}(x_0;u_n)| \longrightarrow 0\,. \tag{25}$$

Accepting this, there exists N_ϵ such that for $n > N_\epsilon$ we have

$$I_{\hat{\tau}}(x_0) \leq J_{\hat{\tau}}(x_0;u_n) < \beta'$$

and this is a contradiction because we assumed $\beta' < I_{\hat{\tau}}(x_0)$.

In order to give an estimate of the absolute value in (25) (recall, $s \to \hat{\tau}-$) we represent

$$\begin{aligned}
&J_{\hat{\tau}}(x_0;u_n) - J_{s_n}(x_0;u_n) &(26)\\
&= \int_{\hat{\tau}}^{T} \left\{ \langle \begin{bmatrix} x(t;\hat{\tau},x_0,u_n) \\ u_n(t) \end{bmatrix}, Q \begin{bmatrix} x(t;\hat{\tau},x_0,u_n) \\ u_n(t) \end{bmatrix} \rangle + |u_n(t)|^2 \right\} \mathrm{d}t &(27)\\
&+\Phi_{\hat{\tau}}(x(\tau_0;\hat{\tau},x_0,u),u(\tau_0)) + \Phi_T(x(T;\tau,x_0,u),u(T))\\
&- \int_{s_n}^{T} \left\{ \langle \begin{bmatrix} x(t;s_n,x_0,u_n) \\ u_n(t) \end{bmatrix}, Q \begin{bmatrix} x(t;s_n,x_0,u_n) \\ u_n(t) \end{bmatrix} \rangle + |u_n(t)|^2 \right\} \mathrm{d}t &(28)\\
&+\Phi_{\hat{\tau}}(x(\tau_0;s_n,x_0,u),u(\tau_0)) - \Phi_T(x(T;\tau,x_0,u),u(T))
\end{aligned}$$

Now we represent the sum of the integrals in (27) and in (28) as

$$\int_{\hat{\tau}}^{T} = \int_{s_n}^{\hat{\tau}} + \int_{\hat{\tau}}^{T}\,.$$

We note that the quadratic terms which contain only u_n cancel out. Boundedness of the integrand (in L^2) proves that the first integral on the right side converges to zero. The second integral converges to zero because $\{u_n\}$ is bounded in $L^2(0,T;U)$ while

$$||x(\cdot;\hat{\tau},x_0,u_n) - x(\cdot;s,x_0,u_n)||_2$$

converges to zero uniformly.

Analogous argument proves that the contribution of the final and intermediate costs, represented by Φ, tend to zero. We stress the fact that the purely quadratic terms in $u(\cdot)$, $Gu(\hat{\tau})$ and $Gu(T)$ cancel out.

If $s \to \hat\tau_+$ an analogous argument holds, provided that $\hat\tau \neq \tau_0$, by using

$$\tilde u_n(t) = \begin{cases} u_n(t) & \text{if} \quad s_n \le t \le T \\ 0 & \text{if} \quad \tau_0 \le t \le s_n\,. \end{cases}$$

We cannot repeat this last argument for $\hat\tau = \tau_0$ because if $s > \tau_0$ the contribution of Φ_{τ_0} is not accounted for. ∎

The previous result holds for every T_0. Of course, if $T_0 < \tau_0$, the control is not penalized at τ_0 and we have continuity also at τ_0.

Moreover,

Corollary 8 *Let $\tau_0 = 0$. The function $s \to I_s(x_0)$ is continuous on $(0, T]$.*

Corollary 9 *Let $\tau_0 < T$. We have that $I_T(x_0)$ is a quadratic function of $(x_0 - \xi_T)$; i.e. there exists $N = N^* \ge 0$ in $\mathcal{L}(X)$ such that*

$$I_T(x_0) = \inf_u \langle \begin{bmatrix} x_0 - \xi_T \\ Gu \end{bmatrix}, M \begin{bmatrix} x_0 - \xi_T \\ Gu \end{bmatrix} \rangle = \langle (x_0 - \xi_T), N(x_0 - \xi_T) \rangle\,. \tag{29}$$

We note that $G^* M_{22} G$ is not assumed coercive, so that the infimum in (29) is not generally a minimum. However, we note:

Lemma 10 *Let x_0 be an optimizable initial condition. Then, the optimal control is unique. Furthermore, if the optimal control $u^+(\cdot; x_0) \in \mathcal{U}$ exists for the initial condition x_0 then we have*

$$\begin{cases} u^+(\tau_0; x_0) = \arg\min_{u \in \mathcal{U}} \langle \begin{bmatrix} x^+(\tau_0; x_0) \\ Gu \end{bmatrix}, \tilde M \begin{bmatrix} x^+(\tau_0; x_0) \\ Gu \end{bmatrix} \rangle\,, \\ u^+(T; x_0) = \arg\min_{u \in \mathcal{U}} \langle \begin{bmatrix} x^+(T; x_0) \\ Gu \end{bmatrix}, M \begin{bmatrix} x^+(T; x_0) \\ Gu \end{bmatrix} \rangle\,. \end{cases} \tag{30}$$

<u>Proof</u>. Unicity was already proved in Theorem 2. Conditions (30) might be proved from (11). Instead, we give an independent proof. We consider the optimal control and $t = T$. We recall that $u(\cdot) \to x(\cdot; x_0, u)$ is a linear continuous transformation from $L^2(0, T)$ to $C(0, T; X)$. By contradiction, let $u(\cdot)$ does not satisfy the second condition in (30). Then, we can find $\tilde u_0 \in U$ such that

$$\langle \begin{bmatrix} x^+(T; x_0) \\ G\tilde u_0 \end{bmatrix}, M \begin{bmatrix} x^+(T; x_0) \\ G\tilde u_0 \end{bmatrix} \rangle < \langle \begin{bmatrix} x^+(T; x_0) \\ Gu^+(T) \end{bmatrix}, M \begin{bmatrix} x^+(T; x_0) \\ Gu^+(T) \end{bmatrix} \rangle - \epsilon_0\,, \qquad \epsilon_0 > 0\,.$$

We change the definition of $u^+(\cdot)$ in a short time interval $[T - \sigma, T]$,

$$\tilde u(t) = \begin{cases} u^+(t; x_0) & \text{if} \quad t < T - \sigma \\ \tilde u_0 & \text{if} \quad T - \sigma \le t \le T\,. \end{cases}$$

Clearly we can find σ so small that

$$|J_{\text{int}}(x_0, \tilde u) - J_{\text{int}}(x_0, u^+)| < \epsilon_0/8\,,$$

$$\left| \langle \begin{bmatrix} x^+(T; x_0) \\ \tilde u_0 \end{bmatrix}, M \begin{bmatrix} x^+(T; x_0) \\ \tilde u_0 \end{bmatrix} \rangle - \langle \begin{bmatrix} x(T; x_0, \tilde u) \\ \tilde u_0 \end{bmatrix}, M \begin{bmatrix} x(T; x_0, \tilde u) \\ \tilde u_0 \end{bmatrix} \rangle \right| < \epsilon_0/8$$

so that

$$J(x_0; \tilde u) < J(x_0; u) - \epsilon_0/2\,.$$

Hence, $u(\cdot)$ is not the optimal control of x_0. The proof of the first equality in (30) is similar. ∎

4 The Dissipation Inequality, the Riccati equation and the optimal control

As we said, we now consider the case which is most important for the applications to singular systems, $T_0 > \tau = T$ and zero reference signals. Hence, $I_s(x_0) = \langle P(s)x_0, x_0\rangle$ and $P(s) = P^*(s) \in \mathcal{L}(\mathcal{X})$ is non negative.

From (29),

$$\langle x_0, P(T)x_0\rangle = \inf_u \langle \begin{bmatrix} x_0 \\ Gu \end{bmatrix}, M \begin{bmatrix} x_0 \\ Gu \end{bmatrix}\rangle = \langle x_0, Nx_0\rangle \,. \tag{31}$$

In this section we prove that $P(t)$ solves both the usual Dissipation Inequality and also an additional compatibility condition.

We observe that if x_0 is optimizable then dynamic programming shows that the restriction of $u^+(t;x_0)$ to $[s,T]$ is optimal for the initial condition at s given by $x^+(s;x_0)$. Moreover, a standard dynamic programming argument proves that $P(t)$ satisfy the integral form of the Dissipation Inequality. Even more:

Theorem 11 *For every $s > s_0$ and every $u \in \mathcal{U}$ we have*

$$\langle x(s;s_0,x_0,u), P(s)x(s;s_0,x_0,u)\rangle - \langle x_0, P(s_0)x_0\rangle + \int_{s_0}^{s} F(x(r;s_0,x_0,u),u(r))\,\mathrm{d}r \geq 0\,. \tag{32}$$

Let x_0 be an initial condition which admits the optimal control $u^+(\cdot;x_0)$. If $u^+(\cdot;x_0)$ is replaced for u in (32) then equality holds for every s_0 and every $s > s_0$.

Conversely, let $P(s)$ be a solution of the Dissipation Inequality which satisfies condition (31) and let x_0 be given. Let $u(\cdot)$ be an input such that the Dissipation Inequality (with $s_0 = 0$) holds as an equality along $x(\cdot;x_0,u)$ and $u(\cdot)$ and furthermore let $u(T)$ satisfy

$$u(T) = \arg\min_{u\in U} \langle \begin{bmatrix} x(T;x_0,u) \\ Gu \end{bmatrix}, M \begin{bmatrix} x(T;x_0,u) \\ Gu \end{bmatrix}\rangle = \langle x(T;x_0,u), Nx(T;x_0,u)\rangle \tag{33}$$

(i.e. the second condition in (30)). Then, $u(\cdot) = u^+(\cdot;x_0)$ is the optimal control of x_0.

This can be repeated for each initial time s_0.

Proof. In order to prove the Dissipation Inequality we proceed as follows: we fix the initial condition x_0 at a certain initial time s_0 and any input u on $[s_0,T]$. Let $x(s;s_0,x_0,u)$ be the value of the solution at time s and let $\{u_n\}$ be a minimizing sequence on $[s,T]$, for the initial condition $x(s;s_0,x_0,u)$. We consider the new control

$$\tilde{u}_n(t) = \begin{cases} u(t) & \text{if } s_0 \leq t \leq s \\ u_n(t) & \text{if } t > s\,. \end{cases}$$

We have

$$I(x_0) \leq J_{s_0}(x_0,\tilde{u}_n) = \int_{s_0}^{s} F(x(r;s_0,x_0,u),u(r))\,\mathrm{d}r$$
$$+ \int_s^T F(x(r;s_0,x_0,u_n),u(r))\,\mathrm{d}r + \langle \begin{bmatrix} x(T;s_0,x_0,u) \\ Gu(T) \end{bmatrix}, M \begin{bmatrix} x(T;s_0,x_0,u) \\ Gu(T) \end{bmatrix}\rangle\,. \tag{34}$$

We pass to the limit and we find the Dissipation Inequality in integral form.

Now we prove the statement concerning the optimal control.

If an optimal control u exists when x_0 is the initial condition at s_0 then dynamic programming shows that

$$u^+_{s_0}(\cdot;x_0)_{|(s,T)} = u^+_s(\cdot;x^+_{s_0}(s;x_0))\,.$$

We replace $u_s^+(\cdot;x_0)$ in (34) and we see that the Dissipation Inequality holds as an equality.

Now we prove the converse statement. We use both the assumptions and equality (31) and we see that

$$\langle x_0, P(0)x_0\rangle = \int_0^T F(x(r;x_0,u),u(r))\,\mathrm{d}r + \langle x(T;x_0,u), P(T)x(T;x_0,u)\rangle$$
$$= \int_0^T F(x(r;x_0,u),u(r))\,\mathrm{d}r + \langle \begin{bmatrix} x(T;x_0,u) \\ Gu(T) \end{bmatrix}, M \begin{bmatrix} x(T;x_0,u) \\ Gu(T) \end{bmatrix}\rangle\,.$$

Let $\hat{u}(\cdot) \neq u(\cdot)$ be any input in $\mathcal{U}$. Then, we have

$$\langle x_0, P(0)x_0\rangle \leq \int_0^T F(x(r;x_0,\hat{u}),\hat{u}(r))\,\mathrm{d}r + \langle x(T;x_0,\hat{u}), P(T)x(T;x_0,\hat{u})\rangle$$
$$\leq \int_0^T F(x(r;x_0,\hat{u}),\hat{u}(r))\,\mathrm{d}r + \langle \begin{bmatrix} x(T;x_0,\hat{u}) \\ G\hat{u}(T) \end{bmatrix}, M \begin{bmatrix} x(T;x_0,\hat{u}) \\ G\hat{u}(T) \end{bmatrix}\rangle = J(x_0;\hat{u})\,.$$

Hence, $u(\cdot)$ is optimal.

Clearly this argument can be repeated for every initial time $s_0 \in [0,T]$. ∎

Finally:

Theorem 12 *Let $P(s)$ be the operator of the quadratic form I_s. Then, $P(s)$ is maximal among the solutions of the Dissipation Inequality which satisfy the final condition (29).*

Proof. We prove the result with $s = 0$. The same argument can be repeated for every $s \in [0,T]$.

Every solution $\hat{P}(t)$ of the Dissipation Inequality which also satisfies the final condition (29) satisfies

$$\langle x_0, \hat{P}(0)x_0\rangle \leq \int_0^T F(x(r;x_0,u),u(r))\,\mathrm{d}r + \langle x(T;x_0,u), Nx(T;x_0,u)\rangle$$
$$\leq \int_0^T F(x(r;x_0,u),u(r))\,\mathrm{d}r + \langle \begin{bmatrix} x(T;x_0,u) \\ Gu(T) \end{bmatrix}, M \begin{bmatrix} x(T;x_0,u) \\ Gu(T) \end{bmatrix}\rangle = J(x_0;u)$$

for every input $u \in \mathcal{U}$. Hence,

$$\langle x_0, \hat{P}(0)x_0\rangle \leq I(x_0) = \langle x_0, P(0)x_0\rangle\,. \quad ∎$$

4.1 The optimal control and the Riccati equation

As already noted, the optimal control $u^+(\cdot;x_0)$ in general does not exist but, when it exists, it admits a variational characterization. In the special case we are studying now, Theorem 2 takes the following form (the solutions of the differential equations must be intended in the weak sense. In particular, $p(t)$ is now a continuous function).

Theorem 13 *Let $x_0 \in X$ and let us consider the following two-point problem:*

$$\begin{cases} x(0) = x_0 \\ \dot{x} = Ax(t) + Bu\,, \qquad u(t) = -B^*p(t) \\ \dot{p} = A^*p - Qx \\ p(T) = \left\{ M_{11} - M_{12}M_{22}^{-1}M_{12}^* \right\} x(T) \end{cases} \tag{35}$$

and let us consider the compatibility condition

$$Gu(T;x_0) = -M_{22}^{-1}M_{12}^*x(T;x_0)\,. \tag{36}$$

The initial condition x_0 is optimizable if and only if problem (35) has a solution $(x(\cdot),p(\cdot))$ such that the function $u(t)=-B^*p(t)$ satisfies the compatibility condition (36). In this case, the function $-B^*p(t)$ is the optimal control.

Now we note that the two-point problem (without the compatibility condition (36)) is always solvable:

Theorem 14 *We have*

$$M_{11} - M_{12}M_{22}^{-1}M_{12}^* \geq 0 \tag{37}$$

so that the two-point problem (35) is solvable. Moreover, the vector $x(T)$ is a linear and continuous function of x_0.

Proof. The positivity condition (37) follows from $M \geq 0$.

The two-point problem (35) (without the compatibility condition) is the two-point problem of the quadratic cost

$$\tilde{J}(x_0;u) = \int_0^T \Big\{\langle Qx(t),x(t)\rangle + \langle u(t),u(t)\rangle\Big\}\,\mathrm{d}t + \langle \mathcal{M}x(T),x(T)\rangle$$

where $\mathcal{M} = M_{11} - M_{12}M_{22}^{-1}M_{12}^* \geq 0$. Hence this problem admits a unique optimal control $\tilde{u}(t;x_0)$ for every x_0. Furthermore, if $\tilde{x}(t;x_0)$ is the optimal trajectory of x_0, then the transformation $x_0 \to \tilde{x}(t;x_0)$ is linear and continuous. ∎

We use the last statement of the theorem as follows: the compatibility condition (36) can be written as

$$\begin{aligned} &\mathcal{L}x(T) = 0 \\ &\mathcal{L} = M_{22}GB^*\Big[M_{11} - M_{12}M_{22}^{-1}M_{12}^*\Big] + M_{12}^*\,. \end{aligned} \tag{38}$$

Hence:

Theorem 15 *The initial condition x_0 is optimizable if and only if $\mathcal{L}x(T)=0$. In particular, the set of the optimizable initial conditions is a closed subspace of X.*

The second assertion follows since we already noted continuity of the transformation $x_0 \to x(T)$ when $(x(\cdot),p(\cdot))$ solves (35).

Of course, the optimal control $\tilde{u}(t;x_0)$ for $\tilde{J}$ is given by

$$\tilde{u}(\cdot;x_0) = -B^*p(\cdot)\,.$$

It always exists while $u^+(\cdot;x_0)$ exists only if x_0 is optimizable. The previous considerations show:

Theorem 16 *If x_0 is optimizable then $\tilde{u}(\cdot;x_0) = u^+(\cdot;x_0)$.*

The Riccati equation is an important tool in the quadratic regulator problem. Hence we now relate our problem to a differential Riccati equation. It is known that the component p of the two-point problem (35) is expressed as

$$p(t) = \mathcal{P}(t)x(t)$$

where $\mathcal{P}(t)$ solves the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{P}(t)x,y\rangle = -\langle Ax,\mathcal{P}(t)y\rangle - \langle \mathcal{P}(t)x,Ay\rangle - \langle Qx,y\rangle$$
$$+\langle B^*\mathcal{P}(t)x, B^*\mathcal{P}(t)y\rangle\,, \qquad \forall x\,,\ y\in \mathrm{dom}A\,; \qquad \mathcal{P}(T)=\mathcal{M}$$

so that the optimal control of $\tilde{J}(x_0;u)$ is

$$\tilde{u}(t) = B^*\mathcal{P}(t)x(t)$$

where now x solves the closed loop equation

$$\dot{x} = \Big[A - BB^*\mathcal{P}(t)\Big]x\,, \qquad x(0)=x_0\,. \tag{39}$$

As we noted, this is also the optimal control of $J(x_0;u)$ when x_0 is optimizable. Hence,

Theorem 17 *Let x_0 be optimizable. Then, the optimal control has the feedback form*

$$u^+(t;x_0) = B^*\mathcal{P}(t)x^+(t;x_0)\,.$$

5 Conclusions

In this paper we have shown two applications which forces us to study a quadratic control problem which penalizes the value taken by the control at the final time T and at an intermediate time τ_0 (of course similar arguments can be repeated if the values of the control at a finite number of intermediate instants are penalized). An obvious approach to this problem is to assume that $u\in W^{12}(0,T)$ and to take as a "new control" the derivative $u'\in L^2(0,T;U)$. The examples we have presented however show that the regularity assumption on u is not natural, and that we must assume solely that u is square integrable.

The optimal control for the problem under study in general does not exist. We characterized the initial conditions which admit an optimal control (which is then unique) and we characterized the optimal control in terms of a two-point problems and compatibility conditions. In the important case that the reference signal is zero, we expressed the optimal control in terms of a suitable Riccati equation.

The optimal control however in general does not exist. So, we studied the value function and we proved that it is a continuous function of the initial datum and the reference signal, and also a continuous function of the initial time, except at the intermediate points at which the control is penalized. Moreover, we gave a construction for a minimizing sequence and we proved that this construction is robust with respect to the reference signal and the time at which the value of the control is penalized.

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