On-line reconstruction of inputs to distributed parameter systems with accessible sources*

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Abstract. In this paper we consider a distributed system representing a diffusion process. We assume that sources of "signals" for example source of inquinants, have a known and accessible position. We show that it is possible to monitor the concentration of inquinants so to on-line reconstruct the rate of immission.

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1 Introduction and assumptions

Reconstruction of input signals to linear dynamical systems is an important problem. Several algorithms have been proposed to achieve this goal, which can be classified in two large classes: off-line and on-line reconstruction algorithms. Off-line algorithms accumulate all the available pieces of information and, after the process has come to an end, these pieces of information are elaborated so to obtain an estimate of the input signal. An example of this is Tikonov method. On-line methods instead at each time t produce an estimate $v(t)$ of the unknown input u at the time t solely on the basis of observations taken at

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previous times. It is obvious that on-line deconvolution methods will require stronger conditions than those used to justify off-line algorithms and, of course, only on-line algorithms can be used for regulation and control purposes, see applications in [17, 21, 8, 24] for the case of finite dimensional systems. In this paper we extend the ideas in these papers to systems with distributed parameters, described by partial differential equations. The example, described below, which suggests the problem and the assumptions in this paper is the rate of immission of an inquinant in a fluid, a process described by a partial differential equation of parabolic type.

The methods in the cited references have already been extended to particular classes of distributed systems, provided that the inpulse response of the system is known, see [18, 19, 6, 7, 22]. But, the impulse response, equivalently the transfer function, is known only for special geometries. For this reason, in this paper we consider a time invariant system described in state space form, i.e.

$$\dot{x} = Ax + Bu\,, \qquad y = Cx \tag{1}$$

and

$$x(0) = 0\,.$$

Here x, u and y denote the state, input and output variables. They belong to Hilbert spaces X, U and Y (precise assumptions will be specified below. Preliminary work on this problem has considered the non realistic case $y = x$, see [9, 11, 13]). Note that in practice u and y will be finite dimensional vectors (i.e. independent sources and sensors are finite in number).

Input reconstruction is an ill posed problem so that any reconstruction algorithm must introduce a penalization parameter $\alpha > 0$ and reconstruct a candidate approximant v_α of the input u. The point is to prove that, under suitable assumptions, v_α converges to u for $\alpha \to 0+$. This is a consistency result for the algorithm we are going to propose but in any practical case we have to fix a (small) value of α and we then choose the corresponding function $v_\alpha(t)$ as an approximant of the unknown input u. See [24] for an on line adaptive method for the determination of α in the finite dimensional case.

Before stating the precise assumptions of this paper we present an example, upon which the assumptions will be modeled.

1.1 An example and the assumptions in this paper

The example which suggest the assumptions in this paper is the following linear equation for the diffusion of a pollutant in a steady fluid:

$$x_t = \Delta x + b(\xi)u(t) \qquad t > 0\,, \quad \xi \in \Omega \tag{2}$$

where Ω is a known region and $x(t,\xi)$ is the concentation of a pollutant at time t in the position ξ.

Let for simplicity u be 1-dimensional so that $b = b(\xi)$ is an element of the state space X (in this example, $X = L^2(\Omega)$). This equation is complemented with initial conditions and boundary conditions which we assume known so that we can even assume

$$x(0,\cdot) = 0\,, \qquad x(t,\cdot)_{|\partial\Omega} = 0\,.$$

(a similar treatment is possible in the case the flux at the boundary is known).

Note that we are now considering the case of a source distributed within the fluid, a condition which will be removed in Section 4.

System (2) can be abstractly represented in the form (1), with

$$X = L^2(\Omega)\,, \quad \operatorname{dom} A = H^2(\Omega) \cap H^1_0(\Omega)\,, \qquad A\theta = \Delta\theta\,.$$

Hence, $A = A^*$ is the infinitesimal generator of a holomorphic semigroup which is exponentially stable (see [14]).

The fact that the source of pollutant is known is reflected in the fact that $b(\xi)$ is a known function, often the characteristic function of a subregion of Ω.

The key assumption of this paper is that the source of inquinant is accessible to measures so that we can measure the concentration of the inquinant at the source, which of course is not directly the inquinant emitted at the moment since this datum is blurred by the past activity. In the case of slow processes, the concentration due to past activity can be much higher than that produced by the immission, whose rate is the function $u(t)$. Hence, the goal is to identify the function $u(t)$ in real time, so to detect unespected variations as soon as possible.

The output that we approximately measure is

$$y = Cx = \mu\langle x, b\rangle$$

where $\mu \neq 0$ is a constant (we shall assume $\mu = 1$). This means that $C = B^*$ where B is given by $(Bu)(\xi) = b(\xi)u$. In practice, if $b(\xi)$

is the characteristic function of $K \subseteq \Omega$ then y is the average of the concentration $x(t, \cdot)$ on K.

More in general, if the input operator is B, acting from a finite dimensional Hilbert space U to the state space X we shall assume that $C = B^*$ (note that this condition implies $Y = U$); in practice this means that in order to detect a number m of independent pullutants, we perform m independent measures. The term used in control theory to denote such pair of inputs and outputs is "colocated input and output operators".

We sum up: the abstract properties we extract from the previous example are:

Assumption A.

A1) The state space X and the input and output spaces $U = Y$ are Hilbert spaces, $\dim U = m < +\infty$.

A2) A selfadjount operator A is given, the infinitesimal generator of a holomorphic semigroup of contractions e^{At} on X. Note that this implies

$$\langle x, Ax \rangle \leq 0 \qquad \text{for all } x \in \operatorname{dom} A.$$

A3) In section 2 and 3: the input operator $B \in \mathcal{L}(U, X)$ and $C = B^*$.

Assumption **A3)** will be relaxed in Section 4. These assumptions may be realistic for the problem described above but, both for the sake of mathematical elegance and for the applications, also the case of boundary inputs/observations has to be considered. See Section 4 for this.

Now we introduce two assumptions which will be used in the paper and which are not restrictive, since it is always possible to have them satisfied. Clearly, if $B = 0$ then the input u does not act on the output and reconstruction is impossible. Hence, it must be $B \neq 0$ so that we can always assume $\ker B = 0$ (just replace the input space with $[\ker B]^{\perp}$). A second observation is that it is not restrictive to assume that the holomorphic semigroup e^{At} is exponentially stable. In fact, this condition is satisfied in the example above and stability (may be not exponential stability) is a must in any linear practical problem. We can always reduce ourselves to study the case that the semigroup is exponentially stable since $y_1(t) = e^{-st} y(t)$ is the output of

$$\eta' = (-sI + A)\eta + Bu_1(t)\,, \quad y_1(t) = C\eta(t) \qquad \text{when } u_1(t) = e^{-st} u(t)\,.$$

The semigroup generated by $(-sI + A)$ is exponentially stable if s is large enough. Hence it is not restrictive to assume also:
Non restrictive assumption B.

B1) $\ker B = 0$ hence B^* is surjective

B2) $||e^{At}|| < Me^{-s_0 t}\,, \qquad s_0 > 0.$

This implies that the resolvent of A is holomorphic in a halfplane $\Re e\,\lambda > -s_0$, $s_0 > 0$ and that $A = A^*$ is negative definite, i.e.

$$\langle Ax, x\rangle < 0 \qquad \forall x \in \operatorname{dom} A\,. \tag{3}$$

Finally, it is clear that input reconstruction is impossible not only when $B = 0$ but also when $T(\lambda) = B^*(\lambda I - A)^{-1}B = 0$ since in this case the input does not affect the output. More in general, it is known that the presence of unstable transmission zeros is an obstacle to the reconstruction of the input system (see [5] for the blocking properties of the zeros). So we assume
Assumption C.

The system has no transmission zero in $\Re e\,\lambda \geq 0$; i.e.,

$$\det B^*(\lambda I - A)^{-1}B \neq 0 \quad \text{in} \quad \Re e\,\lambda \geq 0\,.$$

The reconstruction algorithm we are going to study is informally derived in Section 2 . The heuristic considerations in Section 2 suggest a formula for v_α, a candidate approximant of the unknown input u. The fact that v_α converges to the unknown input u in $L^2(0,T)$ (for each $T \geq 0$) is proved in Section 3 for the case of distributed inputs and observations. The case of boundary inputs is considered in Section 4.

We now complete our comments on the literature. The output of (1) is given by

$$y(t) = C\int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s$$

so that

$$\dot y(t) = CBu(t) + \int_0^t CAe^{A(t-s)}Bu(s)\,\mathrm{d}s\,.$$

This is a Volterra equation of the second kind (see [15]) and can be solved for $u(t)$. This kind of approach has been taken in [16] and in part also in [20] (see also [23]). However, in order to justify differentiation and the last equality, we need suitable regularity assumptions on $u(t)$ and, more important, on the operator B ($B = b(\xi)$ equal to the characteristic function of a subset of Ω does not fit the assumptions in [16, 20]). But, if these regularity assumptions hold, this approach does not need the assumption of colocated input and output operators.

2 Derivation of the formula for $v_\alpha(t)$

We fix a time interval $[0, T]$ over which the system evolves and we fix a penalization parameter α. We recall our goal: for each value of α we want a function $v_\alpha(t)$, $t \in [0, T]$ which under suitable assumption will converge to the unknown input u for $\alpha \to 0+$. In order to keep the notations to a minimum, the dependence of v on α is not explicitly indicated during the computations.

The crucial idea for the construction of v is as follows: we associate a "model" to system (1), which is a copy of the system. In practice this means that we simulate system (1), often numerically. This model is forced to track the output of the system and we hope in this way that we can track also the input u. So, the model is

$$\dot{w} = Aw + Bv\,, \qquad w(0) = 0\,, \qquad\qquad z = Cw\,. \tag{4}$$

As in [17], we associate a "Liapunov" type function to the system and its model. This is

$$\epsilon(t) = ||Cw(t) - Cx(t)||^2 + \alpha \int_0^t ||v(s)||^2\, \mathrm{d}s\,.$$

Here α is the regularization parameter.

We introduce the error $e = w - x$ which solves

$$\dot{e} = Ae + Bv - Bu\,.$$

The input v does depend on the penalization parameter α. Hence we have also $e = e_\alpha$ although we shall use the simpler notation e.

Note that if the input u is regular then $x(t)$ is a classical solution of (1) but we cannot say that e is a classical solution since v has not yet been determined.

We try to force ϵ to decrease as fast as it can. We compute its derivative and we act on v so to minimize the derivative. We make this computation as if e were a classical solution.

$$\dot{\epsilon}(t) = 2\langle Ce, CAe\rangle + 2\langle C^*Ce, B(v - u)\rangle + \alpha v^2\,.$$

This computation is performed (just formally!) at each time t.

Now we choose v so to keep this derivative as small as we can, i.e.

$$v(t) = v_\alpha(t) = -\frac{1}{\alpha}B^*C^*[z(t) - y(t)] = -\frac{1}{\alpha}B^*BB^*e(t)\,. \qquad (5)$$

Remark 1 Note that the function $e(t)$ is not known. The properties of the function $e(t)$ will be an instrument in the proofs. But, the function $v(t)$ can be computed since the right hand side of (5) depends solely on the measured quantities $z(t)$ and $y(t)$.

Measures are always corrupted by noise. For most of clarity we present the proofs in the ideal case of noiseless measures. The effect of the noise is then taken into account in Section 5. ∎

Once the expression for v is replaced in the equation of w, we find

$$\dot{w} = Aw - \frac{1}{\alpha}BB^*C^*[z(t) - y(t)]$$

so that the unknown and not measurable function $e(t)$ solves ($e(0) = 0$ and)

$$\dot{e} = Ae - \frac{1}{\alpha}B[B^*B]B^*e(t) - Bu(t)\,. \qquad (6)$$

Remark 2 We see from here that if u is smooth, then e is a classical solution. ∎

The goal of this paper is the proof that when u is square integrable then

$$\lim_{\alpha \to 0+} v_\alpha = u \qquad \text{in } L^2(0,T).$$

After that, we are justified in choosing v_α with small α as an approximant of u.

3 The proof that v_α approximates u

We consider here the ideal case that measures are not corrupted by errors and we prove that v_α approximates u for $\alpha \to 0+$. As usual in systems theory, problems can be efficiently attached in the frequency domain, i.e. after Laplace transformation.

We shall use the assumption that e^{At} is exponentially stable and for computational reasons it will be convenient to explicitly use that B^*B is invertible. We noted that these assumptions are not restrictive.

In order to work with the Laplace transform, we need $y(t)$, $z(t)$ and $u(t)$ on $[0, +\infty)$. For this, we extend u to $t > T$ so to obtain an arbitrary $L^2(0, +\infty)$ function. The simplest way is to put $u(t) = 0$ for $t > T$. It might seem that we are doing an off-line reconstruction of u now, since we introduce future values. In fact, we are still doing an on-line reconstruction because $v(t)$, the candidate approximant of $u(t)$, is still given by the recursive formula (5). Hence at every time t, $v(t)$ is computed solely on the basis of the available pieces of information. Laplace transformation is only used in the proofs.

The result that we prove in this section is as follows:

Theorem 3 *Let the unknown input u be locally square integrable on $t > 0$ and let the function $v_\alpha(t)$ be defined in (5). Under assumptions* **A)**, **B)** *and* **C)** *stated in Section 1.1 we have*

$$\lim_{\alpha \to 0+} v_\alpha(t) = u(t)$$

in $L^2(0, T)$, for every $T > 0$.

<u>Proof</u>. For each $\alpha > 0$ the operator $A - \frac{1}{\alpha}[BB^*]^2$ generates a holomorphic semigroup which is exponentially stable and

$$\left(\lambda I - A + \frac{1}{\alpha}[BB^*]^2\right)^{-1}$$

is holomorphic in $\Re e\, \lambda > -s_0$. The number s_0 does not depend on $\alpha > 0$.

As we said, we consider that $u(t)$ has been extended to $[0, +\infty)$ so to have a square integrable function. This implies that for each α the functions $e(t)$ and $v(t)$ are square integrable. Even more, if the Laplace transform of u exists in $\Re e\, \lambda > \omega$ then also $\hat{e}(\lambda)$ and $\hat{v}(\lambda)$ do exist in $\Re e\, \lambda > \min\{\omega, -s_0\}$.

We have

$$\hat{e}(\lambda) = -\left[\lambda I - A + \frac{1}{\alpha}[BB^*]^2\right]^{-1} B\hat{u}(\lambda)\,, \qquad B\hat{v}(\lambda) = -\frac{1}{\alpha}[BB^*]^2 \hat{e}(\lambda)$$

so that $$B\hat{v}(\lambda) = \frac{1}{\alpha}[BB^*]^2 \left[\lambda I - A + \frac{1}{\alpha}[BB^*]^2\right]^{-1} B\hat{u}(\lambda)\,.$$

Hence,

$$B\hat{v}(\lambda) - B\hat{u}(\lambda) = -\alpha \left[\alpha I + [BB^*]^2(\lambda I - A)^{-1}\right]^{-1} B\hat{u}(\lambda)\,. \qquad (7)$$

We recall that B^*B is an invertible finite dimensional operator so that

$$\begin{aligned} & B\hat{v}(\lambda) - B\hat{u}(\lambda) = -\alpha \left[\alpha I + [BB^*]^2(\lambda I - A)^{-1}\right]^{-1} BB^*B[B^*B]^{-1}\hat{u}(\lambda) \\ & = -\alpha BB^* \left[\alpha I + BB^*(\lambda I - A)^{-1}BB^*\right]^{-1} \left\{B[B^*B]^{-1}\hat{u}(\lambda)\right\}\,. \qquad (8) \end{aligned}$$

We recall that $(\lambda I - A)^{-1}$ is well defined in $\Re e\,\lambda > -s_0$ so that $[BB^*](\lambda I - A)^{-1}[BB^*]$ is a holomorphic function of λ on this halfplane. We shall prove below that $\{\alpha I + [BB^*](\lambda I - A)^{-1}[BB^*]\}^{-1}$ is well defined on $\Re e\,\lambda \geq 0$ and that we can find a number $M > 0$ such that if $\Re e\,\lambda \geq 0$ then

$$\alpha \left|\left|\left[\alpha I + [BB^*](\lambda I - A)^{-1}[BB^*]\right]^{-1}\right|\right| < M\,. \qquad (9)$$

Furthermore we shall see that for each fixed λ in $\Re e\,\lambda \geq 0$ and every u we have

$$\lim_{\alpha \to 0+} \alpha BB^* \left[\alpha I + BB^*(\lambda I - A)^{-1}BB^*\right]^{-1} \left\{B[B^*B]^{-1}u\right\} = 0\,. \qquad (10)$$

In particular $\hat{v}(\lambda) - \hat{u}(\lambda)$, hence also $\hat{v}(\lambda)$, belongs to H^2. The space here denoted H^2 is the Hardy space of the image of $L^2(0, +\infty)$ under Laplace transform. Its elements are holomorphic functions in $\Re e\,\lambda > 0$ and we need to know (see [10]): every $\hat{f}(\lambda) \in H^2$ admits a trace on the imaginary axis (defined a.e. as a horizontal limit, $\hat{f}(iy) = \lim_{x \to 0+} \hat{f}(x + iy)$) and the norm of $\hat{f}$ in H^2 is

$$\int_{-\infty}^{+\infty} |\hat{f}(iy)|^2 \, \mathrm{d}y\,.$$

See [10].

Thanks to Parseval identity, the norms in $L^2(0, +\infty)$ and that in H^2 are equivalent so that we can use Parseval identity (on the imaginary axis) in order to prove that $||v - u||_{L^2}$ converges to zero. It is sufficient to note that $B[v - u]$ tends to zero since B, acting on a finite dimensional space, has zero kernel.

In fact, from (8),

$$\begin{aligned}&\int_{-\infty}^{+\infty} ||B\,[\hat{v}(i\omega) - \hat{u}(i\omega)]\,||^2 \,\mathrm{d}\omega \\ &= \int_{-\infty}^{+\infty} \left\{ \left|\left| \alpha BB^* \left[\alpha I + BB^*(i\omega I - A)^{-1} BB^*\right]^{-1} \right.\right.\right. \\ &\left.\left.\left. \left\{ B[B^*B]^{-1}\hat{u}(i\omega)\right\} \right|\right|^2 \right\} \,\mathrm{d}\omega \to 0\end{aligned}$$

for $\alpha \to 0+$, thanks to Lebesgue Theorem, using (9), (10) and the integrability of $||\hat{u}(i\omega)||^2$. This is the result we wanted to achieve.

In order to complete the proof, we prove inequality (9) and (10). We consider inequality (9) first. The operators acts between finite dimensional spaces, so that it is sufficient to prove the existence of the inverse and the inequality. We do both these proofs in one shot, arguing by contradiction. If the inequality does not hold then we can find a sequence $\{x_n\}$ of vectors with unitary norm and sequences $\{\lambda_n\}$ in $\Re e\,\lambda \geq 0$ and $\alpha_n \to 0+$ such that

$$[I + \frac{1}{\alpha_n}[BB^*](\lambda_n I - A)^{-1}[BB^*]\,]x_n \to 0\,.$$

The sequences are constant when the inverse does not exists for one value of α and λ.

We take the scalar product with x_n and we get

$$\begin{aligned}&\langle x_n, \left\{ I + \frac{1}{\alpha_n}[BB^*](\lambda_n I - A)^{-1}[BB^*]\, \right\} x_n\rangle \\ &+\langle \left\{ I + \frac{1}{\alpha_n}[BB^*](\lambda_n I - A)^{-1}[BB^*]\, \right\} x_n, x_n\rangle \longrightarrow 0\,.\end{aligned}$$

The left hand side is equal to

$$\begin{aligned}&= \langle x_n, \left\{ I + \frac{1}{\alpha_n}[BB^*](\lambda_n I - A)^{-1}[BB^*]\, \right\} x_n\rangle \\ &+\langle x_n, \left\{ I + \frac{1}{\alpha_n}[BB^*](\bar{\lambda}_n I - A)^{-1}[BB^*]\, \right\} x_n\rangle \\ &= 2||x_n||^2 + \frac{(2\Re e\,\lambda_n)}{\alpha_n}\langle x_n, [BB^*](\lambda_n I - A)^{-1}(\bar{\lambda}_n I - A)^{-1}[BB^*]x_n\rangle \\ &-\frac{2}{\alpha_n}\langle x_n, [BB^*](\lambda_n I - A)^{-1}A(\bar{\lambda}_n I - A)^{-1}[BB^*]x_n\rangle \geq 2\end{aligned}$$

since $||x_n|| = 1$ and both the second and third terms are nonnegative (we recall $A = A^* \leq 0$).

We now prove (10). It is sufficient to prove boundedness of

$$B^*\left[\alpha I - BB^*(\lambda I - A)^{-1}BB^*\right]^{-1}$$
$$= \left[\alpha I + B^*BB^*(\lambda I - A)^{-1}B\right]^{-1} B^*B$$

is bounded for every fixed λ and $\alpha \to 0+$. We proceed by contradiction: if boundedness does not hold then, passing to the inverses, there exists a sequence $\alpha_n \to 0+$ and a sequence of vectors $\{u_n\}$ of unitary norm and such that

$$\left[\alpha_n I + B^*BB^*(\lambda I - A)^{-1}B\right] u_n \to 0\,.$$

We use the fact that u_n belongs to a finite dimensional space and we extract a subsequence which converges to a vector u_0 of norm 1 (hence $u_0 \neq 0$).

$$0 = B^*BB^*(\lambda I - A)^{-1}Bu_0 \qquad \text{so that} \qquad B^*(\lambda I - A)^{-1}Bu_0 = 0$$

(we use again invertibility of B^*B). The last equality is impossible, see Assumption **C**.

Remark 4 We examine the role of the operator B^*B in the previous computations. This operator is selfadjoint positive. Hence there exists an ortogonal transformation of coordinates in the space $U = Y$ under which it takes a diagonal form,

$$B^*B = \mathrm{diag}\left[\begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{array}\right] \qquad \lambda_i > 0\,.$$

It is seen from here that the role of $[B^*B]^{-1}$ in formula (8) is to assign different weights to different entries of the vector u. These different weights are all of the same order for $\alpha \to 0+$ and it is to be expected that they do not have an influence in the proof of the consistency result in Theorem 3. This can be easily checked: even the expression of v_α which is obtained by replacing B^*B with I converges to u (of course, the error committed when a value of α is fixed is not the same). We shall make use of this observation in Section 4. ∎

4 Boundary inputs and boundary observations

Pollutants always enter from a certain layer and measures are a spatial average so that the setup we have considered up to know is realistic for the problem under consideration. But this layer can be very thin and it might be considered as a part of the surface bounding the body. For example, if the pollutant enters a lake from a pipe, we can describe the mouth of the pipe as a geometrical surface, a part of the boundary of the lake. The previous arguments can be adapted to cover this case too, as we indicate now.

We first describe the general setup and then we show that the example in Section 1.1 (with boundary input and observation) is a special case of this. The setup is as follows: we have the generator A of a holomorphic semigroup on a Hilbert space X (we can reduce ourselves to the case that the semigroup is exponentially stable) and an operator $N \in \mathcal{L}(U, X)$. We assume that there exists $\gamma \in [0, 1)$ such that $(-A)^\gamma N \in \mathcal{L}(U, X)$ (the case $\gamma = 0$ is the case considered in the previous sections).

The system is now

$$\dot{x} = Ax - ANu\,, \quad x(0) = 0\,, \qquad y = N^* x. \tag{11}$$

The differential equation (11) is to be read in $(\mathrm{dom}\, A)'$ and defines the function

$$x(t) = -A \int_0^t e^{A(t-s)} Nu(s)\, \mathrm{d}s \tag{12}$$

for every $T > 0$. Thanks to Young inequalities, it is possible to prove that the transformation $u(\cdot) \to x(\cdot)$ defined by (12) belongs to $\mathcal{L}\left(L^2(0, T; U), L^2(0, T, X)\right)$ (see [12, p. 194]) so that the observation y now makes sense since $C = N^* \in \mathcal{L}(X, U)$. In fact, the function $x(t)$ is even more regular, but we are not interested in this.

Now we see that this model can be used in order to represent the diffusion of pollutants from the boundary.

As in [12], we consider the Hilbert space $X = H^1(\Omega)$ and the operator A in (11) is

$$\mathrm{dom}\, A = \{x \in H^2(\Omega)\,,\ (x_\nu)_{|\Gamma} = 0\} \qquad Ax = \Delta x\,.$$

Here ν is the exterior normal to $\Gamma = \partial\Omega$ and $(x_\nu)_{|\Gamma}$ is the exterior normal derivative $(\partial/\partial\nu)x$ computed on Γ (assumed sufficiently regular. For simplicity we consider the case of one kind of pollutant).

The operator N is defined by $\tilde{N}\tilde{B}u$ where $\tilde{B} \in \mathcal{L}(\mathbb{R}^m, L^2(\Gamma))$ (recall the standing assumption that $u(t)$ takes values in $\mathbb{R}^m$) and $\tilde{N}$ is defined by $\tilde{N}\omega = \phi$ where ϕ is the solution of the Neumann problem

$$\Delta\phi = 0 \quad \text{in } \Omega, \qquad \phi_\nu = \omega \quad \text{on } \Gamma.$$

The definition of the weak solution $\phi = \tilde{N}\omega$ is

$$\langle \omega, \gamma_0\phi\rangle_{L^2(\Gamma)} = \int_\Gamma \omega\psi \, \mathrm{d}\Gamma = \int_\Omega (\nabla \tilde{N}\omega)\cdot(\nabla\psi)\, \mathrm{d}\xi = \langle \tilde{N}\omega, \psi\rangle_{H^1(\Omega)}$$

for every $\psi \in H^1(\omega)$ so that $\tilde{N}^* = \gamma_0$ is the trace on Γ and Eq. (11) represents the problem

$$x_t = \Delta x\,, \qquad x_{\nu_{|_\Gamma}} = \tilde{B}u\,, \qquad y = \tilde{B}^* x_\Gamma$$

(and $x(0) = 0$). Formally, $B = AN$, see [12, Sect. 3.3] for details.

The important fact to be noted is that $\operatorname{im} N \subseteq H^{3/2}(\Omega) \subseteq \operatorname{dom}(-A)^\gamma$ with $\gamma < 3/4$, i.e. $\gamma < 1$.

Now we go back to the general formulation and we adapt the algorithm in the previous sections to the case of system (11). We consider the generator A of a holomorphic exponentially stable semigroup and we assume $\operatorname{im} N \subseteq \operatorname{dom}(-A)^\gamma$, without further references to the specific operators in the example. (The semigroup in this example is not exponentially stable but, as we noted, we can always reduce ourselves to this case).

We first consider the definition of $v = v_\alpha$. We introduce the model of the system, which is now

$$\dot{w} = Aw - ANv\,, \qquad z = N^* w\,. \tag{13}$$

Formula (5) cannot be used since $CB = -N^*AN$ is not defined. In fact, AN does not take values in X. However, following the suggestion in Remark 4, we prove that the simpler formula

$$v = v_\alpha = -\frac{1}{\alpha}[z(t) - y(t)] = -\frac{1}{\alpha}N^*[w(t) - x(t)] \tag{14}$$

will do. Namely, we prove a statement analogous to that of Theorem 3. The assumptions are the obvious modifications of those in Theorem 3:

Theorem 5 *Let the unknown input u be locally square integrable on $t > 0$ and let the function $v_\alpha(t)$ be defined in (14). Assumptions are: Assumption* **A)** *with* **A3)** *replaced by $B = AN$, $\operatorname{im} N \subseteq \operatorname{dom}(-A)^\gamma$, $C = N^*$; assumption* **B)** *and*

$$\det N^* A(\lambda I - A)^{-1} N \neq 0 \qquad \text{in} \quad \Re e\, \lambda \geq 0\,.$$

Under these conditions we have

$$\lim_{\alpha \to 0+} v_\alpha(t) = u(t)$$

in $L^2(0, T)$, for every $T > 0$.

Note that the closed loop of (13) with the feedback (14) is

$$\dot{w} = A\left(I + \frac{1}{\alpha}NN^*\right)w - \frac{1}{\alpha}ANN^*x \tag{15}$$

For the moment this formula relies on a formal computation, whose significance depends on the following Lemma:

Lemma 6 *Both the operator*

$$\begin{array}{l} \operatorname{dom} A_\alpha = \{x \in X \;:\; (I + (1/\alpha)NN^*)x \in \operatorname{dom} A\}\;, \\ A_\alpha x = A(I + (1/\alpha)NN^*)x \end{array} \tag{16}$$

and the operator

$$(I + NN^*)A \tag{17}$$

define holomorphic exponentially stable semigroups on X.

In order not to interrupt our arguments now, the proof of the Lemma and the justification of formula (15) are given in the Appendix.

We accept formula (15) and we consider the error function $e(t) = w(t) - x(t)$. Using formula (15) we see that $e(t)$ solves

$$\dot{e} = A\left(I + \frac{1}{\alpha}NN^*\right)e + ANu\,. \tag{18}$$

The arguments presented in the Appendix show that we can compute the Laplace transforms:

$$\hat{e}(\lambda) = \left[\lambda I - A - \frac{1}{\alpha}ANN^*\right]^{-1} AN\hat{u}(\lambda) \tag{19}$$

$$\begin{aligned} \hat{v}(\lambda) &= -\frac{1}{\alpha}N^*\left[I - \frac{1}{\alpha}(\lambda I - A)^{-1}ANN^*\right]^{-1}(\lambda I - A)^{-1}AN\hat{u}(\lambda) \\ &= -\frac{1}{\alpha}\left[I - \frac{1}{\alpha}N^*A(\lambda I - A)^{-1}N\right]^{-1}N^*A(\lambda I - A)^{-1}N\hat{u}(\lambda)\,. \end{aligned} \tag{20}$$

As explained in the appendix, these computations are made in $(\operatorname{dom} A)'$ and it is seen from the last line that $\hat{v}(\lambda) \in X$.

Now:

$$\begin{aligned} &\hat{v}(\lambda) - \hat{u}(\lambda) \\ &= \left\{-\frac{1}{\alpha}\left[I - \frac{1}{\alpha}N^*A(\lambda I - A)^{-1}N\right]^{-1}N^*A(\lambda I - A)^{-1}N - I\right\}\hat{u}(\lambda) \\ &= -\left[I - \frac{1}{\alpha}N^*A(\lambda I - A)^{-1}N\right]^{-1}\hat{u}(\lambda)\,. \end{aligned}$$

We now prove the existence of the inverse and of a number $M>0$ such that

$$\left\|\left[I-\frac{1}{\alpha}N^*A(\lambda I-A)^{-1}N\right]^{-1}\right\|<M \tag{21}$$

for each $\alpha>0$ and λ with $\Re e\,\lambda>0$. Furthermore, precisely the same proof as in Theorem 3 shows that for every fixed λ we have

$$\lim_{\alpha\to 0+}\left[I-\frac{1}{\alpha}N^*A(\lambda I-A)^{-1}N\right]^{-1}=0\,.$$

Once that this is known, the same argument as in Section 3, based on Parseval identity, implies that $v_\alpha\to u$ in square norm.

In order to see the existence of the inverse we note that if K is a bounded operator then (recall $A=A^*$)

$$\left[KA(\lambda I-A)^{-1}\right]^*=A(\bar\lambda I-A)^{-1}K^*\,, \tag{22}$$

$$[KA]^*=AK^*\,,\qquad \mathrm{dom}\,AK^*=\{x\ :\ K^*x\in\mathrm{dom}\,A\}\,. \tag{23}$$

Moreover, we need to compute

$$\begin{array}{l}\Re e\,\langle N^*A(\bar\lambda I-A)^{-1}Nv,v\rangle=-||A(\lambda I-A)^{-1}Nv||^2\\ -2(\Re e\,\lambda)\langle(-A)^{1-2\gamma}(\lambda I-A)^{-1}(-A)^\gamma Nv,(\lambda I-A)^{-1}(-A)^\gamma Nv\rangle\end{array}$$

so that when $\Re e\,\lambda>0$ we have

$$\Re e\,\langle N^*A(\bar\lambda I-A)^{-1}Nv,v\rangle\le 0\,. \tag{24}$$

The computation is as follows:

$$\begin{array}{l}\langle N^*A(\bar\lambda I-A)^{-1}Nv,v\rangle+\langle v,N^*A(\bar\lambda I-A)^{-1}Nv\rangle\\ =\langle N^*A(\bar\lambda I-A)^{-1}Nv,v\rangle+\langle[N^*A(\bar\lambda I-A)^{-1}]^*v,Nv\rangle\\ =\langle N^*A(\bar\lambda I-A)^{-1}Nv,v\rangle+\langle A(\lambda I-A)^{-1}Nv,Nv\rangle\qquad\text{using (22)}\\ =\langle N^*A(\bar\lambda I-A)^{-1}\left\{2\Re e\,\lambda-2A\right\}(\lambda I-A)^{-1}Nv,v\rangle\\ =2\Re e\,\lambda\langle N^*A(\bar\lambda I-A)^{-1}(\lambda I-A)^{-1}Nv,v\rangle\\ -2\langle A(\bar\lambda I-A)^{-1}A(\lambda I-A)^{-1}Nv,Nv\rangle\\ =-2\Re e\,\lambda\langle N^*(-A)^\gamma(-A)^{1-2\gamma}(\bar\lambda I-A)^{-1}(\lambda I-A)^{-1}(-A)^\gamma Nv,v\rangle\\ -2||A(\lambda I-A)^{-1}Nv||^2\,.\end{array}$$

Equality follows since, using (23), we have

$$\begin{array}{l}\langle N^*(-A)^\gamma(-A)^{1-2\gamma}(\bar\lambda I-A)^{-1}(\lambda I-A)^{-1}(-A)^\gamma Nv,v\rangle\\ =\langle(\bar\lambda I-A)^{-1}(-A)^{1-2\gamma}(\lambda I-A)^{-1}(-A)^\gamma Nv,(-A)^\gamma Nv\rangle\\ =\langle(-A)^{1-2\gamma}(\lambda I-A)^{-1}(-A)^\gamma Nv,(\lambda I-A)^{-1}(-A)^\gamma Nv\rangle\,.\end{array}$$

Now we prove (21). We show first that the operator $I - \frac{1}{\alpha} N^* A(\lambda I - A)^{-1} N$ has dense image if $\Re e\,\lambda > 0$. If v_0 is orthogonal to the image then, using (22), v_0 satisfies

$$v_0 = \frac{1}{\alpha} N^* A(\bar{\lambda} I - A)^{-1} N v_0 \,.$$

We compute the inner product of both sides with v_0 and we take the real part. We have

$$||v_0||^2 = \frac{1}{\alpha} \Re e \,\langle N^* A(\bar{\lambda} I - A)^{-1} N v_0, v_0 \rangle \le 0$$

from (24). This shows $v_0 = 0$

Now we prove in one shot the existence of the inverse and the inequality in (21). We proceed by contradiction, as in Section 3. If the number M cannot be found then there exist sequences $\alpha_n \to 0+$, $\{\lambda_n\}$ with positive real part and $\{u_n\}$, $||u_n|| = 1$, such that (the sequences are constant when the inverse does not exist)

$$\left[I - \frac{1}{\alpha_n} N^* A(\lambda_n I - A)^{-1} N \right] u_n \to 0 \,.$$

Then, the real part of the inner product with u_n tends to zero too. This is not possible because the real part is

$$1 - \frac{1}{\alpha_n} \Re e \,\langle N^* A(\lambda_n I - A)^{-1} N u_n, u_n \rangle \ge 1$$

from (24) with λ replaced by $\bar{\lambda}$.

This completes the proof of Theorem 5.

5 Noisy observation

Finally, we examine the effect of the noise in the observation. This is quite standard and we sketch the usual idea. In general the observation is corrupted by errors of known tolerance h. Hence the observation is

$$y = Cx + \theta \qquad ||\theta|| < h \qquad 0 \le t \le T \,.$$

The norm of θ is often the $L^\infty(0,T)$ norm, sometime the $L^2(0,T)$ norm. A bound on the L^∞ norm implies a bound on the L^2 norm, so that we can consider this last more general case.

The function v, the candidate approximant of u, is still given by formula (5) (alternatively, it is given by the formula (14) in the boundary case. We here comment on the distributed case leaving the simple adaptation of the arguments to the distributed case to the reader), but we have to take into account the effect of the noise in the measures of y so that we now have

$$v = -\frac{B^*C^*}{\alpha}[z-y] = -\frac{B^*C^*C}{\alpha}[w-x] + \frac{B^*C^*}{\alpha}\theta \tag{25}$$

(we recall $C = B^*$).

Although this function v is a function of α and θ, it is a common practice to denote it $v_{\alpha,h}$.

In the presence of noise in the observation we cannot expect $\lim_{\alpha\to 0+} v = u$, see [4]. Hence, the consistency result to be proved now is that $\lim v = u$ when both α and the tolerance h tends to $0+$ while respecting suitable consistency conditions.

We replace the expresion (25) of v in the definition of w, hence of e and we note that all the computations in Section 3 can be repeated, with u replaced by $u - (1/\alpha)B^*B\theta$. So, the following additional term appear in formula (8) :

$$-BB^*[\alpha I + BB^*(\lambda I - A)^{-1}BB^*]^{-1}B\theta$$

and, from (9),

$$\left|\left|[\alpha I + BB^*(\lambda I - A)^{-1}BB^*]^{-1}\right|\right| < \frac{M}{\alpha}\,.$$

Hence we get the following consistency result:

Theorem 7 *If $\alpha \to 0+$ and $h \to 0+$ while respecting the condition*

$$\frac{h}{\alpha} \to 0\,,$$

then we have $\lim v_{\alpha,h} = u$ in $L^2(0,T)$ for each=h $T > 0$.

6 Simulations

We present now few simple simulations. In the application we presented the output signal of interest is slow varying with possible sudden changes and the focus is the fast reconstruction of such sudden variations. This is the case considered in Figures 1 and 2. The example in 3 is not realistic and it is presented just to underline the problems with fast varying signals.

We consider a system (described as in Eq. (2)) in 1 space dimension, i.e. $\Omega = (0, 1)$ and $b = b(x)$ is the characteristic function of the subinterval $[2/15, 8/15]$.

The length of the time interval is 10 units, $T = 10$. The tolerance h is kept fixed, $h = 0.1$ in 1 and 2, so to show that the smaller values of the penalization parameter α amplify the effect of the noise.

The parameter α is $\alpha = 0.1$ on the right and $\alpha = 0.01$ for the left figures in the figures 1 and 2. The "unknown" functions to be reconstructed is $u(t) = \{1 - 2[1 - \text{sgn}(5 - t)]\} / (t + 1)$ (in Fig. 1), the case of a slowly varying function with a sudden change, and, in Fig. 2, $u(t) = \arctan(10(t - 5))$ (simulating a saturation). Fig. 3 presents the non realistic example $f(t) = \sin 10t$. Due to the fact that this last function is zero at $t = 0$ as $v(t)$, there is no transient at $t = 0$ but, in order to obtain comparable results, we must take small values of α hence of h, due to the oscillating nature of the signal. Namely, $\alpha = 0.01$ and $h = 0.1$ (left) and $\alpha = 0.001$ and $h = 0.01$.

The plots of the input function is dotted.

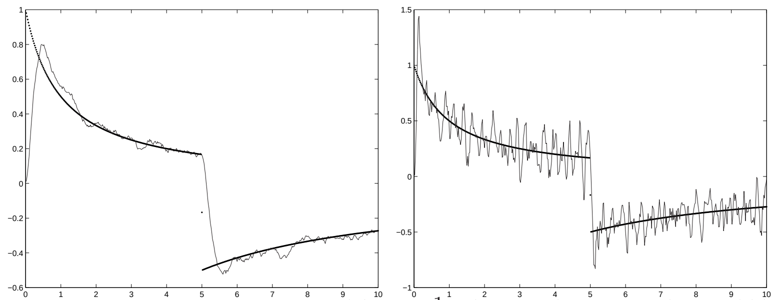

Figure 1: The function $u(t) = \frac{1}{t+1} (1 - 2(1 - \text{sgn}(5 - t)))$.

The fact to be noted is that the abrupt change is easily detected.

Finally we note that a complete analysis of the numerical issue of the method is not the focus of this paper and will be studied elsewhere. This analysis should take into account several aspects of the problem. In particular, the fact that the observations are usually performed, or elaborated, at discrete times and the fact that $w(t)$ is numerically computed with a finite numbers of time steps and space discretization. Note that in the simulation also $x(t)$ is replaced by a numerical approx-

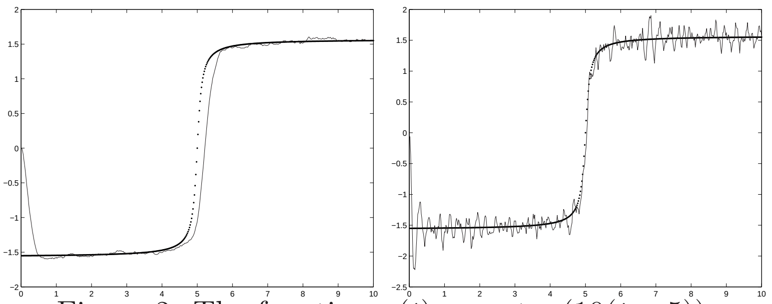

Figure 2: The function $u(t) = \arctan(10(t-5))$.

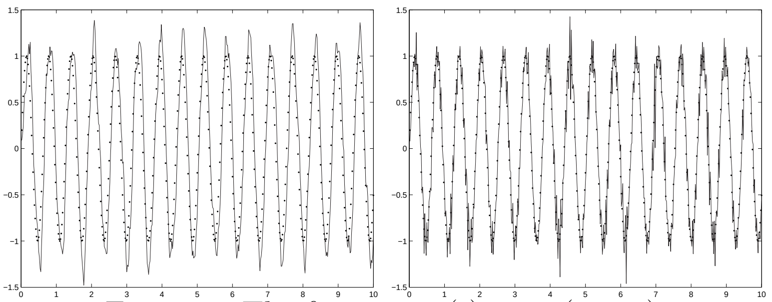

Figure 3: The function $u(t) = \sin(10 * t)$.

imation but this is not so important since this amount to increase the tolerance of the error. Instead, numerical approximation of w might interfere with the step of the observation. The delicate analysis of the interactions of these different parameters can be found, for finite dimensional systems, in [17].

Appendix

In this Appendix we first prove Lemma 6 and then we give a meaning to the formulas (15) and (18).

Proof of Lemma 6. In this proof α is fixed so that, upon redefining $(1/\sqrt{\alpha})N = N$, we can put $\alpha = 1$. Moreover we recall $A = A^*$ is negative definite, see (3). We first note that $(I + NN^*)$ is a selfadjoint bounded and boundedly invertible operator so that, using (23),

$$[(I + NN^*)A]^* = A(I + NN^*)\,.$$

The domain of this last operator is

$$\{x \,:\, (I + NN^*)x \in \operatorname{dom} A\}\,,$$

as required in Lemma 6. We prove that $\tilde{A} = (I + NN^*)A$ generates a holomorphic semigroup which is exponentially stable. It then follows that the operator $A(I + NN^*)$ generates a holomorphic exponentially stable semigroup too.

The proof that $\tilde{A}$ generates a holomorphic semigroup is based on [14, Theorem 3.7.23], i.e. on the following result. Let A generate a holomorphic semigroup. Let the linear operator $\mathcal{B}$, defined on $\operatorname{dom} A$, satisfy the following property: for every $\chi > 0$ there exists $\nu \geq 0$ such that

$$||\mathcal{B}x|| \leq \chi ||Ax|| + \nu ||x|| \text{ for every } x \in \operatorname{dom} A. \tag{26}$$

Then the operator $A + \mathcal{B}$, defined on $\operatorname{dom} A$, generates a holomorphic semigroup too. I.e., the required condition is that the A-bound of $\mathcal{B}$ should be zero.

In order to use this result, we consider

$$\mathcal{B} = NN^*A = -N[N^*(-A)^\gamma](-A)^{1-\gamma}\,.$$

We must prove that the A-bound of this operator is zero. We prove that the operator $N^*(-A)^\gamma$ has a bounded extension. Once that this is known, the A-bound of $\mathcal{B}$ is that of $(-A)^{1-\gamma}$. In order to see that $N^*(-A)^\gamma$ has a bounded extension, we prove that its adjoint is a bounded operator. This is easily seen since (23) gives

$$[N^*(-A)^\gamma]^* = (-A)^\gamma N\,,$$

a bounded operator. Now we see that the A-bound of $(-A)^{1-\gamma}$ is 0. We use inequality [3, pag. 73]: for every positive ρ we have

$$||(-A)^{1-\gamma}x|| \leq C_0 \left(\rho^{1-\gamma}||x|| + \frac{1}{\rho^\gamma}||Ax||\right) \qquad \forall x \in \operatorname{dom} A\,.$$

The constant C_0 does not depend on ρ. This implies inequality (26), as wanted. See [2] for a different proof.

Once we know that $e^{\tilde{A}t}$ is a holomorphic semigroup, in order to prove exponential stability, we prove that the spectrum of $\tilde{A} = (I + NN^*)A$ is in $\Re e\,\lambda < 0$, see [1].

We fix any λ with non negative real part. We show first that λ cannot be either an eigenvalue or an element of the continuous spectrum. In fact, in both these cases we could find a sequence $\{x_n\}$ in $\operatorname{dom} A$, $||x_n|| = 1$, such that

$$\lambda x_n - Ax_n + NN^*(-A)^\gamma(-A)^{1-\gamma}x_n \to 0$$

(if λ would be an eigenvalue then (x_n) would be a stationary sequence). We take the scalar product with $(-A)x_n$. We get

$$\lambda\langle(-A)x_n, x_n\rangle + \langle(-A)^{1-\gamma}x_n, (-A)^{\gamma}NN^*(-A)^{\gamma}(-A)^{1-\gamma}x_n\rangle + ||Ax_n||^2 .$$

Each of the three terms has non negative real part (in fact the second and third terms are real). So, if this sum tends to zero then in particular the last addendum tends to zero. This is not possible since e^{At} is exponentially stable, so that $0 \in \rho(A)$. Hence, λ is neither an eigenvalue nor a member of the continuous spectrum.

Now we prove that λ does not belong to the residual spectrum. By contradiction, let it be possible to find $y_0 \neq 0$ which is orthogonal to the closure of the image of $(\lambda I - \tilde{A})$. In this case, for every $x \in \operatorname{dom} A$ we have

$$0 = \langle y_0, [\lambda I - A - NN^*A]x\rangle = \langle y_0, [I - NN^*A(\lambda I - A)^{-1}](\lambda I - A)x\rangle .$$

Here λ is fixed while x is any element in $\operatorname{dom} A$. Hence, using the fact that A is negative definite and $\Re e\, \lambda \geq 0$, $\xi = (\lambda I - A)x$ runs over all of H when x runs over the domain of A so that

$$y_0 \in \ker[I - NN^*A(\lambda I - A)^{-1}]^* = \ker[I - A(\bar{\lambda} I - A)^{-1}NN^*]$$

i.e.

$$y_0 = A(\bar{\lambda} I - A)^{-1}NN^*y_0 . \tag{27}$$

This shows that $NN^*y_0 \neq 0$.

We take the inner product of both the sides of (27) with NN^*y_0 and we get

$$\langle NN^*y_0, A(\bar{\lambda} I - A)^{-1}NN^*y_0\rangle = ||N^*y_0||^2 > 0$$

We show that this equality is impossible if $\Re e\, \lambda \geq 0$. We compute the real part of the left hand side, which is

$$\langle NN^*y_0, A(\bar{\lambda} I - A)^{-1}\left[2\Re e\, \bar{\lambda} - 2A\right](\lambda I - A)^{-1}NN^*y_0\rangle \leq 0 .$$

The contradiction shows that λ does not belong to the residual spectrum.

We now recall that a holomorphic semigroup is exponentially stable when its spectrum is contained in $\Re e\,\lambda < 0$, and this completes the proof. ∎

Now we consider formula (15) for every fixed $\alpha > 0$. Without restriction we could put $\alpha = 1$. We keep the notation α for consistency with Section 4. But, we repeate that the computations below are made for every fixed $\alpha > 0$. Lemma 6 shows that the formula for $A^{-1}w$ makes sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{-1}w\right) = (I + \frac{1}{\alpha}NN^*)A\left(A^{-1}w\right) - \frac{1}{\alpha}NN^*x$$

so that

$$\left(A^{-1}w\right)(t) = -\int_0^t e^{(I+\frac{1}{\alpha}NN^*)A(t-s)}\frac{1}{\alpha}NN^*x(s)\,\mathrm{d}s\,.$$

Hence,

$$w(t) = -A\int_0^t e^{(I+\frac{1}{\alpha}NN^*)A(t-s)}\frac{1}{\alpha}NN^*x(s)\,\mathrm{d}s\,, \tag{28}$$

a formula which makes sense in $(\mathrm{dom}\,A)'$. A similar expression defines $e(t)$ in $(\mathrm{dom}\,A)'$ and this is sufficient to justify the computations with the Laplace transform. However, it has an interest to note that the functions $w(t)$ in (28) and $e(t)$ in (19) are X-valued functions. This is most easily seen from the expressions of the Laplace transforms. In fact equality (19) can be rewritten as

$$\begin{aligned}\hat{e}(\lambda) &= \left[\lambda I - A - \frac{1}{\alpha}ANN^*\right]^{-1}AN\hat{u}(\lambda)\\ &= \left[I - \frac{1}{\alpha}(\lambda I - A)^{-1}ANN^*\right]^{-1}(\lambda I - A)^{-1}AN\hat{u}(\lambda)\,.\end{aligned}$$

In order to prove that $e(t) \in L^2_{\mathrm{loc}}(0, +\infty; X)$, it is sufficient to prove the existence of a constant M such that $||\hat{e}(\lambda)|| < M||\hat{u}(\lambda)||$ in a suitable right half plane $\Re e\,\lambda > k$.

We note that $(\lambda I - A)^{-1}A$, as an operator on X, admits a bounded extension equal to $A(\lambda I - A)^{-1}$. Exponential stability of the semigroup e^{At} implies that we can find numbers $M > 0$ and $\sigma > 0$ such that for every λ with $\Re e\,\lambda > 0$ we have

$$\left\|A(\lambda I - A)^{-1}N\right\| = \left\|(-A)^{1-\gamma}(\lambda I - A)^{-1}(-A)^{\gamma}N\right\| \le \frac{M}{|(\lambda + \sigma)|^{\gamma}}\,.$$

Hence, the norm of $[I - \frac{1}{\alpha}(\lambda I - A)^{-1}ANN^*]^{-1}$ is bounded in a right half plane: there exists a number M_1 such that

$$\left|\left|\left[I - \frac{1}{\alpha}(\lambda I - A)^{-1}ANN^*\right]^{-1}\right|\right| < M_1$$

(note that we are studying the regularity of $e(t) = e_\alpha(t)$ with a fixed value of α. Uniformity respect to α of the estimate is not an issue now). Analogously,

$$\left|\left|(\lambda I - A)^{-1}AN\hat{u}(\lambda)\right|\right| \leq \frac{M_2}{|(\lambda+\sigma)|^\gamma}||\hat{u}(\lambda)||\,.$$

Hence, $\hat{e}(\lambda)$ is the Laplace transform of an X-valued (locally square integrable) function. The analogous statement for $w(t)$ can be derived similarly from its Laplace transform, or from $w(t) = e(t) + x(t)$.

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