Chapter 5
Systems with Persistent Memory: Controllability via Moment Methods—Problems and Solutions

Facts that are recalled in the problems

\[ z_n' = 2\alpha z_n - \lambda_n^2 \int_0^t N(t-s)z_n(s)ds, \quad z_n(0) = 1. \]  (5.1)

The laplacian \( \Delta \) (with homogeneous Dirichlet conditions in a region \( \Omega \)) has a sequence of eigenvalues, that we denote \( \{-\lambda_n^2\} \).

If \( \dim \Omega = d \) then there exist \( m_0 > 0 \) and \( m_1 > 0 \) such that

\[ m_0 \left( n^{2/d} \right) < \lambda_n^2 < m_1 \left( n^{2/d} \right). \]  (5.2)

**Theorem 5.1.** The following properties hold:

1. if \( \{e_n\} \) is a Riesz sequence in \( H \) and \( T \in \mathcal{L}(H,Y) \) has a bounded inverse, then \( \{Te_n\} \) is a Riesz sequence in \( Y \). In particular, \( \{e_n\} \) is a Riesz sequence if and only if it is the image of an orthonormal basis \( \{\chi_n\} \) of any Hilbert space, under a linear and continuous transformation, which has continuous inverse. So, if \( T \in \mathcal{L}(H,Y) \) is bounded surjective and boundedly invertible and if \( \{e_n\} \) is a Riesz basis of \( H \) then \( \{Te_n\} \) is a Riesz basis of \( Y \).

The Problems

**5.1.** Let \( u(t) \) be a continuous functions. Assume that on \([0, +\infty)\) we have

\[ 0 \leq u(t) \leq a + b \int_0^t u(s)ds. \]  (5.3)

Prove that for every \( T > 0 \) there exists \( M_T \) such that

\[ 0 \leq u(t) \leq (M_T) a. \]
Let \( z_n(t) \) and \( \zeta_n(t) \) solve the Volterra integrodifferential equations (5.4) below. Use the inequality in Problem 5.1 and prove that the sequences \( \{z_n\} \) and \( \{\zeta_n\} \) are quadratically close on every interval \([0, T]\). The equations are

\[
\begin{align*}
z'_n &= -n^2 \int_0^t z_n(s) \, ds, \quad z_n(0) = 1, \\
\zeta'_n &= -n^2 \int_0^t \left[1 - (t - s)^2\right] \zeta_n(s) \, ds, \quad \zeta_n(0) = 1.
\end{align*}
\] (5.4)

5.3. Let \( N(t) \in C^1([0, 2\pi] \times [0, 2\pi]) \) (possibly \( N(t, t) = 0 \)). Prove that the sequence \( \{g_n(t)\}_{n \in \mathbb{N}} \) where \( g_n(t) = e^{\alpha t} + \int_0^t N(t, s) e^{\alpha s} \, ds \) is a Riesz basis in \( L^2(0, 2\pi) \) (use Bari Theorem i.e. Problem 3.15).

5.4. Let \( \lambda_n^2 = n^2 \). Consider the functions \( z_n(t) \) in (5.1) (for simplicity, with \( \alpha = 0 \)). Prove that the sequence \( \{n \int_0^\infty z_n(s) \, ds\} \) is quadratically close to \( \{\sin nt\} \) on every interval \((0, T)\).

5.5. Consider the same sequence \( \{z_n(t)\} \) as in Problem 5.4. Let \( \varepsilon > 0 \). Fix any \( K > 0 \) and prove the existence of \( N = N(\varepsilon, K) > 0 \) such that the following inequality holds for any \( k \leq K \) and any \( n \geq N \):

\[
\left| \int_0^\pi z_n(t) \cos kt \, dt \right| \leq \varepsilon.
\]

5.6. Give a direct proof of the integrodifferential equation (??) of \( Z_n(t) \).

5.7. A kernel \( N(t) \) is a \textit{positive real kernel} when it is integrable on \([0, T]\) for every \( T \) and furthermore

\[
\int_0^t \nu(\tau) \int_0^\varepsilon N(\tau - s) \nu(s) \, ds \, d\tau \geq 0
\] (5.5)

for every \( t \geq 0 \) and for every function \( \nu(t) \in C^\infty([0, +\infty)) \). prove that if (5.5) holds for every function of class \( C^\infty([0, T]) \) it holds also for every \( \nu \) which belongs to \( L^2(0, T) \) for every \( T \).

5.8. Let \( N(t) \) be a positive real kernel. Prove that if

\[
z''_n = -\alpha z_n(t) - \lambda_n^2 \int_0^t N(t - s) z_n(s) \, ds, \quad z_n(0) = 1
\]

with \( \lambda_n^2 \geq 0, \alpha \geq 0 \) then \( z_n(t) \) is bounded on \([0, +\infty)\), uniformly respect to \( n \): \( |z_n(t)| \leq M \) for every \( n \) and every \( t \geq 0 \). If \( \alpha > 0 \) and if there exists \( \varepsilon \in (0, \alpha) \) such that \( e^{\varepsilon t} N(t) \) is positive real then there exists \( M > 0 \) such that \( |z_n(t)| \leq Me^{-\varepsilon t} \) for every \( t \geq 0 \) and every \( n \).
5.9. Let \( N(t) = \sum_{k=0}^{+\infty} a_k e^{-bt} \) with \( a_k \geq 0 \) and \( b_k \geq 0 \). Prove that \( N(t) \) is positive real if the sum is finite and also if \( \{a_k\} \) is bounded and \( \lim b_k \to +\infty \).

5.10. This and the next problem give a hint on the derivation of the estimate (5.2). Consider the operator \( A \) on the square \( Q = (0, \pi) \times (0, \pi) \). An orthonormal basis of \( L^2(Q) \) whose elements are eigenvalues of \( A \) was found in Problem 2.4: the elements are \((4/\pi^2)\sin nx \sin ny\), identified by the pairs \((n, m)\) of positive integers. Order these elements in a sequence as follows: first in increasing order of \( R^2 = n^2 + m^2 \) and if two pairs \((n, m)\) are on the same circumference, order with increasing \( m \). Let \( N(R) \) be the number of the eigenvalues such that \( n^2 + m^2 \leq R^2 \), i.e. the eigenvalues such that \((n, m) \in B(0, R)\) where \( B(0, R) \) is the closed disk of center \((0, 0)\) and radius \( R \). Let \( U_R \) be the union of the squares of opposite vertices \((n - 1, m - 1)\) and \((n, m)\) with \((n, m) \in B(0, R)\). Compare the areas and give an estimate for \( N(R) \).

5.11. We continue with the eigenvalues of the problem 5.10. Let \( K \) be the index of one eigenvalue \( \lambda = \lambda_K \). Let \( \nu_K \) be the number of the eigenvalues \( \lambda \) such that \(|\lambda| = |\lambda_K|\). Give an estimate of \( \nu_K \) and prove that \( \nu_K \leq 16|\lambda_K| \).

Note that
\[
|\lambda_K|^2 - \nu_K \leq N(\lambda_K) \leq |\lambda_K|^2 + \nu_K
\]
and use this observation to give an estimate of \( \lambda_K / K \).

The Solutions

Solution of Problem 5.1 Let \( v(t) = \int_0^t u(s) \, ds \). The function \( v(t) \) is differentiable and we have
\[
v' = u \leq a + bv \quad \text{so that} \quad \frac{d}{dt} e^{-bt} v(t) \leq ae^{-bt}.
\]
Using \( v(0) = 0 \) we see that
\[
0 \leq v(t) \leq \frac{a}{b} \left( e^{bt} - 1 \right)
\]
and
\[
0 \leq u(t) \leq a + bv(t) \leq ae^{bt}.
\]

Solution of Problem 5.2 Note that \( z_0(t) = \cos nt \).

Let \( e_n(t) = z_n(t) - \tilde{z}_n(t) \) so that \( e_n(0) = 0 \) and
\[
e'_n(t) = -n^2 \int_0^t \left[ 1 - (t-s)^2 \right] e_n(s) \, ds - n^2 \int_0^t (t-s)^2 \cos ns \, ds.
\]
Then we have
\[
e''_n = -n^2 e_n + 2n^2 \int_0^t (t-s) e_n(s) \, ds - 2n^2 \int_0^t (t-s) \cos ns \, ds
\]
so that
\[
e_n(t) = 2n \int_0^t \sin n(t - \tau) \left[ \int_0^{\tau} (\tau - s) e_n(s)ds - \int_0^{\tau} (\tau - s) \cos ns ds \right] d\tau =
\]
\[
= 2 \left\{ \int_0^t \left[ (t - \tau)e_n(\tau) - \cos n(t - \tau) \int_0^{\tau} e_n(s)ds \right] d\tau \right\} -
\]
\[
- 2 \int_0^t (t - \tau) \cos n\tau d\tau + 2 \int_0^t \cos n(t - \tau) \int_0^\tau \cos ns ds d\tau.
\]

We fix a time \( T > 0 \). Compute explicitly (or estimate) the integrals in the last line and show the existence of a number \( a > 0 \) which depends on \( T \) and such that
\[
2 \left\{ \int_0^t \left[ (t - \tau)e_n(\tau) - \cos n(t - \tau) \int_0^{\tau} e_n(s)ds \right] d\tau \right\} -
\]
\[
- 2 \int_0^t (t - \tau) \cos n\tau d\tau + 2 \int_0^t \cos n(t - \tau) \int_0^\tau \cos ns ds d\tau \leq a/n.
\]

Hence we have also, for a suitable number \( b \),
\[
0 \leq |e_n(t)| \leq \frac{a}{n} + b \int_0^t |e_n(s)|ds.
\]

The inequality in in Problem 5.1 shows that on \([0, T]\) the following holds (the number \( M \) depends on \( T \)):
\[
|e_n(t)| \leq \frac{M}{n}.
\]

The result follows since \( \sum_{n=1}^{\infty} (1/n^2) < +\infty \).

**Solution of Problem 5.3** Integrate by parts and prove that the following inequality holds on \([0, 2\pi]\):
\[
|e^{i\omega t} - g_n(t)| \leq M \frac{1}{n}
\]

Hence \( \{g_n(t)\}_{n\in\mathbb{Z}} \) is quadratically close to \( \{e^{i\omega t}\}_{n\in\mathbb{Z}} \) which is an orthogonal basis in \( L^2(0, 2\pi) \) whose elements have constant norm. So, \( \{e^{i\omega t}\}_{n\in\mathbb{Z}} \) is a Riesz basis. Now prove that \( \{g_n(t)\} \) is a Riesz basis as follows: let \( y(t) = \sum_{n\in\mathbb{Z}} \alpha_n g_n(t) \). If \( y(t) = 0 \) then we have
\[
y(t) + \int_0^t N(t, s)y(s)ds = 0. \quad (5.6)
\]

This is a Volterra integral equation (not of convolution type). It has the unique solution \( y = 0 \). So we have also
\[
\sum_{n\in\mathbb{Z}} \alpha_n e^{i\omega t} = 0 \quad \text{and so} \quad \{\alpha_n\} = 0.
\]

This implies that \( \{g_n(t)\} \) is \( \omega \)-independent and quadratically close to the *Riesz basis* \( \{e^{i\omega t}\} \) of \( L^2(0, 2\pi) \). Hence it is a Riesz basis too, see Problem 3.15.

The fact that the Volterra integral equation (5.6) has only the solution \( y = 0 \) follows because \( |N(t, s)| \leq M \) for \( 0 \leq s \leq t \leq T \) and so
\[
0 \leq |y(t)| \leq \int_0^t M|y(s)|ds.
\]
This is inequality (5.3) with \( a = 0 \) and so \( y = 0 \).

A different proof is as follows. Note that

\[
g(t) = f(t) + \int_0^t N(t,s)f(s)ds
\]

is a Volterra integral equation not of convolution type. It is still true that the transformation

\[
f(t) \mapsto \left[ f(t) + \int_0^t N(t,s)f(s)ds \right]
\]

is surjective, bounded and boundedly invertible in \( L^2(0,2\pi) \). Using this observation and item 1 in Theorem 5.1 we get that \( \{g_n(t)\} \) is a Riesz basis of \( L^2(0,2\pi) \).

**Solution of Problem 5.4** The equation of \( z_n(t) \) is (5.1) (with \( \alpha = 0 \)) and so \( z_n(t) \) solve

\[
z_n(t) = \cos nt - \int_0^t N'(t-r)z_n(r)dr + \int_0^t \cos ns \int_r^t N''(r)z_n(t-s-r)dr \, ds.
\]

Gronwall inequality shows that \( \{z_n(t)\} \) is bounded on every interval \([0,T]\). Introduce the notations

\[
Z_n(t) = n \int_0^t z_n(s)ds = H * z_n, \quad N_1(t) = N'(t), \quad C_n(t) = \cos nt, \quad S_n(t) = \sin nt.
\]

Then we have

\[
Z_n = S_n - N' * Z_n + N'' * C_n * Z_n. \tag{5.7}
\]

Gronwall inequality implies that \( \{Z_n(t)\} \) is bounded on \([0,T]\).

Rewrite (5.7) as

\[
(Z_n - S_n) + N' * (Z_n - S_n) = -N' * S_n + N'' * C_n * Z_n. \tag{5.8}
\]

Integrate by parts integrals in the right hand side and show that

\[
| -N' * S_n - N'' * C_n * Z_n | \leq M/n.
\]

Use

\[
(Z_n - S_n) = -N' * S_n + N'' * C_n * Z_n - L * \left( -N' * S_n + N'' * C_n * Z_n \right)
\]

(where \( L \) is the resolvent kernel of \( -N' \)) to get that \( |Z_n - S_n| \leq M/n \).

**Solution of Problem 5.5**

Use the orthogonality of the sequence \( \{\cos nt\} \) and note that when \( n \geq k + 1 > k \)

\[
\int_0^\pi z_n(t) \cos kt \, dt = \int_0^\pi (\cos kt)(z_n(t) - \cos nt) \, dt + \int_0^\pi \cos kt \cos nt \, dt
\]

\[
= \int_0^\pi (\cos kt)(z_n(t) - \cos nt) \, dt.
\]
Here we used
\[ \int_0^\infty \cos kt \cos nt \, dt = 0 \]
and this requires \( n > K \). Now use boundedness of \( \{ \cos kt \} \) and the property \( |z_n(t) - \cos nt| \leq M/n, \) an inequality with holds on \( (0, \pi) \). The number \( M \) does not depend on \( n \). The number \( N = N(\varepsilon, K) \) is any number such that
\[ N > K \quad \text{and} \quad \frac{M}{N} < \varepsilon. \]

**Solution of Problem 5.6** Compute the derivative of \( Z_n(t) \) and use the equation of \( Z_n'(t) \), as follows:

\[
Z_n'(t) = z_n'(t) + K_n(t) + \int_0^t K_n(s)z_n'(t-s) \, ds = 2\alpha z_n(t) - \lambda_n^2 \int_0^t N(t-s)z_n(s) \, ds + K_n(t) + \\
+ \int_0^t K_n(s) \left( 2\alpha z_n(t-s) - \lambda_n^2 \int_0^{t-s} N(t-s-r)z_n(r) \, dr \right) \, ds.
\]

**Solution of Problem 5.7**
We use Young inequality: if \( N \in L^1(0,T) \) and \( v \in L^2(0,T) \) then
\[ N \ast v \in L^2(0,T), \quad |N \ast v|_{L^2(0,T)} \leq |N|_{L^1(0,T)} |v|_{L^2(0,T)}. \]
In particular, the transformation \( v \mapsto N \ast v \) is continuous from \( L^2(0,T) \) to itself.
So,
\[ |v \ast N \ast v - v_0 \ast N \ast v_0|_{L^2(0,T)} \leq |v \ast N \ast (v - v_0)|_{L^2(0,T)} + |(v - v_0) \ast N \ast v_0|_{L^2(0,T)}. \]
This inequality implies continuity of the transformation
\[ v \mapsto v \ast N \ast v, \quad L^2(0,T) \mapsto L^2(0,T). \]
The result follows because \( C^\infty([0,T]) \) is dense in \( L^2(0,T) \) for every \( T > 0 \).

**Solution of Problem 5.8** Multiply both the sides with \( z(t) \) to get
\[
\frac{1}{2} \frac{dz_n^2(t)}{dt} = -\alpha z_n^2(t) - \lambda_n^2 z_n(t) \int_0^t N(t-s)z_n(s) \, ds.
\]
Let \( \alpha \geq 0 \). Integrate on \([0,T]\) both the sides to get
\[
\frac{1}{2} z_n^2(T) - \frac{1}{2} = -\alpha \int_0^T z_n^2(s) \, ds - \lambda_n^2 \int_0^T z_n(t) \int_0^t N(t-s)z_n(s) \, ds \, dt \leq 0
\]
so that for every \( T \geq 0 \) we have \( 0 \leq z_n^2(T) \leq 1. \)

If \( \alpha > 0 \) and \( e^{\alpha N(t)} \) is positive real we first write
So we have

\[
\frac{d}{dt} z_n^2(t) + 2 \varepsilon z_n^2(t) = -2(\alpha - \varepsilon) z_n^2(t) - 2 \lambda_n^2 z_n(t) \int_0^t N(t-s)z_n(s)ds.
\]

This equality can be written as

\[
\frac{d}{dt} e^{2\varepsilon t} z_n^2(t) = -2(\alpha - \varepsilon) [e^{2\varepsilon t} z_n^2(t)] - 2 \lambda_n^2 [e^{\varepsilon t} z_n(t)] \int_0^t e^{(t-s)} N(t-s)e^{\varepsilon t}z_n(s)ds.
\]

We integrate both the sides and, thanks to \(\alpha - \varepsilon > 0\), we get

\[
\frac{d}{dt} e^{2\varepsilon t} z_n^2(t) \leq 0.
\]

so that

\[
z_n^2(t) \leq e^{-2\varepsilon t}.
\]

**Solution of Problem 5.9** We consider the case that the sum is finite. Of course it is sufficient to show that \(N(t) = e^{-bt}\) is a positive real kernel when \(b \geq 0\). We study the special case \(b = 0\), i.e. \(N(t) \equiv 1\). In this case,

\[
\int_0^t v(\tau) \left[ \int_0^\tau N(\tau - s)v(s)ds \right] d\tau = \int_0^t v(\tau) \left[ \int_0^\tau v(s)ds \right] d\tau = \int_0^t y(\tau)y(\tau)d\tau
\]

where

\[
y(\tau) = \int_0^\tau v(s)ds.
\]

So we have

\[
\int_0^t v(\tau) \left[ \int_0^\tau v(s)ds \right] d\tau = \frac{1}{2} \int_0^t \frac{dy^2(\tau)}{d\tau} d\tau = \frac{1}{2} y^2(t) - \frac{1}{2} y^2(0) = \frac{1}{2} y^2(t) \geq 0.
\]

Now we consider \(N(t) = e^{-bt}\) with \(b > 0\). In this case,

\[
\int_0^t v(\tau) \left[ \int_0^\tau N(\tau - s)v(s)ds \right] d\tau = \int_0^t v(\tau) \left[ \int_0^\tau e^{-b(\tau-s)}v(s)ds \right] d\tau.
\]

Let

\[
y(t) = \int_0^t e^{-b(t-s)}v(s)ds \implies v(t) = y'(t) + by(t)
\]

so that, using \(N(t) = e^{-bt}\) and \(b > 0\),

\[
\int_0^t v(\tau) \left[ \int_0^\tau N(\tau - s)v(s)ds \right] d\tau = \int_0^t y'(\tau)y(\tau)d\tau + b \int_0^t y^2(\tau)d\tau \geq 0.
\]

The case of the series: under the stated condition,

\[
\sum_{n=1}^{\infty} a_n e^{-bt} = \lim_{N \to \infty} \sum_{n=0}^{N} a_n e^{-bt}
\]
and the limit is uniform on every interval $[0, t]$ (in fact also on $[0, +\infty)$). So, for every fixed $t$,

$$
\int_0^t v(\tau) \left[ \sum_{n=1}^{+\infty} a_n e^{-b_n (\tau - s)} \right] v(s) ds d\tau = \lim_{N \to +\infty} \int_0^t v(\tau) \left[ \sum_{n=0}^{N} a_n e^{-b_n (\tau - s)} \right] v(s) ds d\tau \geq 0 .
$$

**Solution of Problem 5.10** It is convenient to ignore the minus sign and to intend $\lambda_{n,m}^2 = n^2 + m^2$. Hence the eigenvalues are in correspondence with the point of integer coordinates (both positive).

The eigenvalue $\lambda_{n,m}^2$ will have a certain index in the sequence of the eigenvalues, ordered as stated. We denote

$$
\mathcal{I} (\lambda_{n,m})
$$

its index.

The number $N(R)$ is equal to the number of the vertices $(n, m)$ which are used in the construction of $U_R$ and this number in turn is the area of $U_R$ since every component square has area equal 1. So,

$$
N(R) \leq \pi R^2 .
$$

From the other side, when $R$ is sufficiently large, we have $B(0, R/2) \subseteq U_R$ and so we have also $\pi (R^2 / 16) \leq N(R)$.

In conclusion,

$$
\frac{R^2}{16} \leq \frac{1}{16} R^2 \leq N(R) \leq \pi R^2 \leq 4R^2 .
$$

Let us fix our attention to a certain eigenvalue $\lambda_{n_0,m_0}^2$, which is the first one which lays on the circumference of radius $\sqrt{n_0^2 + m_0^2}$. Its index is 1 plus the number of the eigenvalues with $(n, m)$ in the interior of the disk, i.e.

$$
\mathcal{I} (\lambda_{n_0,m_0}) = N(\sqrt{n_0^2 + m_0^2}) + 1
$$

and so, for $n_0$, $m_0$ large,

$$
\frac{1}{16} (n_0^2 + m_0^2) \leq \mathcal{I} (\lambda_{n_0,m_0}) \leq 5 (n_0^2 + m_0^2) .
$$

But, $\lambda_{n_0,m_0}^2 = n_0^2 + m_0^2$ so that

$$
\frac{1}{16} \lambda_{n_0,m_0}^2 \leq \mathcal{I} (\lambda_{n_0,m_0}^2) \leq 5 \lambda_{n_0,m_0}^2 .
$$

Hence we have also
This is the asymptotic estimate (5.2) in the case \( d = 2 \), for the first eigenvalue which lays on its circumference. The next exercise extends the estimate to every eigenvalue on the circumference.

**Solution of Problem 5.11** The notations are as in Problem 5.10. We must give an estimate of

\[
\frac{I(\lambda_{n,0}; m_0)}{\lambda_{n,0}^2} \leq \frac{\mathcal{I}(n^2 + m_0^2)}{\lambda_{n,0}^2}.
\]

when

\[
\lambda_{n,0}^2 = n^2 + m_0^2 = n_0^2 + m_0^2 = \lambda_{n_0, m_0}^2.
\]

The number of these eigenvalues on the circumference of radius \( \lambda_{n_0, m_0} \) is the area of the squares within the circumference and with the “right up” vertex at \((n, m)\) on the circumference. This number is less than the area enclosed by the two circumferences of radius \( \lambda_{n_0}^2 + m_0^2 \) and \( \lambda_{n_0}^2 + m_0^2 - 2 \), hence it is less then

\[
4\pi \lambda_{n_0}^2 - 4.
\]

So we have

\[
\mathcal{I}(n^2 + m_0^2) \leq \mathcal{I}(\lambda_{n,0}^2) \leq \mathcal{I}(\lambda_{n_0, m_0}^2) + 4\pi \lambda_{n_0} - 4.
\]

We divide with \( \lambda_{n_0, m_0}^2 = \lambda_{n,0}^2 \) and, using the estimate in Problem 5.10, we find the required asymptotic estimate

\[
\frac{1}{16} \leq \frac{I(\lambda_{n_0, m_0})}{\lambda_{n_0, m_0}^2} \leq \frac{\mathcal{I}(\lambda_{n,0}^2)}{\lambda_{n_0, m_0}^2} \leq \frac{\mathcal{I}(\lambda_{n_0, m_0})}{\lambda_{n_0, m_0}^2} + \frac{4\pi}{\lambda_{n_0, m_0}} - \frac{4}{\lambda_{n_0, m_0}^2} \leq 6.
\]

This inequality holds when \( \lambda_{n_0, m_0} \) is large enough (in fact if \( \lambda_{n_0, m_0} > 4\pi \)).

The required inequalities follow.