Chapter 1
An Example—Problems and Solutions

Facts that are recalled in the problems

\[
\begin{align*}
    u'' &= u_{xx} + F(x,t), \quad t > 0, \quad x > 0 \quad (1.1) \\
    u_1(x,t) &= \frac{1}{2} \left[ \phi(x+t) + \phi(x-t) \right] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \quad (1.2) \\
    w'' &= w_{xx} + \int_0^t K(t-s) w(x,s) ds, \quad x > 0, \quad t > 0. \quad (1.3)
\end{align*}
\]

We put

\[
y(x,t) = \psi(x,t) + \frac{1}{2} \int_{D(x,t)} e^{-\gamma(t-\tau)} \int_0^T \left[ e^{-\gamma(t-s)} K(\tau-s) \right] y(\xi,s) ds d\xi d\tau. \quad (1.4)
\]

Then \( y \) solves

\[
y(x,T) = \psi(x,T) + \frac{1}{2} \int_0^T \int_0^T G_1(T,x,s,\xi) \left( \sum_{k=0}^{\infty} \frac{1}{2^k} G^k \psi \right) d\xi ds. \quad (1.5)
\]

The kernel \( G_1 \) is bounded on bounded sets.

So, a target \( \xi \) can be reached (by \( y \) at time \( T \)) if and only if the following equality holds:

\[
    \xi(x) = y(x,T) = \psi(x,T) + \frac{1}{2} \int_0^T \int_0^T G(T,x,s,\xi) \left( \tilde{G} \psi \right) d\xi ds. \quad (1.6)
\]

The explicit formula of \( G_1 \) is

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Consider Eq. (1.1) with 

\[ w(t) = G(t, x, s, \xi) y(\xi, s) ds = \]

\[ = \int_0^t \int_0^{x+t} H(t, x, s, \xi) y(\xi, s) ds d\xi \]

\[ = \int_0^t \int_0^{x+t-\tau} H(t, x, s, \xi) y(\xi, s) ds d\xi d\tau = \]

\[ = \int_0^t \int_0^{x+t} H(x+t-\tau-\xi) H(\xi-|x-t+\tau|) \int_0^\tau H(\tau-s) \tilde{K}(t, \tau, s) y(\xi, s) ds d\xi d\tau = \]

\[ \int_{D(x,t)} \int_0^\tau \tilde{K}(t, \tau, s) y(\xi, s) ds d\xi d\tau \]  

(1.7)

where 

\[ \tilde{K}(t, \tau, s) = e^{-\gamma(t-s)} K(\tau-s) . \]

**The problems**

1.1. Let \( \xi(x) \) and \( \eta(x) \) be smooth in \([0, +\infty)\) and let \( f \) be smooth in \( t \geq 0 \). Prove that

\[ u(x, t) = f(t-x) H(t-x) + \frac{1}{2} [ \tilde{\xi}(x+t) + \tilde{\xi}(x-t) ] + \frac{1}{2} [ \tilde{\eta}(x+t) - \tilde{\eta}(x-t) ] \]  

(1.8)

is the mild solution of (1.1) with \( w(0, t) = f(t) \) and \( w(x, 0) = \tilde{\xi}(x) \), \( w'(x, 0) = \tilde{\eta}(x) \), \( F = 0 \) if \( \xi \) and \( \eta \) are suitable extensions of \( \xi \) and \( \eta \). Explain which extension.

Give conditions under which \( u(x, t) \) is continuous in \( x \geq 0 \) and \( t \geq 0 \).

1.2. It is known that the derivative is a continuous transformation from \( L^2(0, +\infty) \) to \( H^{-1}(0, +\infty) \). Use formula (1.8) to prove that the transformations \( (\xi, \eta, f) \mapsto u \) and \( (\xi, \eta, f) \mapsto u_x \) are continuous transformations from \( L^2(0, +\infty) \times L^2(0, +\infty) \times L^2(0, +\infty) \) to, respectively, \( C([0, T], L^2(0, T)) \) and \( C([0, T], H^{-1}(0, T)) \), for every \( T > 0 \).

1.3. Study the controllability of system (1.1) when the boundary control \( f \) is more regular, for example when \( f \in C([0, +\infty)) \). What can be deduced for the controllability properties of system (1.3)?

1.4. Consider Eq. (1.1) with \( u_0 = 0 \), \( v_0 = 0 \) and \( F = 0 \), but impose the condition \( u_x(0, t) = f(t) \) (Neumann boundary condition). Describe the reachable set at time \( T \) when \( f(t) \) is continuous or square integrable.

1.5. Use the expression for \( u(x, t) \) found in Problem 1.4, to give a formula analogous to (1.5) when the boundary condition of (1.3) is \( w_x(0, t) = f(t) \).

1.6. Study the controllability of system (1.3) with the control \( w_x(0, t) = f(t) \).
1.7. Prove that if \( \{ e_n \} \) is an orthonormal basis of a Hilbert space, then \( e_n \rightharpoonup 0 \).

1.8. Find a sequence \( \{ x_n \} \) in a Hilbert space such that \( x_n \rightharpoonup x_0 \) and such that for no subsequence we have \( \lim_{n \to \infty} |x_n| = |x_0| \).

1.9. Use the fact that every weakly convergent sequence is bounded to prove that every compact operator is continuous.

1.10. Let \( H \) be a Hilbert space and let \( \{ e_n \} \) be an orthonormal basis of \( H \). Let \( T \) be the operator

\[
T \left( \sum_{n=1}^{\infty} \alpha_n e_n \right) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e_n.
\]

Prove that this operator is continuous.

1.11. Let \( \{ e_n \} \) be an orthonormal basis of a Hilbert space \( H \). Note that if \( \{ x_n \} \) is a sequence in \( H \) such that \( x_n \rightharpoonup 0 \) then \( \lim_{n \to \infty} \langle x_n, e_k \rangle = 0 \) for every \( k \). Use this observation to prove that the linear operator \( T \) in problem 1.10 is compact.

1.12. Prove that the Volterra operator on \( L^2(0, T) \):

\[
x \mapsto \int_0^T F(t-s)x(s)\,ds
\]

with \( F(t) \in C^1([0, T]) \) is compact.

1.13. Prove that the following operators \( A : L^2(0, 1) \) to itself are not closed:

1. \( Ax = x' \) with domain \( C^1(-1, 1) \).
2. \( Ax = x'' \) with domain \( C^2(-1, 1) \).
3. \( Ax = x(0) \) with domain \( C(-1, 1) \).

1.14. Prove that the following operators \( A : L^2(0, 1) \) to itself are closed:

1. \( A_1 x = x' \) with domain \( H^1(0, 1) \).
2. \( A_2 x = x'' \) with domain \( H^2(0, 1) \cap H^0_0(0, 1) \).

Compute the resolvent and the spectrum of these operators.

1.15. Prove that the second operator in Problem 1.14 has compact resolvent.
The Solutions

Solution of Problem 1.1 The function $\xi(x)$ is the odd extension while $\eta(x)$ is the even extension. With the notation of (4.1),

$$\eta(x) = \int_0^x \psi(s)ds.$$  \hfill (1.9)

We require continuity of $\xi(x)$, $\eta(x)$ for $x \geq 0$ and of $f(t)$ for $t \geq 0$. In order that the odd extension of the continuous function $\xi(x)$ defined in $x \geq 0$ exist we must require $\xi(0) = 0$. The odd extension $\tilde{\xi}$ is continuous.

The first addendum $f(t-x)H(t-x)$ in general is not continuous for $t = x$. It is continuous if we require $f(0) = 0$. So we have also the compatibility condition $f(0) = \xi(0)$. Instead, we don’t need to impose conditions to $\eta$ (a part continuity), since the even extension of a continuous function is continuous.

Of course, further conditions have to be imposed if we require the existence of the partial derivative of $u$.

Solution of Problem 1.2 We consider the dependence on $f$. Dependence on $\xi$ and $\eta$ is analogous.

If $\xi = 0$, $\eta = 0$ then $u(x,t) = f(t-x)H(t-x)$. Recall that this is an abuse of notation for $u(x,t) = \tilde{f}(t-x)$ where $\tilde{f}(t) = f(t)$ if $t \geq 0$, $\tilde{f}(t) = 0$ for $t < 0$.

The transformation $f \mapsto \tilde{f}$ is continuous from $L^2(0, +\infty)$ to $L^2(0, T)$. The transformation $t \mapsto f(t-x)$ belongs to $C([0,T];L^2(\mathbb{R}))$ for every $T > 0$ (continuity of the shift is a theorem due to Lebesgue).

So, the map $f \mapsto u$ is continuous from $L^2(0, +\infty)$ to $C([0,T];L^2(0,T))$ for every $T$. The composition with $D_x$ gives that $u_x \in C([0,T];H^{-1}(0,T))$ and continuous dependence on the data $f$, $\xi$ and $\eta$ is preserved.

Solution of Problem 1.3 We consider system (1.1) with a continuous control $f$. In order to study reachability we impose that the initial conditions are zero and $F = 0$ so that $u(x,t) = f(t-x)H(t-x)$. Let us consider the system at the time $T$. A target $\xi(x)$ defined for $x \in (0,T)$ is reachable if we can solve

$$\xi(x) = f(T-x)$$

and this is possible if and only if $\xi(x)$ is continuous. The steering control is $f(t) = \xi(T-t)$ for $t \in (0,T)$ (and it is arbitrary if $t > T$ but of course if $f$ is a continuous control we must have continuity at $t = T$).

Now, still with $F = 0$ and null initial conditions, we consider the system (1.3). We see from the left hand side of (1.7) that the integral term in (1.5) is a continuous function of $x$, even if $f$, hence $\psi$, is only square integrable (according to the fact that the memory does not add discontinuities to the solution!). So, if $\psi$ is continuous only continuous targets can be reached.

It is easy to see that it is possible to choose $\gamma$ so large that $\|G\| < 1/4$ also if $G$ is considered as a transformation in $C(Q)$ where
$Q = \{(x,t) \mid 0 < x < T, \ 0 < t < T\}$.

So every $\xi \in C(0,T)$ can be reached using continuous controls, for every $T > 0$.

**Solution of Problem 1.4** It is easily seen that

$$u(x,T) = -H(T-x)\int_0^{T-x} f(s)\, ds.$$  \hfill (1.10)

When $f$ is continuous the reachable targets are the functions continuous on $[0, +\infty)$, continuously differentiable a part that for $x = T$, and such that $u(x,T) = 0$ for $x \geq T$. If $f \in L_{\text{loc}}^2(0, +\infty)$ then the reachable targets are the functions $\xi(x) \in H^1([0,S])$ for every $S \geq 0$, with $\xi(x) = 0$ for $x \geq T$.

The following observations have an interest:

- The elements of $H^1([0,T])$ are continuous functions and so the targets $\xi(x)$, $x \in [0,T]$ which are reachable under square integrable controls in the Neumann boundary condition is contained in the set of the continuous functions which are 0 for $t = T$. Note that this set is not dense in $C([0,T])$. In contrast with this, the reachable set under boundary controls in the Dirichlet condition contains $C([0,T])$.

- Use Leibniz formula to compute the derivative of the right hand side of (1.10).

The derivative of $H(T-x)$ is $-\delta(T-x)$ (Dirac delta) but it is multiplied with a continuous factor which tends to zero for $x \to T$ and so the second addendum in the derivative of the product on the right side of (1.10) is zero:

$$-\delta(T-x)\int_0^{T-x} f(s)\, ds = 0.$$  

**Solution of Problem 1.5** The formula is the same, but now $\psi$ is

$$\psi(x,T) = -H(T-x)e^{-\gamma T}\int_0^{T-x} f(s)\, ds.$$  \hfill (1.11)

**Solution of Problem 1.6** The formula for the reachable targets is the same as (1.6), with $\psi(x,T)$ given by (1.11). So, for every $\xi \in L^2(0,T)$ a solution $\psi$ of (1.6) exists, but in order to have a control $f \in L^2$ we must have that $\psi$ and so also the target $\xi$ belongs to $H^1(0,T)$ and it has to be 0 for $t = T$.

**Solution of Problem 1.7** For every $h \in H$ we have $\sum |\langle h, e_n \rangle|^2 < +\infty$ and so $\langle h, e_n \rangle \to 0$ for every $h \in H$.

**Solution of Problem 1.8** An example is an orthonormal basis $\{e_n\}$. We have seen $e_n \to x_0 = 0$ but $|e_n| \equiv 1$ and so $\lim |e_n| = 1 > 0 = |x_0|$.

**Solution of Problem 1.9** If an operator $T$ is not continuous, then
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\[ \text{sup} \{|Tx|, \ |x| = 1\} = +\infty. \]

Hence, there exists a sequence \( \{x_n\} \) with \( |x_n| = 1 \) for every \( n \) and \( |Tx_n| \rightarrow +\infty \).

A bounded sequence in a Hilbert space admits weakly convergent subsequence. Let \( \{x_n\} \) be a weakly convergent subsequence of \( \{x_n\} \). It must be \( \lim |Tx_n| = +\infty \) and so \( \{Tx_n\} \) cannot be norm convergent. It follows that an operator which is not continuous cannot be compact.

**Solution of Problem 1.10**

Note that \( \{\alpha_n\} \in l^2 \) and

\[ +\infty \sum_{n=1}^{+\infty} \alpha_n^n e_n^n = +\infty \sum_{n=1}^{+\infty} |\alpha_n|^2 n^2. \]

We have

\[ T \left( \sum_{n=1}^{+\infty} \alpha_n^n e_n^n \right)^2 = +\infty \sum_{n=1}^{+\infty} |\alpha_n|^2 n^2 < +\infty \sum_{n=1}^{+\infty} |\alpha_n|^2. \] (1.12)

So, the operator is continuous and its norm is less then 1.

**Solution of Problem 1.11**

Let \( x_k = +\infty \sum_{n=1}^{+\infty} \alpha_k^n e_n \rightarrow 0 \).

Weak convergence implies boundedness, so that there exists \( M \) such that for every \( k \) we have

\[ |x_k|^2 = +\infty \sum_{n=1}^{+\infty} |\alpha_k^n|^2 < M. \] (1.13)

We use this observation to improve the estimate (1.12) and we prove \( Tx_k \rightarrow 0 \).

Let \( \epsilon > 0 \) be fixed and note the following equality, which holds for every \( N > 1 \):

\[ T \left( \sum_{n=1}^{+\infty} \alpha_k^n e_n \right)^2 = +\infty \sum_{n=1}^{N} |\alpha_k^n|^2 n^2 + +\infty \sum_{n=N+1}^{+\infty} |\alpha_k^n|^2 n^2. \]

Using (1.13) we see that the following holds for every \( k \):

\[ +\infty \sum_{n=N+1}^{+\infty} |\alpha_k^n|^2 n^2 \leq \frac{1}{(N+1)^2} + \infty \sum_{n=N+1}^{+\infty} |\alpha_k^n|^2 \leq M \frac{1}{(N+1)^2}. \]

The right hand side is less then \( \epsilon/2 \) if \( N \) has a suitable value, \( N = N_\epsilon \). This value of \( N = N_\epsilon \) is kept fixed.

We note that

\[ \lim_{k \rightarrow +\infty} \alpha_k^n = \lim_{k \rightarrow +\infty} \langle x_k, e_j \rangle = 0 \]

and so
\[ 0 = \lim_{k \to +\infty} \sum_{n=1}^{N_k} \left| \frac{\alpha^k_n}{n} \right|^2 = 0 \]

Hence, there exists \( K_\varepsilon \) such that if \( k > K_\varepsilon \) then we have

\[ \sum_{n=1}^{N_\varepsilon} \left| \frac{\alpha^k_n}{n} \right|^2 < \frac{\varepsilon}{2} . \]

In conclusion, for \( k > K_\varepsilon \) we have

\[ |T x_k|^2 \leq \varepsilon . \]

**Solution of Problem 1.12**

We use this consequence of Ascoli-Arzelà Theorem: let the sequence \( \{ f_n(t) \} \) be weakly convergent in \( L^2(0, T) \) (and so it is bounded). If every function \( f_n(t) \) is of class \( H^1(0, T) \) and if the sequence \( \{ f'_n \} \) is bounded in \( L^2(0, T) \) then the sequence \( \{ f'_n \} \) is norm convergent in \( L^2(0, T) \).

The transformation \( V: L^2(0, T) \to L^2(0, T) \) defined by

\[ V x = \int_0^T F(t-s)x(s)ds \]

is continuous. In fact, let \( |x|_{L^2(0, T)} = 1 \). Then we have:

\[ |Vx|^2_{L^2(0, T)} = \int_0^T \left( \int_0^t F(t-s)x(s)ds \right)^2 dt \leq T \left( \int_0^T |F(s)|ds \right)^2 . \]

So, every weakly convergent sequence \( \{ x_n \} \) is transformed to the weakly convergent sequence \( \{ y_n \} \) where

\[ y_n(t) = \int_0^t F(t-s)x_n(s)ds . \]

Every function \( y_n(t) \) is differentiable and

\[ y'_n(t) = F(0)x_n(t) + \int_0^t F'_n(t-s)x_n(s)ds . \]

Boundedness of the weakly convergent sequence \( \{ x_n \} \) implies that \( \{ y'_n \} \) is bounded in \( L^2(0, T) \). Hence, \( \{ y_n \} \) is norm convergent in \( L^2(0, T) \).

**Solution of Problem 1.13** We consider the first operator.

Let \( \{ y_n \} \) be any sequence of smooth functions such that

\[ \lim_{n \to +\infty} y_n(t) = \text{sgn}(t) \]

(the limit is in \( L^2(-1, 1) \)).

Let
\[ x_n(t) = 1 + \int_{-1}^{t} y_n(s) \, ds . \]

Then,
\[ \lim_{n \to +\infty} x_n(t) = |t| . \]

We have
\[ (x_n, Ax_n) \to (|t|, \text{sgn}(t)) \notin \mathcal{G}(A) \]
since \( |t| \notin \text{dom}A \).

The second example is similar.

We suggest to treat the third example in a similar way, but here we present a different argument: the operator in this example is a functional, i.e. takes values in the scalar field (say \( \mathbb{R} \)). If the operator \( A \) is closed then \( A^* \) must have dense domain. In the case of a functional, the domain of \( A^* \) must be dense in \( \mathbb{R} \). Hence the domain must be \( \mathbb{R} \). Every adjoint operator is closed so that if it is everywhere defined it must be continuous. And so the original operator \( A \) has to be continuous.

It is easily seen that this is not the case. Let
\[ x_n(t) = \begin{cases} \sqrt{n/2} & \text{if } -\frac{1}{n} < x < \frac{1}{n}, \\ 0 & \text{otherwise} . \end{cases} \]

Clearly, \( |x_n|_{L^2(-1,1)} = 1 \) but its image is the unbounded sequence \( \{ \sqrt{n/2} \} \).

Note that this argument applies to every functional: a linear functional either is continuous or it is not closed.

**Solution of Problem 1.14** We prove that the first operator is closed. Let \((x_n, Ax_n) = (x_n, x_n') \to (y_0, z_0)\). An element of \( H^1(0,1) \) can be recovered from its derivative, so that
\[ x_n(t) = x_n(0) + \int_{0}^{t} x_n'(s) \, ds . \] (1.14)

The assumption that \( x_n'(t) \to z_0 \) implies that
\[ \int_{0}^{t} x_n'(s) \, ds \to \int_{0}^{t} z_0(s) \, ds \]
(the convergence is even uniform). The left hand side of (1.14) converges by assumption and we have seen that the integral converges. So also the numerical sequence \( \{x_n(0)\} \) converges, and so it converges to to \( x_0(0) \) and we get
\[ x_0(t) = x_0(0) + \int_{0}^{t} z_0(s) \, ds . \]

This shows that \( x_0(t) \in H^1(0,1) = \text{dom}A \), with \( Ax_0 = z_0 \) i.e. \((x_0, z_0) \in \mathcal{G}(A)\).

The second operator is analogous.

We compute the resolvents. The resolvent set of the first operator is empty, and the resolvent operator does not exist, since every \( \lambda \in \mathbb{C} \) is an eigenvalue of \( A_1 \). In fact,
\[ x(t) = e^{\lambda t} \quad \Rightarrow \quad Ax = \lambda x. \]

In order to compute the resolvent of \( A \) we must study the equations

\[ (\lambda I - A) x = f \quad \text{i.e.} \quad x'' = \lambda x - f. \]

The solutions of this ordinary differential equations are

\[ x(t) = A \cos \sqrt{\lambda} t + B \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} t - \frac{1}{\sqrt{\lambda}} \int_0^t \sin \sqrt{\lambda} (t-s)f(s)ds. \]

Note that \( \lambda \) is a complex number so that it has two square roots, but they leads to the same solutions.

Now we must impose \( x \in H^1_0(0,1), \) i.e. both \( x(0) = 1 \) and \( x(1) = 0. \)

The condition \( x(0) = 0 \) gives \( A = 0 \) while the condition \( x(1) = 0 \) gives

\[ B = \frac{1}{\sin \sqrt{\lambda}} \int_0^1 \sin \sqrt{\lambda} (1-s)f(s)ds. \]

The number \( B \) does not exist if \( \lambda = k^2\pi^2 \) (it is easily seen that these are the eigenvalues of \( A \)). Otherwise,

\[ (\lambda I - A)^{-1} f = \]

\[ = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} t \left( \frac{1}{\sin \sqrt{\lambda}} \int_0^t \sin \sqrt{\lambda} (1-s)f(s)ds \right) - \frac{1}{\sqrt{\lambda}} \int_0^t \sin \sqrt{\lambda} (t-s)f(s)ds. \]

It is easily seen that \( f \mapsto (\lambda I - A)^{-1} f \) is continuous and so every \( \lambda \) which is not an eigenvalue belongs to the resolvent set of \( A. \)

**Solution of Problem 1.15**

The resolvent operator is the sum of two operators. The second one,

\[ f \mapsto -\frac{1}{\sqrt{\lambda}} \int_0^t \sin \sqrt{\lambda}(t-s)f(s)ds \]

is compact. This has been proved in Problem 1.12. The first operator is a **continuous functional**:

\[ f \mapsto \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} t \left( \frac{1}{\sin \sqrt{\lambda}} \int_0^1 \sin \sqrt{\lambda} (1-s)f(s)ds \right). \]

Every continuous linear functional is compact. In fact, it transforms weakly convergent sequences to weakly convergent sequences, hence to norm convergent sequences, since weak and norm convergence coincide in finite dimension.

Hence the resolvent is compact because it is the sum of two compact operators.
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