

Riesz bases for heat equations with memory and controllability problems

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The equation

$$\theta_t = \int_0^t N(t-s) \Delta \theta(s) \, ds$$

is used to model

- heat equation with finite diffusion speed (at very low temperatures, or close to phase transition)
- non-fickian diffusion of solutes in solvents (as in case of high concentration or complex molecular structure)
- viscoelasticity

Introduced when?

- heat equation with memory was first studied by Cattaneo (1948, but Maxwell had already hinted at this equation)
- nonfickian diffusion was considered in the same years by several authors, when it was discovered that diffusion in polymers presents a sharp separation between wet and non-wet regions, a kind of wavefront
- viscoelasticity was studied first by Maxwell and Kirchoff (mid 19-th century)

After that

Only special forms of the equations had been studied, essentially equivalent to telegraph equation. Gurtin and Pipkin introduced the general form in 1966, then studied by many people, especially from the point of view of

- consistency with thermodynamics (this led to identify suitable assumptions on the kernel)
- stability theory
- controllability
- kernel identification

Overview on this and related equations by Joseph and Preziosi.

Few recent names

Recent results concerns mostly stability and controllability.
We can cite

- (stability) Fabrizio, Muñoz-Rivera, Naso, Morro, Renardy. Essentially final results in recent important papers by V. Pata
- (Controllability) Kim, Leugering, Loreti, L.P., Renardy, Sforza, Zhang.
- (kernel identification) Lorenzi.

Remark-1

Related papers of mine on my WEB page

<http://calvino.polito.it/~lucipan/>

With memory *versus* without memory

In order to fix our ideas, we concentrate on heat problems. In this case, the usual heat equation is transformed to heat equation with memory when the usual Fourier law (q is density of heat flux)

$$q(x, t) = -\nabla\theta(x, t)$$

is replaced by the law

$$q(x, t) = -\nabla \int_0^t N(t-s)\theta(x, s) ds.$$

Our goal

To present our recent results on Riesz sequences associated to the equation

$$\theta_t = \int_0^t N(t, s) \Delta \theta(s) \, ds$$

$$\theta = \theta(x, t), \quad x \in (0, \pi).$$

For this, it is convenient to think at a control problem, with boundary control

$$\theta(0, t) = u(t), \quad \theta(\pi, t) = 0$$

and null initial condition, $\theta(x, 0) = 0$.

The FIRST problem

To identify the reachable subspace

$$R_T = \{\theta(\cdot, T), \quad u \in L^2(0, \pi)\}$$

at least for T sufficiently large.

We can prove exact controllability in L^2 , i.e. $R_T = L^2(0, \pi)$ if $T \geq \pi$, using moment methods and Riesz sequences.

The **SECOND** problem

Using moment methods it is also possible to study the reachable set

$$\{\theta(\cdot, T), \theta_t(\cdot, T)\}$$

FACT: The first and second problems have been already studied using several different method.

BUT

the moment approach to the solution of the previous PROBLEMS pave the way to the solution of a THIRD problem which is more interesting for thermodynamic applications: we shall clarify the relation between temperature and flux in the presence of memory.

The THIRD problem

To identify the reachable set of the pairs

$$(\theta(\cdot, T), q(\cdot, T)) .$$

We recall:

$$q(\cdot, t) = -\frac{\partial}{\partial x} \int_0^t N(t-s)\theta(\cdot, s) \, ds .$$

In particular: in what space should the reachable set be studied?

This control problem was considered (in a special case) by Renardy.

THIRD PROBLEM: Interpretation

To understand how **strongly related** are temperature and flux.

For the standard Fourier Law,

$$q(x, t) = -\nabla\theta(x, t) :$$

temperature uniquely identify the flux.

And, in the presence of memory?

How strong is the relation?

This problem had already been raised by Cattaneo in his 1949 paper.

Note: definition of **strong relation** is part of the problem.

Dependence/independence

We interpret Cattaneo problem as follows: we identify a natural space in which the pair (temperature/flux) lives. We consider the

reachable set R_T of the pairs
(temperature/flux)
in this space.

The reachable set in general does depend on T . If R_T is “small” this means that temperature and flux are strongly related.

“Large” R_T is interpreted as a weak relation.

Our solution

Using moment methods and Riesz sequences, we proved:

- square integrable boundary input $\implies t \rightarrow \theta(\cdot, t)$ and $t \rightarrow q(\cdot, t)$ BOTH belong to $C(0, +\infty; L^2(0, \pi))$
- if $T \geq 2\pi$ then $R_T = L^2(0, \pi) \times L^2(0, \pi)$ **WEAK RELATION, IN FACT INDEPENDENT**
- if $T < 2\pi$ **STRONGLY RELATED**

(S. Avdonin, L.P., in print Quart. Appl. Math.)

Plan of this talk

We present the details of a simplified moment problem, in order to sketch the methods we are using. Then we give more details on PROBLEM 3. Namely:

- we shall reduce PROBLEM 1 to a moment problem
- we shall study a **simplified version** of this moment problem in order to sketch the ideas
- we present the results on flux dependence/independence

Control to moment problem

Assume null initial condition.

We project the equation on the eigenfunctions

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$$

and we get

$$\theta(\cdot, t) = \sum_{n=1}^{+\infty} [\phi_n'(0)\phi_n(x)] \left[\int_0^t z_n(t-s)v(s) \, ds \right]$$

$$\text{where } \begin{cases} z_n' = -n^2 \int_0^t N(t-s)z_n(s) \, ds & z_n(0) = 1 \\ v(t) = \int_0^t N(t-s)u(s) \, ds. \end{cases}$$

Controllability and moment problem

A vector $\sum_{n=1}^{+\infty} c_n \phi_n(x) \in R_T$ if and only if there exists $u \in L^2(0, T)$ which solves

$$\int_0^T u(t) g_n(t) dt = c_n, \quad g_n(t) = \left[n \int_0^t N(t-r) z(r) dr \right].$$

Here, $\{c_n\} \in l^2$ is arbitrary if we want that every element of $L^2(0, \pi)$ be reachable.

This is an example of a **Moment Problem**.

Remark

Controllability/observability problems can often be reduced to moment problems but usually the sequence $g_n(t)$ is

1) a sequence of exponentials

which are

2) eigenfunctions of a certain operator.

Neither 1) nor 2) in the case under study!

Fact

A moment problem

$$\int_0^T g_n(t)u(t) dt = c_n$$

is solvable with $u \in L^2$ if and only if $\{c_n\} \in l^2$ when the sequence $\{g_n\}$ is a Riesz sequence.

And so

Our control problem is solvable in time T if and only if

$$\left\{ n \int_0^t N(t-s)z_n(s) ds \right\}$$

is a Riesz sequence in $L^2(0, T)$. It turns out that it is so, for

every $T > \pi$.

A simplified moment problem

In order to sketch the ideas, we study a simpler moment problem:

$$\int_0^T z_n(s)v(s) \, ds = c_n$$

We sketch the proof that $\{z_n\}$ is Riesz in $L^2(0, T)$ for $T \geq \pi$, so that

- the moment problem is solvable if and only if $\{c_n\} \in l^2$ and the transformation $v(\cdot) \rightarrow \{c_n\}: \mathcal{L}(L^2(0, T), l^2)$ is continuous and continuously invertible.

Riesz sequences

Let H be a Hilbert space and $\{z_n\}$ be a sequence in H . The sequence $\{z_n\}$ is Riesz when there exists a linear bounded boundedly invertible transformation $L: H \rightarrow \text{cl span } \{z_n\}$ which transforms an orthonormal basis of H to $\{z_n\}$. If $\{z_n\}$ is complete, then it is a Riesz basis.

How can we see that $\{z_n\}$ is a Riesz sequence?

Several tests are available. The most useful for us are **Paley-Wiener test** and **Bari Theorem**.

Paley-Wiener

Adapted to Hilbert spaces, it is as follows: Let $\{e_n\}$ be a Riesz sequence and let $\{z_n\}$ satisfy

$$\sum \|e_n - z_n\|^2 < 1.$$

Then, $\{z_n\}$ is Riesz too.

A CONSEQUENCE we shall use:

Let $\{z_n\}$ be quadratically close to $\{e_n\}$, i.e.

$$\sum \|e_n - z_n\|^2 < +\infty.$$

Then, there exists N such that $\{z_n\}_{n>N}$ is a Riesz sequence (not a basis).

Bari Theorem

First, a FACT: *if $\{z_n\}_{n>N}$ is Riesz in a Hilbert space H and if $\sum \alpha_n z_n$ converges in H , then $\{\alpha_n\} \in l^2$.*

The, a DEFINITION: $\{z_n\}$ is ω -independent in H if

$$\sum \alpha_n z_n = 0 \implies \{\alpha_n\} = 0.$$

Now, **Bari Theorem**: $\{e_n\}$ Riesz, $\{z_n\}$ quadratically close to $\{e_n\}$ and ω -independent implies $\{z_n\}$ Riesz.

The Proof that $\{z_n\}$ is Riesz—1

$z_n(t)$ satisfies

$$z_n' = -n^2 \int_0^t N(t-s) z_n(s) ds$$

hence

$$z_n'' = -n^2 z_n - n^2 \int_0^t N'(t-s) z_n(s) ds$$

and conditions $z_n(0) = 1$, $z_n'(0) = 0$. Hence,

$$\begin{aligned} z_n(t) &= \cos nt - n \int_0^t \sin ns \left[\int_0^{t-s} N'(t-s-r) z_n(r) dr \right] ds \\ &= \cos nt - \int_0^t z_n(r) \left[n \int_0^{t-r} \sin ns N'(t-r-s) ds \right] dr . \end{aligned}$$

The Proof that $\{z_n\}$ is Riesz—2

Integration by parts can absorb the factor n and a second integration by parts introduces a factor $1/n$. This can be used to prove

$\{z_n(t)\}$ is quadratically close to $\{\cos nt\}$.

Hence the tail $\{z_n(t)\}_{n>N}$ (large N) is Riesz

and we wonder whether $\{z_n(t)\}_{n\geq 1}$ is Riesz. This is true since we can prove that $\{z_n\}$ is ω -independent. This requires three steps.

ω -independence-step 1

We first prove that $\{z_n\}$ is **linearly independent**. Let

$$\sum_{k=1}^N \alpha_n z_n = 0, \quad \alpha_N \neq 0.$$

compute the derivative and find

$$0 = \int_0^t N(t-s) \left[\sum_{k=1}^N n^2 \alpha_n z_n(s) \right] ds = 0 \implies \sum_{k=1}^N n^2 \alpha_n z_n(s) = 0.$$

Combine to remove the **FIRST** element: $\sum_{k=2}^N \beta_n z_n = 0$. After a **finite number** of steps we get $z_N = 0$ **CONTRADICTION.**

Alternative method

By contradiction, let N be the **SMALLEST** index such that

$$\sum_{k=1}^N \alpha_n z_n = 0, \quad \alpha_N \neq 0.$$

Using

$$\sum_{k=1}^N n^2 \alpha_n z_n(s) = 0$$

we can remove the **LAST** element, and we find that $\{z_k\}_{k \leq N-1}$ linearly dependent, **CONTRADICTION** with the **definition of N** .

This alternative method cannot be used for series.

ω -independence-step 2-informal—A

We consider

$$\sum_{n=1}^{+\infty} \alpha_n z_n(t)$$

and we attempt a similar procedure. **Formally**

$$0 = \sum_{n=1}^{+\infty} \alpha_n z'_n(t) = - \int_0^t N(t-s) \left[\sum_{n=1}^{+\infty} (\alpha_n n^2) z_n(s) \right] ds$$

implies

$$\sum_{n=1}^{+\infty} (\alpha_n n^2) z'_n(t) = 0.$$

ω -independence-step 2-informal-B

So we have:

$$0 = \sum_{n=1}^{+\infty} n^2 \alpha_n z_n(t) = \sum_{n=1}^{+\infty} \alpha_n z_n(t).$$

Here we can remove the first element and we get

$$\sum_{n=2}^{+\infty} (1 - n^2) \alpha_n z_n(t) = 0.$$

This can be iterated, but we apparently **we have an endless process**. And, the “alternative” route is not usable here since the series does not have a “last” element.

In fact, **the process ends since:**

ω -independence-step 2-informal–C

after a finite number of steps, we get

$$\sum_{n=N}^{+\infty} \tilde{\alpha}_n z_n(t) = 0.$$

We have $\tilde{\alpha}_n = 0$ if and only if $\alpha_n = 0$ and it must be $\tilde{\alpha}_n = 0$, thanks to the corollary of Paley Wiener which implies $\{z_n\}_{n \geq N}$ is Riesz.

So, the original equality is

$$0 = \sum_{n=1}^{+\infty} \alpha_n z_n(t) = \sum_{n=1}^{N-1} \alpha_n z_n(t)$$

So $\alpha_n = 0$ ($1 \leq n \leq N - 1$) since $\{z_n(t)\}$ is linearly independent.

ω -independence-step 2—A

We have to justify

$$0 = \frac{d}{dt} \left[\sum_{n=1}^{+\infty} \alpha_n z_n(t) \right] = \sum_{n=1}^{+\infty} \alpha_n z'_n(t).$$

This is a consequence of the following important fact

If $\{\alpha_n\} \in l^2$ satisfies $\sum_{n=1}^{+\infty} \alpha_n z_n(t) = 0$
Then there exists $\{\gamma_n\} \in l^2$ such that

$$\alpha_n = \frac{\gamma_n}{n^2}.$$

ω -independence-step 2—B

The proof is technical, based on the following idea: After some elaboration,

$$z_n(t) = \cos nt + \dots$$

where \dots does depend on n in a harmless way, and is smooth. So, we have also

$$\sum \alpha_n \cos nt = \sum \alpha_n \dots$$

The crux of the matter now is to prove that $\sum \alpha_n \dots$ is smooth enough and so to derive from the smoothness of the right hand side the “regularity” property of $\{\alpha_n\}$, using known properties of the COSINE Fourier series.

In fact

Using the methods just outlined, we can prove that $\{z_n(t)\}$ is a Riesz sequence in $L^2(0, T)$ if $T \geq \pi$.

In fact, the sequence of interest for control problems is not $\{z_n(t)\}$ but similar ideas can be used.

Note that the controllability time for the pair $\theta(\cdot, T), \theta_t(\cdot; T)$ (in $L^2(0, \pi) \times H^{-1}(0, \pi)$) is $2T$.

More in general

In fact, it is possible to prove that the following sequences are Riesz in $L^2(0, T)$, $T > \pi$:

- $\{z'_n(t)/n\}$

- $\left\{ \frac{1}{n} \int_0^t z_n(s) \, ds \right\}, \quad \left\{ \frac{1}{n} \int_0^t N(t-s) z_n(s) \, ds \right\}$

while

- $\left\{ z_n(t) + i \frac{1}{n} z'_n(t) \right\}$ (here also $n < 0$, z_{-n} suitably defined) is Riesz in $L^2(0, T)$, $T > 2\pi$.

The goal–2

Temperature flux independence

Cattaneo in his **1948 paper** already posed the problem to understand the degree of dependence/independence of flux and temperature in case of heat equations with memory.

As we noted, the interpretation of the terms “dependence/independence” is part of the problem.

Interpretation as controllability

We proceed as already described: we identify a “natural” space in which the pairs $(\theta(\cdot, t), q(\cdot, t))$ live when the boundary temperature is square integrable. Then we study “how large” the “reachable set” is.

I.e.

in order to study dependence/independence of flux and temperature, we must study the reachable pairs $(\theta(\cdot, t), q(\cdot, t))$ and this is a typical control problem.

We interpret “independence” as a kind of “controllability”.

We know that $\theta(\cdot, t) \in C(0, T; L^2(0, \pi))$ so that $q(\cdot, t) \in C(0, T; H^{-1}(0, \pi))$. We can first study the flux in this space.

A more precise result

In fact, a more precise result is

the flux is smoother: in fact, we shall see that

$$q(\cdot, t) \in C(0, +\infty; L^2(0, \pi))$$

So, we are going to prove the following result (we assume $\theta(x, 0) = 0$ for simplicity).

The result

Theorem 1 *Let the boundary condition be*

$$\theta(0, t) = u(t), \quad \theta(\pi, t) = 0.$$

For every $u(\cdot) \in L^2_{\text{loc}}(0, +\infty)$ we have:

$$\theta \in C(0, +\infty; L^2(0, \pi)), \quad q \in C(0, +\infty; L^2(0, \pi)).$$

Moreover, if $T \geq 2\pi$, then

$$\{(\theta(\cdot, T), q(\cdot; T)), \quad u(\cdot) \in L^2(0, T)\} = \boxed{??}.$$

Controllability

When $T \geq 2\pi$ we have

$$\begin{aligned} & \{(\theta(\cdot, T), q(\cdot; T)), \quad u(\cdot) \in L^2(0, T)\} \\ & = \boxed{??} = L^2(0, \pi) \times L^2(0, \pi). \end{aligned}$$

This is a controllability result and this shows that $\theta(T, \cdot)$ and $q(T, \cdot)$ (same time T) are not related in the case of heat equations with memory provided that T is large enough.

For small time

An example will show that the relation is **strict** if T is **small**.

PLAN

We first study the property of the flux. So doing, we shall find a new **Riesz system** related to the **heat equation with memory**.

A **second Riesz system** is then used to prove **controllability**.

Representation of the flux—1

We recall the formula for the solution θ :

$$\theta(x, t) = \sum_{n=1}^{+\infty} \phi_n(x) \phi_n'(0) \int_0^t z_n(t-s) v(s) \, ds .$$

In this formula,

$$v(t) = \int_0^t N(t-s) u(s) \, ds , \quad \phi_n(x) = \sqrt{2/\pi} \sin nx$$

$$z_n'(t) = 2\alpha z_n - n^2 \int_0^t N(t-s) z_n(s) \, ds , \quad z_n(0) = 1 .$$

Representation of the flux—2

So we have

$$q(x, t) = \sum_{n=1}^{+\infty} \phi'_n(x) \phi'_n(0) \int_0^t N(t-s) \left[\int_0^r z_n(s-r) v(r) \, dr \right] \, ds .$$

Facts to be noted here: the derivative $\phi'_n(x)$ **reduces regularity** but **a new integration in time might increase regularity!**

Reduction to a moment problem-1

For the moment we work in $L^2 \times H^{-1}$. Facts:

- $\{\phi_n(x)\}_{n>0}$ is an orthonormal basis of $L^2(0, \pi)$ while $\{\phi'_n(x)\}_{n>0}$ is a Riesz basis of $H^{-1}(0, \pi)$.
- So, every target $(\xi, \eta) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ can be represented as

$$\xi = \sum_{n=1}^{+\infty} \xi_n \phi_n(x), \quad \eta = \sum_{n=1}^{+\infty} \eta_n \phi'_n(x).$$

- Here, $\{\xi_n\} \in l^2$ and $\{\eta_n\} \in l^2$.
- we have $\eta \in L^2(0, \pi)$ if $\{n\eta_n\} \in l^2$

Reduction to a moment problem-2

The target is reachable at a certain time T if and only if we can find a control $u \in L^2(0, T)$ such that the following moment problem can be solved:

$$\left\{ \begin{array}{l} \int_0^T z_n(r)v(T-r) \, dr = \frac{\xi_n}{\phi'_n(0)} \sim \frac{\xi_n}{n} \\ \int_0^T \left\{ \phi'_n(0) \int_0^r N(r-s)z_n(s) \, ds \right\} v(T-r) \, dr \\ = - \left(\eta_n - 2\alpha \frac{\xi_n}{n} \right) . \end{array} \right.$$

where

$$v(t) = \int_0^t N(t-s)u(s) \, ds$$

Reduction to a moment problem-3

Here v is “smooth”: $v(t) = \int_0^t N(t-s)u(s) ds$. and $\{\xi_n\}, \{\eta_n\}$ (hence also $-\left\{\eta_n - 2\alpha\frac{\xi_n}{n}\right\}$) are arbitrary in l^2 , so that this moment problem, which corresponds to controllability in $L^2(0, \pi) \times H^{-1}(0, \pi)$ **is not solvable** because the following one a Riesz sequence in $L^2(0, \pi)$:

$$\left\{ \phi'_n(0) \int_0^r N(r-s)z_n(s) ds \right\}$$

(as noted above).

Reduction to a moment problem-4

For the following we note that

$$q(\cdot, T) \in L^2(0, \pi) \Leftrightarrow \{n\eta_n\} \in l^2.$$

So, **controllability in** $L^2(0, \pi) \times L^2(0, \pi)$ is the moment problem with $\{n\eta_n\} \in l^2$ and we shall see also with a weaker condition:

$$\eta_n = \frac{c_n}{n} - 2\frac{\gamma}{n}, \quad \{c_n\} \in l^2, \quad \gamma \in \mathbb{R}.$$

With this, the second line of the moment problem becomes

$$\begin{aligned} \int_0^T \left\{ n\phi'_n(0) \int_0^r N(r-s)z_n(s) ds \right\} v(T-r) dr \\ = - (c_n - 2\gamma - 2\alpha\xi_n). \end{aligned}$$

Remark

Note the right hand side

$$c_n - 2\gamma - 2\alpha\xi_n$$

where γ is constant. So the right hand side is not in l^2 : this is not a standard moment problem for a Riesz sequence. But, using a certain trick which we see below, it can be reduced to a standard moment problem!

Reduction to a moment problem-5

This (“vector”) moment problem (for the pair $(\theta(x, t), q(x, t))$) can be written in a more standard form: Let

$$c_n = \left[\frac{\xi_n}{\phi'_n(0)} - i\eta_n \right] \quad n \geq 1$$

$$\hat{\zeta}_n(t) = z_n(t) - i\frac{1}{n}z'_n(t), \quad n \geq 1,$$

$$\hat{\zeta}_n(t) = \overline{\zeta_{-n}(t)}, \quad c_n = \bar{c}_{-n}, \quad n \leq -1.$$

Note that $n = 0$ is not used.

Moment problem: final form

The moment problem takes the form

$$\int_0^T \hat{\zeta}_n(r) v(T-r) \, dr = c_n \quad n \in \mathbb{Z} - \{0\} = \mathbb{Z}'$$

We then prove:

- The sequence $\{\hat{\zeta}_n(t)\}$ is a Riesz sequence in $L^2(0, T)$,
 $T \geq 2\pi$

This is a new Riesz sequence associated to the heat equation with memory.

Formula for the flux

We consider more in detail

$$q(x, t) = \sum_{n=1}^{+\infty} \phi'_n(x) \int_0^t v(t-r) \times \\ \times \left\{ \phi'_n(0) \int_0^r N(r-s) z_n(s) ds \right\} dr .$$

This series converges in the sense of distributions (it converges in $H^{-1}(\mathbb{R})$) but it turns out that that **the support of the singular part IS NOT in $(0, \pi)$** : the flux is a regular distribution on $(0, \pi)$ **identified by a square integrable function.**

What we do

We show the idea on a simple example: the case $N(t) = 1$ (i.e. wave equation.)

Byproduct of this: we shall see that temperature and flux are strongly related for “small” T .

The computations in the general case are a bit involved, but are a variant of the ideas that we are going to see now.

Example-1

We consider the time interval $[0, \pi]$ and the kernel $N(t) \equiv 1$. Furthermore, we let the initial condition to be 0. So, our equation

$$\theta_t(x, t) = \int_0^t \theta_{xx}(x, s) \quad \theta(0, t) = u(t), \quad \theta(\pi, t) = 0$$

is nothing else than the wave equation

$$\theta_{tt}(x, t) = \theta_{xx}(x, t) \quad \theta(0, t) = u(t), \quad \theta(\pi, t) = 0$$

with null initial condition, which we can solve explicitly.

Example-2

Let $t \leq \pi$, so there is no reflection from the right end of the interval, and we know that

$$(1) \quad \theta(x, t) = u(t - x)H(t - x)$$

where $H(t)$ is the Heaviside function.

Note that this is not the unique representation of the temperature: the series we computed is a **different representation**. But this formula has its interest:

Example-3

It is easily seen, using this formula, that if $t < \pi$ then

$$q(x, t) = \theta(x, t).$$

So, flux and temperature are **strongly related if T is “small”**.
From a control point of view, **the pair (temperature/flux) can't be controlled in time $T < \pi$** .

Let us now look at the series expansions we wrote before,
for $\theta(x, t)$ and $q(x, t)$ **adapted to this example**.

Example-4

Let us introduce the function

$$f^t(x) = u(t - x)H(t - x).$$

Written down explicitly for this example, the series of the flux is

$$\begin{aligned} q(x, t) &= -\frac{2}{\pi} \sum_{n=1}^{+\infty} \left[\int_0^t \int_0^r n \sin n(r - s) f^t(s) \, ds \, dr \right] \cos nx \end{aligned}$$

Example-5

The elements of the series of $q(x, t)$ are periodic on \mathbb{R} and the series converges in the sense of the distributions on \mathbb{R} . It's sum is a distribution with a singular part.

We manipulate this series till we extract the singular part of the distribution using the expansion

$$\sum_{n=1}^{+\infty} \cos nx = \pi \left(\sum_{k=-\infty}^{+\infty} \delta(x - 2k\pi) \right) - \frac{1}{2}$$

Note the regular part $-1/2$

Example-6

Then, with

$$f(t) = u(t - x)H(t - x) \quad H(t) = \text{heaviside}$$

we can proceed as follows:

$$\begin{aligned} q(x, t) &= \\ &= -\frac{2}{\pi} \left[\int_0^t f(s) ds \right] \boxed{\sum_{n=1}^{+\infty} \cos nx} \longrightarrow \left\{ \begin{array}{l} -1/2 \\ + \sum_{k=-\infty}^{+\infty} \delta(x - 2k\pi) \end{array} \right. \\ &+ \sum_{n=1}^{+\infty} \frac{2}{\pi} \cos nx \int_0^t f(t - s) \cos ns ds \end{aligned}$$

Example-7

So, we have

$$\begin{aligned} q(x, t) = & -2 \left(\sum_{k=-\infty}^{+\infty} \delta(x - 2k\pi) \right) \int_0^t f(s) ds \\ & + \left\{ \frac{1}{\pi} \int_0^\pi f^t(s) ds \right. \\ & \left. + \sum_{n=1}^{+\infty} \frac{2}{\pi} \cos nx \int_0^\pi f^t(s) \cos n(s) ds \right\} . \end{aligned}$$

Example-8

The brace is a cosine expansion and it is equal to $f^t(t - x)$ for $x \in (0, \pi)$. So, we see that $q(x, t)$ is a distribution with singular support $x = 0$ plus a regular part: as an element of $H^{-1}(0, \pi)$ the delta function supported at 0 has no effect, and we see that **the distribution $q(\cdot, t) \in H^{-1}(0, \pi)$ is identified by a square integrable function for every t .**

In conclusion

Using that $\{\hat{\zeta}_n(t)\}$ is a Riesz sequence, the previous computations **can be adapted to the general case** and this

- prove the stated **regularity of the flux**.
- **gives force to the problem we have put forward: The pair temperature/flux is identified by elements of $L^2(0, \pi) \times L^2(0, \pi)$. Is it true that every pair $(\xi, \eta) \in L^2(0, \pi) \times L^2(0, \pi)$ can be reached in suitable time T ?**

Answer

Answer is positive as we already asserted.

In the proof, a new sequence of functions appears naturally, and it has an interest to see that also this new sequence is a Riesz sequence in $L^2(0, \pi)$.

The moment problem to be solved is non standard and we need few details to understand how it can be solved.

Back to the moment problem-1

We recall

$$\theta(x, T) = \sum_{n=1}^{+\infty} \xi_n \phi_n(x) \in L^2(0, \pi),$$

$$q(x, T) = \sum_{n=1}^{+\infty} \eta_n \phi'_n(x) \in H^{-1}(0, \pi)$$

so that $\{\xi_n\} \in l^2$, $\{\eta_n\} \in l^2$. Furthermore, $q(\cdot, T) \in L^2(0, \pi)$ for “smoother” $\{\eta_m\}$: i.e. with the properties we are going to see.

Note $\phi'_n(x) \sim n \cos nx$.

Back to the moment problem-2

Combining previous results: if the moment problem can be solved with **arbitrary** $\{\xi_n\} \in l^2$ and

$$\eta_n = \frac{k_n}{n} - 2\frac{\gamma}{n}, \quad \text{arbitrary } \{k_n\} \in l^2, \quad \gamma \in \mathbb{R}$$

then (for general $N(t)$!)

$$q(x, T) = -2\pi\gamma\delta(x) + \left\{ \gamma + \sum_{n=1}^{+\infty} k_n \cos nx \right\}.$$

The brace is an **arbitrary** in $L^2(0, \pi)$.

We sum up-1

So, in order to prove controllability **in $[L^2(0, \pi)]^2$** , it is sufficient that we prove that we can arbitrarily assign the coefficients ξ_n , k_n and γ .

Let

$$\chi_n = \frac{k_n - \gamma}{n}, \quad \xi_n - i\chi_n = d_n - i\frac{\gamma}{n}.$$

The sequence $(\xi_n - i\chi_n) - i\gamma = d_n - i\gamma$ has to solve the following moment problem.

We sum up-2

$$d_n - i\gamma = \int_0^T u(T-t) \left\{ n \int_0^t H_n(t-s) z_n(s) ds \right. \\ \left. - [i z_n(t) - i N(t)] \right\} dt ,$$
$$H_n(t) = N(t) - i \frac{1}{\lambda_n} N'(t)$$

Important fact:

$$n \in \mathbb{Z}' = \mathbb{Z} \setminus \{0\} .$$

New Riesz system

We introduce the sequence $\{R_n(t)\}_{n \in \mathbb{Z}}$

$$R_0(t) = iN(t)$$

$$R_n(t) = n \int_0^t H_n(t-s) z_n(s) \, ds - iz_n(t),$$

$$n \in \mathbb{Z} - \{0\}, \quad H_n(t) = N(t) - i \frac{1}{n} N'(t).$$

New form of the moment problem

With these notations controllability in $[L^2(0, \pi)]^2$ is reduced to solve

$$d_n - \left\{ i\gamma + \int_0^T R_0(t)u(T-t) dt \right\} = \int_0^T R_n(t)u(T-t) dt$$

i.e.

$$\begin{cases} \int_0^T R_0(t)u(T-t) dt = -i\gamma \\ \int_0^T R_n(t)u(T-t) dt = d_n. \end{cases}$$

CONCLUSION

The introduction of $R_0(t)$ reduces the nonstandard moment problem to a moment problem in l^2 .

We prove that the sequence $\{R_n(t)\}_{n \in \mathbb{Z}}$ is a Riesz sequence, if $T > 2\pi$. Hence the moment problem is solvable if $T > 2\pi$ and we have:

If $T > 2\pi$ temperature and flux are not related in $L^2(0, \pi) \times L^2(0, \pi)$.

Otherwise, they are strongly related.