

Riesz systems and the controllability of heat equations with memory

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Abstract

The main result we derive is the proof that a particular set of functions related to the controllability of the heat equation with memory and finite signal speed, with suitable kernel, is a Riesz system. Riesz systems are important tools in applied mathematics, for example for the solution of inverse problems. In this paper we show that the Riesz system we identify can be used to give a constructive method for the computation of the control steering a given initial condition to a prescribed target.

1 Introduction

Let us consider the following heat equation with memory in one space dimension,

$$\theta_t = \int_0^t N(t-s)\Delta\theta(s) ds. \quad (1)$$

Here, $\theta = \theta(t, x)$ with $x \in (0, \pi)$ and $t > 0$. We associate the following initial and boundary conditions to Eq. (1):

$$\begin{aligned} \theta(0) = \theta(0, x) &= \xi(x) \quad x \in (0, \pi), \\ \theta(t, 1) = 0, \quad \theta(t, 0) &= u(t), \quad t > 0. \end{aligned} \quad (2)$$

The function $u(\cdot)$ is locally square integrable for $t > 0$ and the initial condition ξ belongs to $L^2(0, \pi)$. The operator Δ in Eq. (1) is the laplacian in one space dimension,

$$\Delta\theta(x) = \theta_{xx}(x).$$

Assumptions on the kernel $N(t)$ are described below.

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Equation (1) has been independently proposed by several authors as a version of the “heat equation” with finite signal speed, noticeably in [8] and [14].

The goal of this paper is the identification of a special sequence $\{z_n(t)\}$ of functions, related to Eq. (1), which forms a *Riesz system* in $L^2(0, \pi)$ (the definition is in Section 1.1). The functions $z_n(t)$ are the solution of the integrodifferential equation

$$z_n'(t) = -n^2 \int_0^t N(t-s)z_n(s) ds, \quad z_n(0) = 1.$$

The introduction of these functions is suggested by a control problem (described below) and in this paper we use these functions (and known results on the controllability of Eq. (1)) in order to represent a function u , a “control”, which drives the initial condition ξ to a final target $\eta \in L^2(0, \pi)$ in time T (it will be $T \geq \pi$); i.e. a control which solves the problem

$$\theta(T) = \eta. \tag{3}$$

So, we can interpret the results in this paper as a “constructive” solution of the control problem (1)–(3). In fact, this is only a part of the story since moment problems appear often in applied mathematics, for example in the solution of inverse problems, see [1, 4], so that the identification of a suitable Riesz system which is naturally associated to Eq. (1) is interesting by itself, in particular because the solution of moment problems posed with respect to Riesz systems are well posed. Algorithms for the solution of moment problems are described in [1].

Applications of the Riesz system introduced here to the solution of inverse problems will appear elsewhere, see [22].

Let us now comment on the solvability of the problem (1)–(2). Existence and unicity of solution for every locally square integrable control u is proved in [20], provided that the kernel $N(t)$ is twice differentiable and $N(0) > 0$ (so we can assume $N(0) = 1$.) In that paper it is proved that for every square integrable control u , the solution is a continuous $L^2(0, \pi)$ -valued function, so that evaluation of the solution at a certain time T is permissible.

The controllability problem for Equation (1) has been studied and solved firstly in [5]. It was proved in that paper that the controllability problem is solvable under an additional regularity condition on η , provided that $T > \pi$. The solution rests on Carleman estimates and it can't be considered constructive. The case that the space variable belongs to \mathbb{R}^n has been considered in [27], still using Carleman estimates, and in [20]. The proof in this last paper (see also [21]) is based on Baire theorem and compactness arguments, and it is not constructive. In this paper we first show that the controllability problem can be reduced to a suitable moment problem with respect to the *Riesz system* $\{z_n\}$.

The known results on the controllability problem shortly recalled above (and described with more details in Section 3.2) are derived under the assumption that $N(t)$ is of class C^3 . So, as in the previous papers on controllability, *the standing assumptions in this paper are that the kernel $N(t)$ is of class C^3 and $N(0) = 1$* . Controllability for a different class of kernels is studied in [19].

Finally, let us note that even if $\eta = 0$ then we don't have controllability to rest: we construct a control which forces the solution to hit the 0 target at a certain time T but we cannot force the solution to remain at rest in the future: in fact, in the case of heat equations with memory, controllability to rest can be achieved only in exceptional cases, see [15].

1.1 Preliminaries on the abstract moment problem

Key references for this section are [1, 2, 12, 28].

We consider a *separable Hilbert space H* . A sequence $\{z_n\}_{n \geq 1}$ in H is a *Schauder basis* when every element $h \in H$ can be represented in a unique way as

$$h = \sum_{n=1}^{+\infty} \alpha_n z_n. \quad (4)$$

The convergence of the series is in the norm of H . A Schauder basis has the additional property of being a *minimal basis*, i.e. for every j we have

$$z_j \notin \text{clspan}\{z_r, r \neq j\},$$

see [12, p. 312].

An abstract moment problem in H is as follows. Let z_n be elements of H and let $\{c_n\} \in l^2$. We want to know whether it is possible to find $v \in H$ such that

$$\langle v, z_n \rangle = c_n \quad (5)$$

(the inner product is that of H .) A moment problem like (5) can be defined in a Banach space too and in this case the crochet represents duality.

Moment problems are mostly studied in the case that the functions z_n have a special form (polynomials or exponentials) and this study has been at the core of functional analysis. However, even if the functions z_n do not have any special form, conditions for solvability are in [2, Theorem I.2.1]. In particular, if the moment problem is solvable for every element $\{c_n\}$ of a dense subset of l^2 then the sequence $\{z_n\}$ is ω -*(linearly) independent*, i.e. the following property holds:

$$\{\alpha_n\} \in l^2 \quad \text{and} \quad \sum \alpha_n z_n = 0 \implies \{\alpha_n\} = 0. \quad (6)$$

If $\{z_n\}$ is a minimal basis, it is possible to give a formula for the coefficients associated to h in (4). It is possible to construct a *biorthogonal basis* of

$\{z_n\}$. I.e. it is possible to find a basis $\{\zeta_n\}$ such that

$$\langle z_n, \zeta_k \rangle = \delta_{n,k}$$

($\delta_{n,k}$ is the Kronecker delta) and the (unique) element α_n in the representation (4) of h is

$$\alpha_n = \langle h, \zeta_n \rangle.$$

So, when $\{z_n\}$ is a minimal basis and problem (5) is solvable, its solution is

$$v = \sum_{n=1}^{+\infty} c_n \zeta_n.$$

In spite of this, practical computations with minimal basis do not lead to well posed problems, since the biorthogonal basis can be unbounded.

A more restrictive condition, under which practical computations are feasible, is that the sequence $\{z_n\}$ be a *Riesz sequence* (or “Riesz system”). A Riesz sequence is called an *L-sequence* in [2].) The meaning of this is as follows: let $L \subseteq H$ be the closed linear space spanned by the vectors z_n . The sequence $\{z_n\}$ is a Riesz sequence when there exists an orthogonal basis $\{\epsilon_n\}$ (of a Hilbert space H') and a linear bounded operator T' from H' onto L which is boundedly invertible and such that $T'\epsilon_n = z_n$. If $L = H$ then the sequence $\{z_n\}$ is called a Riesz basis of H . Riesz systems share the following properties with Fourier series: **1)** every Riesz system is bounded; **2)** there exist positive numbers m and M such that

$$m \sum |\alpha|^2 \leq \left\| \sum \alpha_n z_n \right\|_H^2 \leq M \sum |\alpha|^2.$$

Note that the biorthogonal sequence of a Riesz basis is a Riesz basis too, hence it is bounded.

If a system is a Riesz system in the Hilbert space H , which is not a basis, then it may have unbounded “biorthogonal system” due to the fact that the set of the projections of ζ_n on $[\text{span}\{z_n\}]^\perp$ can be unbounded but it is possible to find such biorthogonal sets which are bounded. In particular, the one which belongs to $\text{clspan}\{z_n\}$ is bounded and it is a Riesz system too. Moreover, when $\{z_n\}$ is a Riesz system, then the series

$$\sum \alpha_n z_n$$

converges if and only if $\{\alpha_n\} \in l^2$ and every $f \in L = \text{clspan}\{z_n\}$ can be represented as

$$f = \sum \alpha_n z_n, \quad \alpha_n = \langle f, \zeta_n \rangle$$

where $\{\zeta_n\}$ is biorthogonal to $\{z_n\}$.

So, we wish not only that our moment problem is solvable, but also that our sequence $\{z_n\}$ is a Riesz sequence. Several tests have been given for this. We shall use the following one:

Theorem 1.1 (Bari Theorem) *If $\{\epsilon_n\}$ is a Riesz sequence in H and if the sequence $\{z_n\}$ satisfies conditions (6) and*

$$\sum_{n=1}^{+\infty} \|z_n - \epsilon_n\|^2 < +\infty \quad (7)$$

then the sequence $\{z_n\}$ is a Riesz basis too.

See [28, p. 45] and [12, p. 322] for the proof (the proof in [28, p. 45] assumes that $\{\epsilon_n\}$ is an orthonormal basis.)

As we said, if condition (6) holds, then the sequence $\{z_n\}$ is called ω -independent; if condition (7) holds, the sequence $\{z_n\}$ is called *quadratically close* to $\{\epsilon_n\}$ (when $H = L^2(0, T)$ we also say “ L^2 -close” to $\{\epsilon_n\}$) and the Riesz basis $\{z_n\}$ is called in particular a “Bari basis”.

In fact, we need a minor variant of Bari Theorem, which is as follows:

Theorem 1.2 *Let $\{\epsilon_n\}_{n \geq 0}$ be a Riesz basis of H . If $\{z_n\}_{n \geq 1}$ is ω -independent and quadratically close to $\{\epsilon_n\}_{n \geq 1}$, then $\{z_n\}_{n \geq 1}$ is a Riesz system in H .*

The proof is outlined in [12, Remarque 2.1, p. 323] and it is reported in the Appendix for the sake of completeness.

A different version of Bari theorem used in control theory is in [13].

1.2 Preliminaries on the wave equation

In order to put this paper in the proper setting, we recall here the very well known problem of the controllability of the wave equation in one space dimension,

$$w_{tt} = w_{xx} \quad 0 < x < \pi, \quad t > 0 \quad (8)$$

with conditions

$$w(t, 0) = u(t), \quad w(t, \pi) = 0, \quad \begin{cases} w(0, x) = \xi_0 \\ w_t(0, x) = \xi_1. \end{cases}$$

The problem we consider is as follows: a target $\eta \in L^2(0, \pi)$ is given and we want to find a suitable time $T > 0$ and a square integrable control $u(\cdot)$ such that $w(T, \cdot) = \eta(\cdot)$. Note that we are controlling the final shape but not the final velocity. We shall see that it is possible to choose $T = \pi$, the same time for every initial conditions and target (if instead we want to control both the configuration and the velocity then it must be $T = 2\pi$.) The proof of this result goes as follows: we consider the functions $\phi_n(x) = \sin nx$, $n \in \mathbb{N}$, which solve the eigenvalue problem

$$\phi_n''(x) = -n^2 \phi_n(x), \quad \phi_n(0) = \phi_n(\pi) = 0. \quad (9)$$

We compute the scalar product (in $L^2(0, \pi)$) of both the sides of (8) with ϕ_n . Integration by parts in x gives the equality

$$\frac{d^2}{dt^2} \langle w, \phi_n \rangle = -n^2 \langle w, \phi_n \rangle + \phi_n'(0)u(t)$$

so that

$$\langle w(t), \phi_n \rangle = A_n \cos nt + B_n \sin nt + \int_0^t [\sin ns]u(t-s) ds.$$

The coefficients A_n and B_n are given by

$$A_n = \langle \xi_0, \phi_n \rangle, \quad B_n = \frac{1}{n} \langle \xi_1, \phi_n \rangle$$

(so that $w(t, \cdot) \in H^1(0, \pi)$ if $\xi_0(\cdot) \in H^1(0, \pi)$ and $\xi_1(\cdot) \in L^2(0, \pi)$.) The condition $w(\pi, \cdot) = \eta(\cdot)$ is then equivalent to the moment problem

$$\int_0^\pi [\sin ns]v(s) ds = \langle \eta, \phi_n \rangle - (-1)^n \langle \xi_0, \phi_n \rangle = c_n \quad (10)$$

where $v(s) = u(\pi - s)$: the control $u(\cdot)$ exists if and only if there exists a function v which solves the equalities (10) for every n . The known theory of the Fourier series shows that this problem is very easily solved:

$$v(t) = \frac{2}{\pi} \sum_{k=1}^{+\infty} c_k \sin kt. \quad (11)$$

We can go the opposite way: if controllability of the wave equation has been independently proved, then the previous arguments are a proof of the fact that the moment problem (10) is solvable. This is the turn of ideas we follow in this paper: we first prove that the sequence $\{z_n\}$ is L^2 -close to a Riesz system; we then use the fact that control problem (1)-(3) is known to be solvable in order to prove that the sequence $\{z_n\}$ is ω -independent so that Bari Theorem can be applied.

So, it will turn out that the sequence $\{z_n\}$ is a Riesz sequence. We then use this property in order to derive a formula for the control which drives the initial condition ξ to the target η , which extends (11).

1.3 References

It seems that one of the first papers which uses moment problems in control theory is [7], followed by several papers in particular by Russel and Fattorini (see [10, 11, 23].) Among the numerous recent papers we confine ourselves to cite the papers [3, 16]. A part the papers, too numerous to be cited, the applications of the moment problem to control theory has been examined in three books: [2, 17, 18].

As we noted, Riesz systems are crucial in the solution of a large class of inverse problems. As an example of this, we cite the paper [4].

We note that the “classical” moment problems, when $\{z_n\}$ is a sequence of polynomials, do not correspond to Riesz basis and in fact the truncated problems obtained by considering only finitely many equation is severely ill conditioned, see [25]. In contrast with this, moment problems which correspond to Riesz basis are well posed problems.

An application of the results proved here to the solution of an identification problem can be found in [22].

2 Reduction to a moment problem

In this section we are going to prove that the controllability problem we presented for the equation (1) is equivalent to a certain moment problem. As in the case of the wave equation, we consider the functions $\phi_n(x) = \sin nx$, $n \in \mathbb{N}$, which solve the eigenvalue problem (9). We note that $\{\phi_n\}$ is a complete orthogonal system in $L^2(0, \pi)$ (not a normal system: $\|\phi_n\| = \sqrt{\pi/2}$.)

Let λ be one of the numbers $-n^2$ and let ϕ be the corresponding eigenfunction $\sin nx$. Let $h(t)$ solve

$$h'(t) = -\lambda \int_t^T N(r-t)h(r) \, dr, \quad h(T) = 1 \quad (12)$$

(here T is any fixed number. Its value for the controllability problem will be specified later on.) Let the control function $u(t)$ and the initial datum ξ be fixed and let us compute the $L^2((0, \pi) \times (0, T))$ inner product of $\theta(t, x)$ and $h(t)\phi(x)$. Integration by parts gives the following equality:

$$\begin{aligned}
& \int_0^\pi \phi(x) \int_0^T \frac{d}{dt} [h(t)\theta(t, x)] dt dx \\
&= \int_0^\pi \phi(x) \int_0^T h'(t)\theta(t, x) dt dx + \int_0^\pi \phi(x) \int_0^T h(t)\theta_t(t, x) dt dx \\
&= \int_0^\pi \phi(x) \int_0^T \left[-\lambda \int_t^T N(r-t)h(r) dr \right] \theta(t, x) dt dx \\
&+ \int_0^\pi \phi(x) \int_0^T h(t) \left[\int_0^t N(t-r)\theta_{xx}(r, x) dr \right] dt dx \\
&= \int_0^\pi \phi(x) \int_0^T \left[-\lambda \int_t^T N(r-t)h(r) dr \right] \theta(t, x) dt dx \\
&+ \int_0^T h(t) \int_0^t N(t-r) \left[\int_0^\pi \phi(x)\theta_{xx}(r, x) dx \right] dr dt \\
&= \int_0^\pi \phi(x) \int_0^T \left[-\lambda \int_t^T N(r-t)h(r) dr \right] \theta(t, x) dt dx \\
&+ \int_0^T h(t) \int_0^t N(t-r) \left[\phi'(0)u(r) + \lambda \int_0^\pi \phi(x)\theta(r, x) dx \right] dr dt.
\end{aligned}$$

So we have

$$\begin{aligned}
& \int_0^\pi \phi(x)\theta(T, x) dx - h(0) \int_0^\pi \phi(x)\xi(x) dx = \phi'(0) \int_0^T h(r)\tilde{v}(r) dr, \\
& \tilde{v}(t) = \int_0^t N(t-r)u(r) dr. \tag{13}
\end{aligned}$$

Smoothness assumptions needed to justify integration by parts hold provided that u is smooth and certain compatibility conditions between u and ξ hold, see [20]. These conditions are satisfied by the pairs (ξ, u) in a dense subset of $L^2(0, \pi) \times L^2(0, T)$. Both the sides of (13) are continuous functions of u and ξ so that equality (13) holds for every initial condition $\xi \in L^2(0, \pi)$ and every control $u \in L^2(0, T)$. This proves the necessity part of the following result:

Lemma 2.1 *Let $h_n(t)$ be the solution of problem (12) with $\lambda = -n^2$. Let ξ be an initial condition and let η be the prescribed target, both in $L^2(0, \pi)$. A square integrable control u which transfer ξ to η in time T exists if and only if for every n we have*

$$\int_0^T h_n(r)\tilde{v}(r) dr = \frac{1}{\phi'_n(0)} \int_0^\pi \phi_n(x) [\eta(x) - h_n(0)\xi(x)] dx \tag{14}$$

where

$$\tilde{v}(r) = \int_0^r N(r-s)u(s) ds. \tag{15}$$

Proof. We need to prove the sufficiency part. We assume that there exists a control u such that the function \tilde{v} defined in (15) solves (14) for every n . Repeating the computations above we see that

$$\int_0^T h_n(r)\tilde{v}(r) \, dr = \frac{1}{\phi'_n(0)} \int_0^\pi \phi_n(x) [\theta(T, x) - h_n(0)\xi(x)] \, dx.$$

Our assumption is that (14) holds for the function $\tilde{v}(r)$ given by (15) so that we have also

$$\int_0^\pi \phi_n(x)\theta(T, x) \, dx = \int_0^T \phi_n(x)\eta(x) \, dx.$$

The set $\{\phi_n(x)\}$ being complete in $L^2(0, \pi)$, we get $\theta(t, x) = \eta(x)$, as wanted. ■

In the next section, we shall concentrate on the solution of the problem (14) in terms of \tilde{v} and we disregard the fact that \tilde{v} should have a special expression in terms of u . I.e. we study a different problem: the problem of finding a square integrable function \tilde{v} which solves (14). Once this problem is solved and the regularity properties of the solution \tilde{v} have been studied, we shall see that also the original control problem (in terms of u) can be solved.

Let us consider now the right hand side of (14). The integral on the right hand side is

$$c_n = \langle \phi_n, \eta \rangle_{L^2(0, \pi)} - h_n(0) \langle \phi_n, \xi \rangle_{L^2(0, \pi)}.$$

We shall see in section 3 that the sequence $\{h_n(0)\}$ is bounded so that the sequence $\{c_n\}$ belongs to l^2 : the problem of determining a function \tilde{v} which solves (14) is a moment problem in $L^2(0, T)$,

$$\int_0^T h_n(r)\tilde{v}(r) \, dr = \frac{c_n}{n}, \quad \{c_n\} \in l^2. \quad (16)$$

The sequence $\{h_n(0)\}$ being bounded, the sequence $\{c_n/n\}$ fills a dense subspace of l^2 , which is a proper subspace of l^2 .

3 The Riesz system

In this section we prove that the sequence $\{h_n\}$ which appears in (16) is a Riesz system in a suitable bounded interval identified below (and, in the course of the proof, we shall also see boundedness of the sequence of functions $\{h_n(t)\}$ on every bounded interval.) Computations have a more natural appearance if we use the following transformation: $z_n(t) = h_n(T - t)$ (note that T will be π later on. For the moment, T is a any fixed number.) The function $z_n(t)$ solves

$$z'_n(t) = -n^2 \int_0^t N(t-s)z_n(s) \, ds, \quad z_n(0) = 1 \quad (17)$$

and the moment problem takes the form

$$\int_0^T z_n(t)v(t) dt = \frac{1}{n}c_n, \quad c_n = \int_0^\pi \phi_n(x)[\eta(x) - z_n(T)\xi(x)] dx \quad (18)$$

where now $v(t) = \tilde{v}(T-t)$ and $u(t)$ has to be determined from the equation

$$\int_0^t N(t-s)u(s) ds = v(T-t). \quad (19)$$

The proof that $\{z_n(t)\}_{n \geq 1}$ is a Riesz sequence in $L^2(0, \pi)$ is based on Theorem 1.2 and it is divided in two parts: we first prove (in subsection 3.1) that $\{z_n(t)\}_{n \geq 1}$ is quadratically close to the sequence $\{e^{\alpha t} \cos nt\}_{n \geq 1}$, where $\alpha = N'(0)/2$, (the sequence $\{\cos nt\}$ plus the element $1/\sqrt{2}$ is a complete orthogonal system of elements of constant norm in $L^2(0, \pi)$ so that the sequence $\{e^{\alpha t} \cos nt\}_{n \geq 1}$ is a Riesz system.) The property of ω -independence is in subsection 3.2. In order to illustrate the ideas in this paper as clearly as possible, the computations in Section 3.1 uses the restrictive assumption $N'(0) = 0$. At the expenses of more involved computations, the same ideas can be used in the general case, even if $N'(0) \neq 0$, as shown in the Appendix.

3.1 L^2 -closedness to a Riesz sequence

In this section we prove that $\{z_n(t)\}$ is quadratically close to a Riesz sequence. The sole conditions on the kernel $N(t)$ needed in this proof are the standing assumptions of this paper: $N(t)$ is of class C^3 with $N(0) = 1$. The computations in this case are involved and relegated to the appendix. For most of clarity, we here present the computations under the restrictive condition $N'(0) = 0$. In the case $N'(0) = 0$ we prove that $\{z_n(t)\}_{n \geq 1}$ is L^2 -close to the the Riesz sequence $\{\cos nt\}_{n \geq 1}$.

We compute the derivatives of both the sides of (17) and we see that

$$z_n''(t) = -n^2 z_n(t) - n^2 \int_0^t N'(t-s)z_n(s) ds, \quad z_n(0) = 1, \quad z_n'(0) = 0$$

so that

$$z_n(t) = \cos nt - n \int_0^t \sin n(t-s) \int_0^s N'(s-r)z_n(r) dr ds.$$

Integration by parts gives

$$\begin{aligned}
z_n(t) &= \cos nt - \int_0^t \left[\frac{d}{ds} \cos n(t-s) \right] \int_0^s N'(s-r) z_n(r) dr ds \\
&= \cos nt + N'(0) \int_0^t \cos n(t-s) z_n(s) ds - \int_0^t N'(t-r) z_n(r) dr \\
&\quad + \int_0^t \cos n(t-s) \int_0^s N''(s-r) z_n(r) dr ds \\
&= \cos nt + \int_0^t [N'(0) \cos n(t-r) - N'(t-r)] z_n(r) dr \\
&\quad + \int_0^t \left[\int_r^t \cos n(t-s) N''(s-r) ds \right] z_n(r) dr. \tag{20}
\end{aligned}$$

Gronwall inequality shows the existence of a constant M such that

$$|z_n(t)| < M, \quad t \in [0, T].$$

The number M does depend on T but not on n . This shows:

Lemma 3.1 *The sequence $\{c_n\}$ in (18) belongs to l^2 .*

Note that this holds also if $N'(0) \neq 0$. From now on instead we use the restrictive condition $N'(0) = 0$ (to be removed later on.)

Equality (20) suggests comparison of $z_n(t)$ with $\cos nt$. The sequence $\{\cos nt\}$ being orthogonal on $L^2(0, \pi)$, with constant norm, from now on we impose

$$T = \pi.$$

We introduce $e_n(t) = z_n(t) - \cos nt$ and we see that, when $N'(0) = 0$,

$$\begin{aligned}
e_n(t) &= - \int_0^t N'(t-r) e_n(r) dr + \int_0^t \cos n(t-r) \int_0^r N''(r-s) e_n(s) ds dr \\
&\quad - \int_0^t N'(t-s) \cos ns ds + \int_0^t \cos n(t-r) \int_0^r N''(r-s) \cos ns ds dr. \tag{21}
\end{aligned}$$

Both the integrals in the last line can be integrated by parts:

$$\begin{aligned}
\int_0^t N'(t-s) \cos ns ds &= \frac{1}{n} \int_0^t \left[\frac{d}{ds} \sin ns \right] N'(t-s) ds \\
&= \frac{1}{n} \int_0^t N''(t-s) \sin ns ds \asymp \frac{1}{n}, \tag{22}
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \cos n(t-r) \int_0^r N''(r-s) \cos ns ds dr \\
&= \frac{1}{n} \int_0^t \cos n(t-r) \left\{ N''(0) \sin nr \right. \\
&\quad \left. + \int_0^r N'''(r-s) \sin ns ds \right\} dr \asymp \frac{1}{n} \tag{23}
\end{aligned}$$

(the integral in (22) can be integrated by parts once more, and (22) is of the order $1/n^2$). Using again Gronwall inequality we see the existence of a constant M such that

$$|e_n(t)| \leq \frac{M}{n}.$$

This shows that the sequence $\{z_n(t)\}$ is quadratically close to the sequence $\{\cos nt\}$.

The inequality above is sufficient for the application of Bari Theorem, but section 4 will use a refined version of this inequality.

The inequalities in this section have been derived in the simple case $N'(0) = 0$. We state now the general result, proved in the appendix:

Theorem 3.2 *Let $N(t)$ be of class C^3 and let $N'(0) = 1$. We define $\alpha = N'(0)/2$. The sequence $\{z_n(t)\}$ is $L^2(0, T)$ -close to the sequence $\{e^{\alpha t} \cos nt\}$ for every $T > 0$.*

Remark 3.3 *Similar computations show that the sequence $\{z'_n(t)/n\}$ is L^2 -close to the sequence $\{-\sin nt\}$ and the fact that the sequence $\{n \int_0^t z_n(s) ds\}$ is L^2 -close to the sequence $\{\sin nt\}$.*

3.2 The property of ω -independence

In the previous section we stated that the sequence $\{z_n(t)\}$ is $L^2(0, T)$ -close to the Riesz sequence $\{e^{\alpha t} \cos nt\}$ where $\alpha = N'(0)/2$. Here $T > 0$ is arbitrary. In this section we use known controllability results in order to prove that *if $T \geq \pi$, then $\{z_n(t)\}$ is ω -independent* so that, from Theorem 1.2, it is a Riesz system in $L^2(0, \pi)$. This is the point where finite signal speed has to be taken into account. This property, derived by many authors, see for example [6, 9], is recalled in the form we need in this paper. For completeness, a sketch of the proof is given in the Appendix.

We consider system (1) but now

- on the interval $(-\epsilon, \pi)$ with $\epsilon > 0$;
- with homogeneous Dirichlet boundary condition $\theta(t, -\epsilon) = 0$, $\theta(t, \pi) = 0$ and null initial condition, $\theta(0, x) = 0$;
- the system is acted upon by a control distributed on $(-\epsilon, 0)$.

I.e. we consider the system

$$\begin{cases} \theta_t(t, x) = \int_0^t N(t-s) \Delta \theta(s, x) ds + \chi(x) \nu(t, x) & x \in (-\epsilon, \pi) \\ \theta(0, x) = 0 & x \in (-\epsilon, \pi), \quad \theta(t, -\epsilon) = \theta(t, \pi) = 0. \end{cases} \quad (24)$$

The function $\chi(x)$ is the characteristic function of $(-\epsilon, 0)$. The function $\nu(t, x)$ is a ‘‘distributed control’’, and belongs to $L^2_{\text{loc}}((0, +\infty) \times (-\epsilon, 0))$.

The property of “finite propagation” we need is that $\theta(t, x)$ is supported on $x \in [-\epsilon, \pi - \epsilon]$ for every $t < \pi - \epsilon$.

Now we observe that a smooth control $\nu(t, x)$ produces a smooth solution $\theta(t, x)$ so that evaluation of $\theta(t, x)$ at $x = 0$ is possible and, when we restrict the space variable x to $[0, \pi]$, this corresponds to consider the above control problem with “boundary control” at 0 given by $u(t) = \theta(t, 0)$. The conditions under which $\theta(t, x)$ is smooth (so that the computation of $u(t) = \theta(t, 0)$ makes sense) are in [20, Appendix]. The required (smoothness) conditions on $\nu(t, x)$ are satisfied in a dense subset of $L^2((0, T) \times (-\epsilon, 0))$.

Now we recall the known controllability results, from [5] (under more restrictive assumptions) and from [26, 27], under the assumption in this paper (and in fact more general, see below.) These papers study controllability under distributed control. The results proved in these paper cover the case of a space interval $[-\epsilon, \pi]$ with distributed control supported on $[-\epsilon, 0]$ and every initial and final condition in $L^2(-\epsilon, \pi)$ (actually, [5] imposes further regularity to the target.) Exact controllability is proved in time $T \geq \pi + \epsilon$. As described in [5], this result on controllability with control acting on $[-\epsilon, 0]$ implies boundary controllability on $[0, \pi]$. Let for simplicity the initial condition ξ , defined on $[0, \pi]$, be $\xi = 0$. Extend ξ and the target $\eta(x)$ (given on $[0, \pi]$) with 0 to $[-\epsilon, 0]$ and consider the distributed system (24). Construct the steering *distributed* control, in time $T + \epsilon$ for this new function, defined on $(-\epsilon, \pi)$ and then use the trace $u(t) = \theta(t; 0)$ as the boundary control. So, we have exact controllability in every time $T > \pi$; and, more in general, *boundary controllability is possible in an arbitrary time, longer then the width of the interval*. However, the steering control $\nu(t, x)$ so constructed needs not be smooth so that in principle $u(t)$ might not be well defined. In spite of this, we are going to show that the previous argument implies approximate controllability, and this is sufficient for our needs.

Approximate controllability is seen as follows: let $R_T \subseteq L^2(-\epsilon, \pi)$ be the reachable set at time T for the control system (24) and let $R_{T;0} \subseteq L^2(0, \pi)$ be the set of the restrictions to $L^2(0, \pi)$ of the elements of R_T . The previous observations on controllability proves that, *for every $\sigma > 0$* , every function of $L^2(0, \pi)$ whose support is in $[0, \pi - \sigma]$ belongs to $R_{\pi;0}$. I.e. the subspace $R_{\pi;0}$ is dense in $L^2(0, \pi)$. Density is retained if we confine ourselves to consider solely those elements which are reachable by using smooth controls $\nu(t, x)$.

So, we conclude approximate controllability: every $\eta(x)$ in a dense subset of $L^2(0, \pi)$ is a reachable target for the control system (1)-(2).

Let us go back to consider the moment problem (18) with $\xi = 0$. The moment problem is solvable for every reachable target. So, the set of the sequences $\{\eta_n\}$ for which the following moment problem is solvable is a dense subspace of l^2 . The moment problem is

$$\int_0^T z_n(t)v(t) dt = \frac{1}{n}\eta_n, \quad \eta_n = \int_0^\pi \phi_n(x)\eta(x) dx, \quad \eta \in R_{\pi;0}. \quad (25)$$

Also the set of the sequences $\{\eta_n/n\}$ which correspond to solvable moment problems is dense in l^2 . This shows that the sequence $\{z_n(t)\}_{n \geq 1}$ is ω -independent, see [2, Theorem I.2.1 (d)], as we wanted: Theorem 1.2 can be applied and we conclude that the sequence $\{z_n\}$ is a Riesz sequence in $L^2(0, \pi)$.

In conclusion, we have the following result:

Theorem 3.4 *The sequence $\{z_n(t)\}_{n \geq 1}$ is a Riesz system in $L^2(0, \pi)$, provided that $N(t) \in C^3(0, \pi + \epsilon)$, $N(0) = 1$.*

Remark 3.5 *The results on controllability in [5] concerns one dimensional space variable and require that $N(t)$ is continuous for $t \geq 0$ and completely monotonic, i.e. of class C^∞ with derivatives of alternating sign; in particular, $N(t) \geq 0$. The result in [27] considers the case of elliptic operators with variable coefficients in a region Ω of \mathbb{R}^n . The kernel N can depend on x , $N = N(t, x)$. The coefficients and $N(t, x)$ have to be of class C^3 and $N(0, x) > 0$. Both these papers identify the controllability time as a consequence of Carleman estimates. Also the papers [20, 21] studies the problem in a region of \mathbb{R}^n but the controllability time is not identified there.*

Finally, we note that the controllability result which is really needed in this section is approximate controllability.

4 Back to the control problem

In the previous section we proved that the moment problem (18), i.e. (14), is solvable for a suitable square integrable function v . The proof is based on the fact that controllability of our system has been already proved with different methods but, as we noted, these methods do not provide a real construction of the control $u(t)$ which steers the initial datum ξ to the prescribed target η . This problem is examined now.

Moment method gives a formula for the solution $v(t)$ of problem (18), i.e.

$$v(t) = \sum_{n=1}^{+\infty} \frac{c_n}{n} \zeta_n(t) \quad (26)$$

where $\{\zeta_n(t)\}$ is biorthogonal to $\{z_n(t)\}$. The steering control solves the Volterra integral equation of the first kind (19). We are now going to prove that the function $v(t)$ (26) is of class $W^{1,2}(0, \pi)$ so that the control $u(t)$ can be computed from the Volterra integral equation of the second kind

$$u(t) + \int_0^t N'(t-s)u(s) \, ds = v'(t). \quad (27)$$

In order to prove this additional regularity of $v(t)$, we need the following estimate:

Lemma 4.1 *Let $\alpha = N'(0)/2$. There exists a sequence of square integrable functions $\{a_n(t)\}$ such that for $t \in [0, \pi]$ we have:*

$$\left| z_n(t) - e^{\alpha t} \cos nt - \frac{e^{\alpha t}}{2n} [2\alpha + N''(0)t] \sin nt \right| \leq \frac{a_n(t)}{n}, \quad (28)$$

$$\sum_{n=1}^{+\infty} |a_n(t)|^2 < M + \infty, \quad \int_0^\pi \left[\sum_{n=1}^{+\infty} |a_n(t)|^2 \right] dt < +\infty. \quad (29)$$

Proof. We present the proof in the case $N'(0) = 0$ (see the appendix for the general case). We observed already that the row (22) is of the order $1/n^2$. Instead, row (23) gives a term of the order $1/n^2$ plus the contribution

$$\frac{N''(0)}{2n} t \sin nt + \frac{\tilde{a}_n(t)}{n}.$$

For fixed t , $\tilde{a}_n(t)$ is

$$\tilde{a}_n(t) = \int_0^t \cos n(t-r) b_n(r) dr, \quad b_n(r) = \int_0^r N^{(3)}(r-s) \sin ns ds.$$

So, for any r the sequence $\{b_n(r)\}$ is the sequence of the Fourier coefficients (in a sine series and a part the factor $2/\pi$) of a function which is 0 for $s > r$, and $N^{(3)}(r-s)$ otherwise. Hence,

$$\sum_n |b_n(r)|^2 = \frac{\pi^3}{4} \int_0^r |N^{(3)}(s)|^2 ds < +\infty$$

and the series has finite integral on $[0, \pi]$. These properties are then retained by the sequence $\{\tilde{a}_n(t)\}$;

$$\sum_{n=1}^{+\infty} |\tilde{a}_n(t)|^2 \leq M \quad \forall t \in [0, \pi].$$

We introduce

$$\hat{e}_n(t) = [e_n(t) - (N''(0)/2n)t \sin nt]$$

in (21) so that, with certain kernels $M_n(t)$, we get

$$\begin{aligned} \hat{e}_n(t) = & \int_0^t M_n(t-s) \hat{e}_n(s) + \frac{N''(0)}{2n} \left[- \int_0^t N'(t-r) r \sin nr \right. \\ & \left. + \int_0^t \cos n(t-r) \int_0^r N''(r-s) s \sin ns ds \right] dr + \frac{1}{n} \tilde{a}_n(t) + \frac{1}{n^2} \Psi_n(t). \end{aligned}$$

The functions $M_n(t)$ and $\Psi_n(t)$ are bounded on $[0, \pi]$, uniformly with respect to n . Moreover, the integrals in bracket are of the order of $1/n$ so that (with

a suitable constant M) the following inequality holds for every n and every $t \in [0, \pi]$:

$$|\hat{e}_n(t)| \leq \int_0^t M|\hat{e}_n(s)| \, ds + \frac{M}{n^2} + \frac{1}{n}|\tilde{a}_n(t)|. \quad (30)$$

Now we proceed as in the proof of the generalized Gronwall inequality in [24, p. 11]. Multiplying both the sides of (30) with Me^{-Mt} we see that

$$\frac{d}{dt} \left[e^{-Mt} \int_0^t M|\hat{e}_n(s)| \, ds \right] \leq e^{-Mt} \left\{ \frac{M}{n^2} + \frac{1}{n}|\tilde{a}_n(t)| \right\}.$$

So, with suitable constants H and K we get on $[0, \pi]$

$$\int_0^t M|\hat{e}_n(s)| \, ds \leq \frac{H}{n^2} + \frac{K}{n} \int_0^t |\tilde{a}_n(s)| \, ds.$$

We replace this estimate in (30) and we see that

$$|\hat{e}_n(t)| \leq \frac{a_n(t)}{n}, \quad a_n(t) = \frac{M+H}{n} + \left\{ \tilde{a}_n(t) + K \int_0^t |\tilde{a}_n(s)| \, ds \right\}.$$

The sequence $\{a_n(t)\}$ inherits properties (29) from the corresponding properties of $\{\tilde{a}_n(t)\}$. ■

Remark 4.2 Note that when $N(t) \in W^{4,2}(0, \pi)$ then it is easier to find $|\hat{e}_n(t)| < M/n^2$.

Now we present an additional piece of information from the proof of Bari Theorem: let $\epsilon_n = e^{\alpha t} \cos nt$ (this is a Riesz system). We know that $\{z_n(t)\}$ is ω -independent and L^2 -close to $\{\epsilon_n\}$. It is then seen from the proof of Bari Theorem that

$$\zeta_k = \mathcal{T}^* \zeta_k + \epsilon_k$$

where \mathcal{T} is the operator

$$\mathcal{T}f = - \sum_{k=1}^{+\infty} \langle f, \epsilon_k \rangle e_k, \quad \mathcal{T}^*g = - \sum_{k=1}^{+\infty} \epsilon_k \langle g, e_k \rangle.$$

Hence, each function $\zeta_n(t)$ has the following representation

$$\begin{aligned} \zeta_n(t) &= \epsilon_n - \sum_{k=1}^{+\infty} \epsilon_k \int_0^\pi \zeta_n(s) \hat{e}_k(s) \, ds - \sum_{k=1}^{+\infty} \epsilon_k \int_0^\pi \Phi_n(s) \frac{\sin ks}{k} \, ds, \\ \Phi_n(s) &= e^{\alpha s} [\alpha + (N''(0)s)/2] \zeta_n(s). \end{aligned} \quad (31)$$

We recall that the biorthogonal sequence of a Riesz basis is a Riesz basis, hence it is bounded. So, also the sequence $\{\Phi_n(t)\}$ is bounded in $L^2(0, \pi)$.

Finally, we introduce the sequence of the functions

$$r_k(t) = \frac{1}{k} \epsilon_k'(t) = e^{\alpha t} \left[\alpha \frac{\cos kt}{k} - \sin kt \right]. \quad (32)$$

We shall see in Appendix the existence of a number M such that the following inequalities hold, for every $f \in L^2(0, \pi)$ and $\{\gamma_n\} \in l^2$:

$$\begin{cases} \sum_k \left| \int_0^\pi f(s) r_k(s) ds \right|^2 \leq M \int_0^\pi |f(s)|^2 ds, \\ \int_0^\pi \left| \sum_k \gamma_k r_k(t) \right|^2 dt < M \sum_k |\gamma_k|^2. \end{cases} \quad (33)$$

We compute:

$$\int_0^\pi \Phi_n(s) \sin ks ds = - \int_0^\pi [e^{-\alpha s} \Phi_n(s)] r_k(s) ds + \alpha \int_0^\pi \Phi_n(s) \frac{\cos ks}{k} ds$$

so that

$$\left| \int_0^\pi \Phi_n(s) \sin ks ds \right|^2 \leq 2\gamma_{n,k}^2 + 2\frac{M}{k^2} \quad (34)$$

where M does not depend on n and

$$\gamma_{n,k} = \int_0^\pi [e^{-\alpha s} \Phi_n(s)] r_k(s) ds$$

Hence, from (33),

$$\sum_{k=1}^{+\infty} |\gamma_{n,k}|^2 \asymp \int_0^\pi e^{2\alpha s} |\Phi_n(s)|^2 ds < M \quad (35)$$

(M does not depend on n since $\{\zeta_n(t)\}$ is bounded in $L^2(0, \pi)$.)

We are now ready to study the regularity of the function $v(t)$. We first prove a result which has an independent interest:

Lemma 4.3 *We have: $\zeta_n(t) \in W^{1,2}(0, \pi)$ for every n .*

Proof. We need to see that the series of the derivatives of the two series in (31) converge in $L^2(0, \pi)$. Using (32), the derivative of the first series has the following form

$$- \sum_{k=1}^{+\infty} (k\gamma_{n,k}) r_k(t), \quad |k\gamma_{n,k}| = \left| \int_0^\pi \zeta_n(s) k \hat{e}_k(s) ds \right| \leq \|\zeta_n\|_{L^2(0,\pi)} \|a_k\|_{L^2(0,\pi)}.$$

Using (29), we see that $\{\gamma_{n,k}\} \in l^2$ so that the series converges in $L^2(0, \pi)$.

The derivative of the second series has the form $-\sum_{k=1}^{+\infty} \gamma_{n,k} r_k(t)$ and now

$$\gamma_{n,k} = \int_0^\pi \Phi_n(s) \sin ks \, ds,$$

the Fourier coefficients of $\Phi_n(s)$ so that also in this case we have $\{\gamma_{n,k}\} \in l^2$ and the series converges. ■

We now consider the function $v(t)$, which is a linear combination of the three series

$$\sum_{n=1}^{+\infty} \frac{c_n}{n} \epsilon_n \tag{36}$$

$$\sum_{n=1}^{+\infty} \frac{c_n}{n} \left[\sum_{k=1}^{+\infty} \epsilon_k \int_0^\pi \Phi_n(s) \frac{\sin ks}{k} \, ds \right] \tag{37}$$

$$\sum_{n=1}^{+\infty} \frac{c_n}{n} \left[\sum_{k=1}^{+\infty} \epsilon_k \int_0^\pi \zeta_n(s) \hat{e}_k(s) \, ds \right]. \tag{38}$$

We have to prove that the series of the derivatives converge in $L^2(0, \pi)$. This is clear for the series (36).

We consider the series (37) and we prove that the series which is obtained with a formal termwise differentiation converges in $L^2(0, \pi)$. So we consider

$$\begin{aligned} & \int_0^\pi \left| \sum_{n=M}^R \frac{c_n}{n} \left[\sum_{k=1}^{+\infty} \epsilon'_k(t) \int_0^\pi \Phi_n(s) \frac{\sin ks}{k} \, ds \right] \right|^2 dt \\ &= \int_0^\pi \left| \sum_{n=M}^R \frac{c_n}{n} \left[\sum_{k=1}^{+\infty} r_k(t) \int_0^\pi \Phi_n(s) \sin ks \, ds \right] \right|^2 dt \\ &\leq \left(\sum_{n=M}^R |c_n|^2 \right) \left(\sum_{n=M}^R \frac{1}{n^2} \int_0^\pi \left| \sum_{k=1}^{+\infty} r_k(t) \int_0^\pi \Phi_n(s) \sin ks \, ds \right|^2 dt \right) \\ &\leq \left(\sum_{n=M}^R |c_n|^2 \right) \sum_{n=M}^R \frac{M}{n^2} \sum_{k=1}^{+\infty} \left[\gamma_{n,k}^2 + \frac{1}{k^2} \right] \leq (\text{const}) \cdot \left(\sum_{n=M}^R |c_n|^2 \right) \sum_{n=M}^R \frac{1}{n^2} \end{aligned}$$

(here we used inequalities (33)-(35).) This shows $L^2(0, \pi)$ -convergence of the sequence of the partial sums, as wanted.

We now consider the series of the derivatives of (38), i.e. the series

$$\begin{aligned}
& \int_0^\pi \left| \sum_{n=M}^R \frac{c_n}{n} \sum_{k=1}^{+\infty} \epsilon_k'(t) \int_0^\pi \zeta_n(s) \hat{e}_k(s) \, ds \right|^2 dt \\
&= \int_0^\pi \left| \sum_{n=M}^R \frac{c_n}{n} \left(\sum_{k=1}^{+\infty} r_k(t) \int_0^\pi \zeta_n(s) [k \hat{e}_k(s)] \, ds \right) \right|^2 dt \\
&\leq \left(\sum_{n=M}^R |c_n|^2 \right) \left(\sum_{n=M}^R \frac{1}{n^2} \int_0^\pi \left| \sum_{k=1}^{+\infty} \gamma_{n,k} r_k(t) \right|^2 dt \right) \\
&M \left(\sum_{n=M}^R |c_n|^2 \right) \left(\sum_{n=M}^R \frac{1}{n^2} \sum_{k=1}^{+\infty} |\gamma_{n,k}|^2 \right). \tag{39}
\end{aligned}$$

Here we used (33) with

$$\gamma_{n,k} = \int_0^\pi \zeta_n(s) [k \hat{e}_k(s)] \, ds.$$

Hence,

$$\begin{aligned}
|\gamma_{n,k}|^2 &\leq \int_0^\pi |\zeta_n(s)|^2 \, ds \int_0^\pi |k \hat{e}_k(s)|^2 \, ds \\
&\leq \|\zeta_n\|_{L^2(0,\pi)}^2 \int_0^\pi |a_k(s)|^2 \, ds \leq M \int_0^\pi |a_k(s)|^2 \, ds.
\end{aligned}$$

Property (29) shows that

$$\sum_{k=1}^{+\infty} |\gamma_{k,n}|^2 < M$$

and the number M does not depend on n .

Going back to (39), we see that the partial sums of the series of the derivatives of (38) converges in $L^2(0, \pi)$, as wanted.

This concludes the proof that the solution to the control problem is the solution $u(t)$ of (27), where $v(t)$ is given by (26). In this sense, this paper provides a constructive approach to the computation of the solution u of the control problem (1)–(3). We have also a minor improvement on the existing controllability results: the sequence $\{z_n\}_{n \geq 1}$ being a Riesz sequence in $L^2(0, \pi)$, the moment problem can be solved *for every* sequence $\{c_n\} \in l^2$ so that, thanks to Lemma 2.1, the controllability time with boundary control is π . This result has been proved in [19] under different assumptions on the kernel $N(t)$.

Remark 4.4 *Note the role of the factor $1/n$ in (18) and see [22] for further applications of the results in this paper.*

Appendix

In this appendix we remove the assumption $N'(0) = 0$, which was used solely to show in a simple case the ideas in this paper and, for completeness, we outline the proof of the finite signal speed and of the preliminary result Theorem 1.2.

Proof of Theorem 1.2

The proof of Theorem 1.2 is as follows: if we can find a vector $z_0 \in H$ which is orthogonal to the sequence $\{z_n\}_{n \geq 1}$, then the new sequence $\{z_n\}_{n \geq 0}$ is ω -independent and quadratically close to the basis $\{\epsilon_n\}_{n \geq 0}$. Theorem 1.1 can be applied and we see that $\{z_n\}_{n \geq 0}$ is a Riesz basis; hence, $\{z_n\}_{n \geq 1}$ is a Riesz system.

So, we have to prove the existence of such orthogonal element z_0 . For this, we add any element \tilde{z} as the first element of the sequence $\{z_n\}_{n \geq 1}$ and we consider this new sequence.

As in the proof of Theorem 1.1, we consider the operator T defined by

$$T \left[\sum_{n=0}^{+\infty} \alpha_n \epsilon_n \right] = \alpha_0 [\tilde{z} - \epsilon_0] + \sum_{n=1}^{+\infty} \alpha_n [z_k - \epsilon_n].$$

It is possible to prove that this operator is compact. We then consider the operator $T' = (I + T)$ which has the following property:

$$T' \epsilon_k = z_k.$$

If T' is boundedly invertible, we are done. Otherwise, thanks to the compactness of T , the rank of T' is closed and different from H . Hence, there exists an element $z_0 \in [\text{im } T']^\perp$. In particular, this element is orthogonal to every z_k and we are done.

The proof of the inequalities (33)

We first give a proof based on direct computations. See Remark 4.5 for a more abstract derivation.

The first inequality follows since

$$\begin{aligned} \sum_k \left| \int_0^\pi f(s) r_k(s) \, ds \right|^2 &= \sum_k \left| \int_0^\pi [f(s) e^{\alpha s}] \left(\alpha \frac{\cos ks}{k} - \sin ks \right) \, ds \right|^2 \\ &\leq 2\alpha^2 \sum_k \left| \int_0^\pi [f(s) e^{\alpha s}] \frac{\cos ks}{k} \, ds \right|^2 + 2 \sum_k \left| \int_0^\pi [f(s) e^{\alpha s}] \sin ks \, ds \right|^2 \\ &\leq M \|f\|_{L^2(0, \pi)}^2 \end{aligned}$$

In order to prove the second inequality we recall:

$$\begin{aligned}\int e^{2\alpha x} \sin kx \, dx &= \frac{e^{2\alpha x} k}{k^2 + 4\alpha^2} \left[\frac{2\alpha}{k} \sin kx - \cos kx \right], \\ \int e^{2\alpha x} \cos kx \, dx &= \frac{e^{2\alpha x} k}{k^2 + 4\alpha^2} \left[\frac{2\alpha}{k} \cos kx - \sin kx \right].\end{aligned}$$

Now, $|\sum \gamma_k r_k(t)|^2$ is equal to

$$e^{2\alpha t} \left[\sum_k \gamma_k \left(\alpha \frac{\cos kt}{k} - \sin kt \right) \right] \left[\sum_n \gamma_n \left(\alpha \frac{\cos nt}{n} - \sin nt \right) \right].$$

We expand the product and we use Werner formulas to convert the products of sine/cosine functions. We get:

$$\begin{aligned}\left| \sum \gamma_k r_k(t) \right|^2 &= e^{2\alpha t} \sum_{k,m} \alpha^2 \frac{\cos(k+m)t + \cos(k-m)t}{2km} \gamma_k \gamma_m \\ &+ e^{2\alpha t} \sum_{k,m} \frac{\cos(k+m)t - \cos(k-m)t}{2} \gamma_k \gamma_m \\ &- 2e^{2\alpha t} \sum_{k,m} \frac{\alpha}{2k} [\sin(k+m)t + \sin(k-m)t] \gamma_k \gamma_m\end{aligned}$$

As an example, we compute

$$\begin{aligned}&\int_0^\pi \left[e^{2\alpha t} \sum_{k,m} \alpha^2 \frac{\cos(k+m)t}{2km} \gamma_k \gamma_m \right] dt \\ &= \sum_{k,m} \alpha^2 \frac{\gamma_k \gamma_m}{2km} \left[\frac{e^{2\alpha x} (k+m)}{(k+m)^2 + 4\alpha^2} \frac{2\alpha}{k+m} \cos(k+m)x \right]_0^\pi \\ &= 2\alpha^3 \sum_{k,m} \frac{\gamma_k \gamma_m}{2km[(k+m)^2 + 4\alpha^2]} \left[(-1)^{k+m} e^{2\alpha\pi} - 1 \right] \\ &\leq 2(e^{2\alpha\pi} + 1) \alpha^3 \sum_m \frac{|\gamma_k|}{k} \left[\sum_m \frac{|\gamma_m|}{2m[(k+m)^2 + 4\alpha^2]} \right] \\ &\leq \hat{M} \sum_k \frac{|\gamma_k|}{k} \sum_m \frac{|\gamma_m|}{m} \leq M \sum_k |\gamma_k|^2.\end{aligned}$$

The remaining terms are treated analogously.

Remark 4.5 *The previous inequalities can be derived in a more abstract way as follows: the sequence $\{r_k(t)\}$ is $L^2(0, \pi)$ -close to the Riesz sequence $\{e^{\alpha t} \sin kt\}$. Using [28, Theorem 13], it is possible to prove the existence of a number N such that $\{r_k(t)\}_{k \geq N}$ is a Riesz system.*

This proves the existence of a number M such that

$$\sum_{k \geq N} \left| \int_0^\pi f(s) r_k(s) \, ds \right|^2 \leq M \|f\|_{L^2(0, \pi)}^2.$$

So, the series is convergent and it is easy to estimate also the first term from above, and to get the required inequality.

The second inequality is obtained as follows:

$$\begin{aligned} \int_0^\pi \left| \sum_{k=1}^{N-1} \gamma_k r_k(t) + \sum_{k \geq N} \gamma_k r_k(t) \right|^2 dt &\leq 2 \int_0^\pi \left| \sum_{k=1}^{N-1} \gamma_k r_k(t) \right|^2 dt \\ + 2 \int_0^\pi \left| \sum_{k \geq N} \gamma_k r_k(t) \right|^2 dt &\leq M \sum \left\{ \sum_{k=1}^{N-1} |\gamma_k|^2 + \sum_{k \geq N} |\gamma_k|^2 \right\}, \end{aligned}$$

the finite sum being directly estimated. The estimate on the series is obtained from the properties of Riesz systems.

The case $N'(0) \neq 0$

It is convenient to introduce the following transformation. We define $\alpha = -N'(0)/2$ and $\tilde{\theta}(t) = e^{2\alpha t} \theta(t)$. Clearly, $\tilde{\theta}(t)$ solves

$$\tilde{\theta}'(t) = 2\alpha \tilde{\theta}(t) + \int_0^t \tilde{N}(t-s) \Delta \tilde{\theta}(s) \, ds, \quad \tilde{N}(t) = e^{2\alpha t} N(t). \quad (40)$$

The new kernel $\tilde{N}(t) = e^{2\alpha t} N(t)$ satisfies $\tilde{N}(0) = 1$, $\tilde{N}'(0) = 0$. Controllability of the original system and the system so modified being equivalent, we study the controllability of system (40). The corresponding moment problem has the same form as in Section 2, i.e.

$$\begin{aligned} \int_0^T \tilde{z}_n(r) v(r) \, dr &= \int_0^T \tilde{z}_n(r) \int_{T-r}^T \tilde{N}(T-r-s) u(s) \, ds \, dr \\ &= \frac{1}{\phi_n'(0)} \int_0^\pi \phi_n(x) [\theta(T, x) - z_n(T) \theta(0, x)] \, dx \end{aligned}$$

where $\phi_n(x)$ still solves (9) while $z_n(t)$ now solves

$$\tilde{z}_n'(t) = 2\alpha \tilde{z}_n(t) - n^2 \int_0^t N(t-s) \tilde{z}_n(s) \, ds \quad \tilde{z}_n(0) = 1. \quad (41)$$

Of course,

$$\tilde{z}_n(t) = e^{2\alpha t} z_n(t).$$

We must prove estimates similar to those in Section 3.1 when $\tilde{z}_n(t)$ solves Eq. (41) and now, after this transformation, $\tilde{N}'(0) = 0$. In fact we shall prove the following estimate for $\tilde{z}_n(t)$:

$$|\tilde{z}_n(t) - e^{\alpha t} \cos nt| = \left| \tilde{z}_n(t) - e^{-N'(0)t/2} \cos nt \right| \leq \frac{M}{n}$$

which is equivalent to

$$|z_n(t) - e^{-\alpha t} \cos nt| = \left| z_n(t) - e^{N'(0)t/2} \cos nt \right| < \frac{M}{n}$$

(with a different constant M , which depends on the interval $[0, T]$ we are considering).

This being understood, we work with Eq. (41) and for simplicity of notations we drop the $\tilde{\cdot}$.

We note that

$$\begin{aligned} z_n''(t) &= 2\alpha z_n'(t) - n^2 z_n(t) - n^2 \int_0^t N'(t-s) z_n(s) \, ds, \\ z_n(0) &= 1, \quad z_n'(0) = 2\alpha. \end{aligned}$$

The zeros of the characteristic polynomial $\lambda^2 - 2\alpha\lambda + n^2$ are

$$\sigma_1 = \alpha + i\beta_n, \quad \sigma_2 = \alpha - i\beta_n, \quad \beta_n = n\sqrt{1 - \alpha^2/n^2}.$$

Note that β_n is real for large n and that

$$|n - \beta_n| \leq \frac{\alpha^2}{n}. \tag{42}$$

The solution $z(t)$ of the problem

$$z'' - 2\alpha z' + n^2 z = f, \quad z(0) = 1, \quad z'(0) = 2\alpha$$

is

$$z(t) = e^{\alpha t} \cos \beta_n t + \frac{\alpha}{\beta_n} e^{\alpha t} \sin \beta_n t + \frac{1}{\beta_n} \int_0^t e^{\alpha(t-s)} \sin \beta_n(t-s) f(s) \, ds.$$

We apply this formula and we find

$$\begin{aligned}
z_n(t) &= e^{\alpha t} \cos \beta_n t + \frac{\alpha}{\beta_n} e^{\alpha t} \sin \beta_n t \\
&- \frac{n^2}{\beta_n} \int_0^t e^{\alpha r} \sin \beta_n r \int_0^{t-r} N'(t-r-s) z_n(s) \, ds \, dr \\
&= e^{\alpha t} \cos \beta_n t + \frac{\alpha}{\beta_n} e^{\alpha t} \sin \beta_n t \\
&+ \frac{n^2}{\beta_n^2} \int_0^t \left[\frac{d}{dr} \cos \beta_n r \right] \left[e^{\alpha r} \int_0^{t-r} N'(t-r-s) z_n(s) \, ds \right] \, dr \\
&= e^{\alpha t} \cos \beta_n t + \frac{\alpha}{\beta_n} e^{\alpha t} \sin \beta_n t - \frac{n^2}{\beta_n^2} \left\{ \int_0^t N'(t-s) z_n(s) \, ds \right. \\
&+ \alpha \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N'(t-r-s) z_n(s) \, ds \, dr \\
&\left. - \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N''(t-r-s) z_n(s) \, ds \, dr \right\}. \tag{43}
\end{aligned}$$

We introduce

$$e_n(t) = z_n(t) - e^{\alpha t} \cos \beta_n t$$

and we see that

$$\begin{aligned}
e_n(t) &= + \frac{\alpha}{\beta_n} e^{\alpha t} \sin \beta_n t - \frac{n^2}{\beta_n^2} \left\{ \int_0^t N'(t-s) e_n(s) \, ds \right. \\
&+ \alpha \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N'(t-r-s) e_n(s) \, ds \, dr \\
&\left. - \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N''(t-r-s) e_n(s) \, ds \, dr \right\} \\
&- \frac{n^2}{\beta_n^2} \left\{ \int_0^t N'(t-s) e^{\alpha s} \cos \beta_n s \, ds \right. \\
&+ \alpha \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N'(t-r-s) e^{\alpha s} \cos \beta_n s \, ds \, dr \\
&\left. - \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N''(t-r-s) e^{\alpha s} \cos \beta_n s \, ds \, dr \right\}. \tag{44}
\end{aligned}$$

We recall that $\beta_n \asymp n$ (see (42)) and we integrate by parts the integrals in the last brace, as in Section 3.1. We see that

$$|e_n(t)| \leq \frac{M}{n} \quad \forall t \in [0, T]. \tag{45}$$

The constant M does depend on T but not on n .

Using (42), we see that there exists a constant M such that

$$|\cos nt - \cos \beta_n t| < \frac{M}{n} \quad t \in [0, T] \tag{46}$$

so that the sequence $\{z_n(t)\}$ is L^2 -close to the sequence $\{e^{\alpha t} \cos nt\}$ as we wanted to prove.

We now sketch the proof of Lemma 4.1 in the general case $N'(0) \neq 0$. The first and a second integral in the second brace of (44) can be integrated twice by parts so that their contribution is of the order $1/n^2$. Instead, partial integration of the third integral gives

$$\begin{aligned} & \frac{1}{\beta_n} N''(0) e^{\alpha t} \int_0^t \cos \beta_n r \sin \beta_n(t-r) dr \\ & + \frac{1}{\beta_n} \int_0^t \alpha e^{\alpha r} \cos \beta_n r \int_0^{t-r} N''(t-r-s) e^{\alpha s} \sin \beta_n s ds dr \\ & - \frac{1}{\beta_n} \int_0^t e^{\alpha r} \cos \beta_n r \int_0^{t-r} N'''(t-r-s) e^{\alpha s} \sin \beta_n s ds dr. \end{aligned}$$

When inserted in (44), the first integral above gives terms of the order $1/n^2$ and the term

$$\frac{N''(0)}{2} \frac{n^2}{\beta_n^3} e^{\alpha t} t \sin \beta_n t,$$

which has a difference of the order $1/n^2$ with

$$\frac{N''(0)}{2n} e^{\alpha t} t \sin nt.$$

The remaining terms are dominated by

$$\frac{a_n(t)}{n} + \frac{M}{n^2}.$$

The sequence $\{a_n(t)\}$ has the properties (29). This (and the first addendum in (44)) suggests the definition

$$\hat{e}_n(t) = e(t) - \frac{e^{\alpha t}}{n} \left[\alpha + \frac{1}{2} N''(0) t \right] \sin nt.$$

Inequality (28) is obtained by adding and subtracting $-\frac{e^{\alpha t}}{n} [\alpha + \frac{1}{2} N''(0) t] \sin nt$ to $e_n(s)$ in (44), integrating by parts and using the method in Lemma 4.1.

Finite signal speed

The following fact, that we enunciate with reference to system (24), has been noted in several paper. We adapt the proof in [6], which consider the case that x is not confined to a bounded set.

Using the formula for the solutions given in [20], we see that when $\xi = 0$, the solution of problem (24) is given by the following Volterra equation of the second kind:

$$\theta(t) = \Psi(t) + \int_0^t L(t-s) \theta(s) ds, \quad \Psi(t) = \int_0^t R_+(t-s) \chi \nu(s) ds. \quad (47)$$

The operator $L(t)$ is defined as follows

$$L(t)\phi = N'(0)R_+(t)\phi - N'(t)\phi + \int_0^t R_+(t-s)N''(s)\phi \, ds$$

for every $\phi \in L^2(-\epsilon, \pi)$. The operator $R_+(t)$ is given by

$$[R_+(t)\phi](\cdot) = w(t, \cdot)$$

where $w(t, x)$ solves

$$w_{tt} = w_{xx} \quad t > 0, \quad x \in (-\epsilon, \pi),$$

with conditions

$$w(t, -\epsilon) = w(t, \pi) = 0, \quad w(0, x) = \phi(x), \quad w_t(0, x) = 0.$$

It is known that when the support of ϕ is in $(-\epsilon, 0)$ then $[R_+(t)\phi](x) = w(t, x) = 0$ for $x \in (\pi - \epsilon, \pi)$ and $t < \pi - \epsilon$. This is the “finite signal speed” property of the wave equation, and we see that it is shared by the solution $\theta(t)$ of (24), i.e. of (47). We proceed as follows: we note that if a function $y(t, \cdot)$ has support in $x \leq \pi - \epsilon$ for every $t \leq \pi - \epsilon$, then the same property is retained by its integrals $\int_0^t y(s, \cdot) \, ds$, $t \leq \pi - \epsilon$. So, $\Psi(t, x) = 0$ for $x \in (\pi - \epsilon, \pi)$ and $t < \pi - \epsilon$ because it is obtained as the integral of functions which have this property for every $s \leq t \leq \pi - \epsilon$. The Volterra integral equation (47) can be solved by successive approximations. If \mathcal{L} is the integral operator in (47), then

$$\theta(t) = \sum_{n=0}^{+\infty} \mathcal{L}^n \Psi.$$

The result follows since each term in the expression of $L(t-s)\theta(s)$ in (47) is supported in $x \leq \pi - \epsilon$ for $t \leq \pi - \epsilon$, a property which is retained by every integral $\mathcal{L}^n \Psi$.

References

- [1] D.D. Ang, R. Gorenflo, L.V. Khoi and D.D. Trong, *Moment theory and some inverse problems in potential theory and heat conduction*. Lecture Notes in Mathematics, 1792. Springer-Verlag, Berlin 2002.
- [2] S.A. Avdonin and S.A. Ivanov, *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*. Cambridge University Press, New York 1995.
- [3] S.A. Avdonin, S.A. Ivanov and D.L. Russell, *Exponential bases in Sobolev spaces in control and observation problems for the wave equation*. Proc. Roy. Soc. Edinburgh **A-130** (2000) 947–970.

- [4] S.A. Avdonin, L. Suzanne and V. Protopopescu, *Solving the dynamical inverse problem for the Schrödinger equation by the boundary control method*. Inverse Problems **18** (2002) 349–361.
- [5] V. Barbu and M. Iannelli, *Controllability of the heat equation with memory*. Diff. Integral Eq. **13** (2000) 1393–1412.
- [6] A. Belloni-Morante, *An integro-differential equation arising from the theory of heat conduction in rigid materials with memory*. Boll. Unione Mat. Ital.(5) **15-B** (1978) 470-482.
- [7] A. G. Butkovski, *The method of moments in optimal control theory with distributed parameter systems*. Avtomat. i Telemekh. **24** (1963) 1217–1225.
- [8] C. Cattaneo, *Sulla conduzione del calore*. Atti del Seminario Matematico e Fisico dell'Università di Modena **3** (1948) 3-21.
- [9] G.W. Desch, R. Grimmer, and R. Zeman, *Wave propagation for abstract integro-differential equations*. In *Infinite-dimensional systems* (Retzhof, 1983), 62–70, Lecture Notes in Math., 1076, Springer, Berlin (1984).
- [10] H. O. Fattorini and D.L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations*. Quart. Appl. Math. **32** (1974/75) 45–69.
- [11] H. O. Fattorini and D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*. Arch. Rational Mech. Anal. **43** (1971) 272–292.
- [12] I.C. Gohberg and M.G. Krejn, *Opérateurs linéaires non auto-adjoints dans un espace hilbertien*. Dunod, Paris 1971.
- [13] B.Z. Guo, *Riesz basis approach to the stabilization of a flexible beam with a tip mass*. SIAM J. Control Optim. **39** (2001) 1736–1747.
- [14] M.E. Gurtin and A.G. Pipkin, *A general theory of heat conduction with finite wave speed*. Arch. Rat. Mech. Anal. **31** (1968) 113–126.
- [15] S.A. Ivanov and L. Pandolfi, *Heat equation with memory: lack of controllability to the rest*. Submitted, *Rapporto interno 35/2007, Politecnico di Torino, Dipartimento di Matematica*.
- [16] S. Jaffard, M. Tucsnak and E. Zuazua, *Singular internal stabilization of the wave equation*. Differential Equations **145** (1998) 184–215.
- [17] W. Krabs, *On moment theory and controllability of one-dimensional vibrating systems and heating processes*. Lecture Notes in Control and Information Sciences, 173. Springer-Verlag, Berlin 1992.

- [18] V. Komornik and P. Loreti, *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York 2005.
- [19] G. Leugering, *time optimal boundary controllability of a simple linear viscoelastic liquid*. Math. Methods in the Appl. Sci. **9** (1987) 413–430.
- [20] L. Pandolfi, *The controllability of the Gurtin-Pipkin equation: a cosine operator approach*. Applied Mathematics and Optimization **52** (2005) 143–165.
- [21] L. Pandolfi, *Controllability of the Gurtin-Pipkin equation*. SISSA, Proceedings of Science, PoS(CSTNA2005)015.
- [22] L. Pandolfi, *Riesz systems and an identification problem for heat equations with memory*. in preparation.
- [23] D. L. Russell, *Nonharmonic Fourier series in the control theory of distributed parameter systems*. J. Math. Anal. Appl. **18** (1967) 542–560.
- [24] G. Sansone, R. Conti, *Non-linear differential equations*, Pergamon Press, Oxford, 1964.
- [25] G. Talenti, *Recovering a function from a finite number of moments*. Inverse Problems **3** (1987) 501–517.
- [26] X. Fu, J. Yong and X. Zhang, *Controllability and observability of the heat equation with hyperbolic memory kernel*. Submitted
- [27] J. Yong and X. Zhang, *Exact controllability of the heat equation with hyperbolic memory kernel*. In *Control of Partial Differential Equations*, Control theory of partial differential equations, 387–401, Lect. Notes Pure Appl. Math., 242, Chapman & Hall/CRC, Boca Raton, FL 2005.
- [28] R.M. Young, *An Introduction to Nonharmonic Fourier Series*. Academic Press, New York 1980.

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