Sections of Grassmann Bundles

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Let $C$ be a smooth complex projective curve of genus $g \geq 0$

**Definition**

A $g_d^r$ on $C$ is a pair $(V, L)$ where

$$
\begin{align*}
L &\in \text{Pic}^d(C) \\
[V] &=\in G(r + 1, H^0(L))
\end{align*}
$$

$P \in C$ is a *ramification point* of $g_d^r$ if and only if

$$\dim(V \cap H^0(L(-(r + 1)P))) > 0$$

If $v = (v_\alpha) \in V$ let $D_h v := (v_\alpha, v'_\alpha, \ldots, v^{(h)}_\alpha)_{\alpha \in A}$ $D_h v \in H^0(J^h L)$, where $J^h L$ is the $h$th jet extension of $L$.

$P \in C$ is a *ramification point* of $g_d^r$ if and only if $\exists v \in V | D_r v(P) = 0$. 
Let $E \to X$ a vector bundle on $X$, a smooth projective variety, s.t. $H^0(X, E) \geq r + 1$. For each $V \in G(r + 1, H^0(E))$ let define

$$\{ \Phi_V : X \times V \to E \quad (P, v) \mapsto v(P). \}$$

If $E = J^r L$ we define

**Definition**

The determinant map

$$\left\{ \mathcal{W}_r : G(r + 1, H^0(J^r L)) \to \mathbb{P}H^0(L \otimes^{r+1} K \otimes^{\frac{r(r+1)}{2}}) \right\}$$

where

$$\mathcal{W}_V = \det(\Phi_V) \in \mathbb{P}H^0(\bigwedge^{r+1} J^r L) \cong \mathbb{P}H^0(L \otimes^{r+1} K \otimes^{\frac{r(r+1)}{2}})$$

is the *Wronski map* associated to $J^r L$. 
The *Wronski map* can be extended on $H^0(J^iL)$, for each $0 \leq i \leq r - 1$:

$$
\begin{cases}
W_{r,i} : G(r + 1, H^0(J^iL)) \rightarrow \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \\
\Lambda \mapsto W_{r,i}(\Lambda) = W_r(D_{r-i}\Lambda).
\end{cases}
$$

(1)

**Remark**

- if $i = r$ then $W_{r,r} = W_r$
- if $i = 0$ then the *Wronski map* is a morphism
- if $r = 1$ it is known in literature as *Whal map*
- if $C = \mathbb{P}^1$ then

$$
W_r : G(r + 1, H^0(J^r O_{\mathbb{P}^1}(d))) \rightarrow \mathbb{P}H^0(O_{\mathbb{P}^1}((r + 1)(d - r)))
$$

is known to be a finite and surjective morphism
- if $C = \mathbb{P}^1$ and $i = 0$ $W_{r,0}$ is the usual *Wronski map*
- if $C$ is hyperelliptic $W_{1,0}$ is trivially not surjective.

What about the surjectivity of $W_1$ if $C$ is hyperelliptic?
Decomposition of $H^0(J^h L)$.

For each $h \geq 0$, we extend the definition of $D_h$ to a map

$$D_h : H^0(J^i L) \longrightarrow H^0(J^{i+h} L)$$

for each $i \geq 0$, as

$$\mu = (\mu_0, \alpha, \mu_1, \alpha, \ldots, \mu_i, \alpha) \mapsto D_h \mu = (\mu_0, \alpha, \mu_1, \alpha, \ldots, \mu_i, \alpha, \mu_i', \alpha, \ldots, \mu_i^{(h-i)}).$$

In particular $D_h(D_i \lambda) = D_{h+i} \lambda$, for each $\lambda \in H^0(C, L)$.

The following truncation exact sequences

$$0 \longrightarrow J^{h-j}(K \otimes^j L) \longrightarrow J^h L \overset{t_{h,j-1}}{\longrightarrow} J^{j-1} L \longrightarrow 0$$

let to prove the following

**Theorem 1.**

*If $g(C) \geq 2$ the following direct sum decomposition holds:*

$$H^0(J^h L) = \bigoplus_{j=0}^{h} 0^j \oplus D_{h-j} H^0(L \otimes K \otimes^j).$$
Let $C$ an hyperelliptic curve, $P \in C$ a Weierstrass point, $L = \mathcal{O}(2P)$, then $K = L^{\otimes g^{-1}}$

$$\mathbb{W}_1 : G(2, H^0(J^1L)) \rightarrow \mathbb{P}H^0(L^{\otimes g+1})$$

by Theorem 1 sends

$$V := [D_1\lambda_0 + \mu_0, D_1\lambda_1 + \mu_1] \mapsto \mathbb{W}_1(V) = (D_1\lambda_0 + \mu_0) \wedge (D_1\lambda_1 + \mu_1) \mod \mathbb{C}^*$$
On the other hand

**Theorem 2.**

For each $a \in \mathbb{Z}$,

$$H^0(L^\otimes a) = \text{Sym}^a H^0(L) \oplus \text{Sym}^{a-g-1} H^0(L) \cdot D_1 \lambda_0 \wedge D_1 \lambda_1.$$  

implies that each $\sigma \in H^0(L^\otimes g+1)$ can be then written in a unique way as

$$\sigma := a_0 \lambda_0^{g+1} + a_1 \lambda_0^g \lambda_1 + \ldots + a_{g+1} \lambda_{g+1} + a_{g+2} D_1 \lambda_0 \wedge D_1 \lambda_1.$$  

As a consequence, if $U$ is the open set of $\mathbb{P}H^0(L^\otimes g+1)$ defined by $a_{g+2} \neq 0$.

**Theorem 3.**

Each element of $U$ has a preimage in $G(2, H^0(J^1L))$.  

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Hyperelliptic curves.
Let \( \pi : F \to X \) a vector bundle on a smooth projective variety \( X \), 
\( \text{rk}(F) = d + 1 \).

If \( 0 \leq r \leq d \)

\( \rho_r : G(r + 1, F) \to X \)
denotes the \textit{Grassmann bundle} of \( r + 1 \)-dimensional subspaces of the 
fibers of \( F \).

It carries a \textit{universal exact sequence}

\[
0 \longrightarrow S_r \longrightarrow \rho_r^*F \longrightarrow Q_r \longrightarrow 0.
\]

Assume \( F \) is filtered by quotients \( Q_iF \) s.t. \( \text{rk}(Q_iF) = i + 1 \). Maps

\( \partial_i : S_r \longrightarrow \rho_r^*F \rho_r^*Q_iF \)

let to define the \textit{Schubert varieties}

\[
\Omega_i(Q\cdot F) = \{ \Lambda \in G(r + 1, F) | \text{rk}_\Lambda \partial_{ij-1} \leq j, 0 \leq j \leq r \}.
\]
Grassmann bundles.

By results of Gatto and Gatto-Santiago, we prove that

**Theorem 4.**

\[
[\Omega_{i_0 i_1 \ldots i_r}(Q \cdot F)] = [\Omega_{i_0}(Q \cdot F)] \wedge [\Omega_{i_1}(Q \cdot F)] \wedge \ldots \wedge [\Omega_{i_r}(Q \cdot F)]
\]

under the isomorphism between \( A_\ast(G(r + 1, F)) \) and \( \bigwedge^{r+1} A_\ast(\mathbb{P}(F)) \).

It is equivalent to the *Determinantal Formula* of Kempf-Laksov.

**Definition**

The \( Q \cdot \)-wronskian subvariety of \( G(r + 1, F) \) is

\[ W_r(Q \cdot F) = \Omega_{(01 \ldots r-1,r+1)}(Q \cdot F). \]

The \( Q \cdot \)-base locus subvariety of \( G(r + 1, F) \) is

\[ B_r(Q \cdot F) = \Omega_{(12 \ldots r+1)}(Q \cdot F); \]

The \( Q \cdot \)-cuspidal locus subvariety is \( C_r(Q \cdot F) = \Omega_{(023 \ldots r+1)}(Q \cdot F). \)

\( W_r(Q \cdot F) \) is by definition the zero scheme of the map

\[ W_\partial := \bigwedge^{r+1} \partial_r \in H^0(X, \bigwedge^{r+1} \rho^*_r Q_r F \otimes \bigwedge^{r+1} S^\vee_r) \]

that will be called *wronskian section*. 
Let $\phi' : E \to F$ be a homomorphism of vector bundles of ranks $r + 1$, $k + 1$ over $X$. If $n := \min(r, k)$

$$X_{n+1-f} := \{ p \in X | \text{rk}(\phi')_p \leq n + 1 - f \}$$

is the support of the $(n + 1 - f)$-th degeneracy locus associated to $\phi'$. When $X_{n+1-f}$ has the right codimension, Porteous Formulas computes its class in terms of Chern classes of $E$ and $F$.

To $\phi := 1 \oplus \phi'$ it is naturally associated a section $\gamma_\phi$ of $G(r + 1, E \oplus F)$. Then if $\epsilon := [\Omega_i(Q \bullet F)] \in A^*(\mathbb{P}(F))$, and $m = \max(r, k)$ we have

$$[X_{n+1-f}] = \gamma_\phi^* (\epsilon^0 \wedge \ldots \wedge \epsilon^{n-f} \wedge \epsilon^{k+1} \wedge \ldots \wedge \epsilon^{m+f})$$

in $A^*(X)$. 
Sections of Grassmann bundles.

\[ \Gamma(\rho_r) = \{ \text{holomorphic functions } \gamma : X \to G(r + 1, F) \mid \rho_r \circ \gamma = \text{id}_X \} \]

\[ \Gamma(\rho_r) \leftrightarrow \{ \text{vector subbundles of rank } r + 1 \} \]

If \( \gamma \in \Gamma(\rho_r) \) we find a formula for the class of the image of \( X \) by \( \gamma \) in \( A^*(G(r + 1, F)) \)

\[ \gamma^*[X] = \sum \rho_r^*(\Delta_{\lambda(l)}(\gamma^* c_t(Q_r - \rho_r^* F)c_t(F)))\mu^l \]

where \( \{\mu^l\} \) is a basis of \( A^*(G(r + 1, F)) \).

Definition

The ramification locus of \( \gamma \in \Gamma(\rho_r) \) is \( \gamma^{-1}(\mathcal{W}_r(Q_r F)) \); its base locus is \( \gamma^{-1}(\mathcal{B}_r(Q_r F)) \) and its \( Q_r \)-cuspidal locus is \( \gamma^{-1}(\mathcal{C}_r(Q_r F)) \).
If $\gamma \in \Gamma(\rho_r)$, the section

$$\gamma^* W_{\theta} \in H^0(X, Q_r F \otimes \gamma^* \bigwedge^{r+1} S_r^\vee)$$

will be said the $\partial$-(or $Q_\bullet$-)wronskian of $\gamma$.

The set $\Gamma(\rho_r)$ is too huge.

Set

$$\Gamma_\xi(\rho_r) := \left\{ \gamma \in \Gamma(\rho_r) \mid \bigwedge^{r+1} \gamma^* S_r^\vee = \xi \in \text{Pic}(X) \right\}.$$

The holomorphic map:

$$\begin{cases}
\Gamma_\xi(\rho_r) & \longrightarrow \mathbb{P}H^0(\bigwedge^{r+1} Q_r F \otimes \xi) \\
\gamma & \longmapsto \gamma^* W_{\theta} \mod \mathbb{C}^*
\end{cases}$$

will be said the Wronski map on $\Gamma_\xi(\rho_r)$. 
Set
\[ \Gamma_t(\rho_r) := \{ \gamma \in \Gamma(\rho_r) \mid \gamma^* S_r \text{ is trivial} \} \]

\( \Gamma_t(\rho_r) \) is certainly more tractable, as \( \Gamma_t(\rho_r) \) can be typically equipped with the structure of an open subset of the grassmannian \( G(r + 1, H^0(F)) \).

If \( C \) is a hyperelliptic curve let consider

\[
\begin{cases}
\mathrm{Wr} & : \Gamma_t(\rho_1) \longrightarrow \mathbb{P}H^0(J^1L) = \mathbb{P}H^0(L \otimes g + 1) \\
\gamma & \longmapsto \mathrm{Wr}(\gamma) := \gamma^* W_{\partial},
\end{cases}
\]

then we prove

**Theorem 5.**
Each element of \( U_{g+2} \subseteq H^0(L \otimes g + 1) \) has a pre-image in \( \Gamma_t(\rho_1) \)
THANK YOU