Schubert Calculus on a Grassmann Algebra

Letterio Gatto

Politecnico di Torino

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2. Set Up
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Newton Formulas in Schubert Calculus

Degree of Schubert Varieties

Rational curves with inflectional tangents at prescribed points.

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References for this talk...

*Schubert Calculus via Hasse–Schmidt Derivations,*


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Let $A := A_0 \oplus A_1 \oplus \ldots$ be a graded ring with $A_0 = \mathbb{Z}$; $X$ an indeterminate over $A$; $p := X^n - e_1 X^{n-1} + \ldots + (-1)^n e_n \in A[X]$, a monic polynomial of degree $n$ such that $e_i \in A_i$; $M := XA[X]$ and $M(p) = M/pM$.

The $A$-module $M(p)$ is generated by $\{i : X^i + pM\}$. In particular:

$\{1, \ldots, n\}$ is an $A$-basis of $M(p)$. One has, e.g.:

$X^{n+1} = e_1 X^n - e_2 X^{n-1} + \ldots - (-1)^n e_n X$. 

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The $A$-module $M(p)$ is generated by $\epsilon^i := X^i + pM$. In particular:

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is an $A$-basis of $M(p)$. One has, e.g.:

$$ \epsilon^{n+1} = e_1 \epsilon^n - e_2 \epsilon^{n-1} + \ldots - (-1)^n e_n \epsilon^1 $$

Let $M(p) := \bigoplus_{k \geq 0} k M(p)$ be the exterior algebra of $M(p)$. Then:

$\bigwedge^{i_1} \ldots \bigwedge^{i_k}$

$1 \leq i_1 < \ldots < i_k \leq n$ is a basis of $\bigwedge^k M(p)$, the $k$th exterior power of $M(p)$.
Let 

\[ M(p) := \bigoplus_{k \geq 0} M(p^k) \]

be the exterior algebra of \( M(p) \). Then: 

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\[ \bigwedge M(p) := \bigoplus_{k \geq 0} \bigwedge^k M(p) \]

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Let

$$\wedge^k M(p) := \bigoplus_{k \geq 0} \wedge^k M(p)$$

be the exterior algebra of $M(p)$.

Then:
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Then:

\[ (\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})_{1 \leq i_1 < \ldots < i_k \leq n} \]

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be the exterior algebra of $M(p)$.

Then:

$$\left( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \right)_{1 \leq i_1 < \ldots < i_k \leq n}$$

is a basis of $\wedge^k M(p)$, the $k^{th}$ exterior power of $M(p)$. 
Example.

Let $M := A \oplus A \oplus A \oplus A$. Then

- $\pi_1 \wedge \pi_2, \pi_3 \wedge \pi_4 = 0$,
- $\pi_1 \wedge \pi_3 = 1$,
- $\pi_1 \wedge \pi_4 = 2$,
- $\pi_2 \wedge \pi_3 = 2$,
- $\pi_2 \wedge \pi_4 = 3$,
- $\pi_3 \wedge \pi_4 = 4$.

is a basis of $\pi_2 M$: $w = (i_1 - 1) + (i_2 - 2)$ is the weight of $\pi_1 \wedge \pi_2$. 

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\[
\begin{align*}
\epsilon_1 \wedge \epsilon_2, \\
w &= 0
\end{align*}
\]
Example.

Let $M := A\epsilon^1 \oplus A\epsilon^2 \oplus A\epsilon^3 \oplus A\epsilon^4$. Then

\[ w = 0 \quad \text{and} \quad w = 1 \]

are weights of $A\epsilon^1 \wedge A\epsilon^2$, $A\epsilon^1 \wedge A\epsilon^3$, $A\epsilon^1 \wedge A\epsilon^4$.
Example.

Let $M := A\varepsilon^1 \oplus A\varepsilon^2 \oplus A\varepsilon^3 \oplus A\varepsilon^4$. Then

$$
\begin{align*}
\varepsilon^1 \wedge \varepsilon^2, & \quad \text{w=0} \\
\varepsilon^1 \wedge \varepsilon^3, & \quad \text{w=1} \\
\varepsilon^1 \wedge \varepsilon^4, & \quad \text{w=2}
\end{align*}
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Example.

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$\epsilon^1 \wedge \epsilon^2, \quad \epsilon^1 \wedge \epsilon^3, \quad \epsilon^1 \wedge \epsilon^4, \quad \epsilon^2 \wedge \epsilon^3,$

$w=0, \quad w=1, \quad w=2, \quad w=2$
Example.

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\epsilon^1 \wedge \epsilon^2, & \quad w = 0 \\
\epsilon^1 \wedge \epsilon^3, & \quad w = 1 \\
\epsilon^1 \wedge \epsilon^4, & \quad w = 2 \\
\epsilon^2 \wedge \epsilon^3, & \quad w = 2 \\
\epsilon^2 \wedge \epsilon^4, & \quad w = 3
\end{align*}
\]
Example.

Let $M := A\mathbf{e}^1 \oplus A\mathbf{e}^2 \oplus A\mathbf{e}^3 \oplus A\mathbf{e}^4$. Then

$\mathbf{w} = \begin{cases} (i_1 - 1) + (i_2 - 2) & \text{weight of } \mathbf{i}_1 \wedge \mathbf{i}_2 \end{cases}$

\[
\begin{array}{cccc}
\mathbf{e}_1 \wedge \mathbf{e}_2, & \mathbf{e}_1 \wedge \mathbf{e}_3, & \mathbf{e}_1 \wedge \mathbf{e}_4, & \mathbf{e}_2 \wedge \mathbf{e}_3, \\
\quad w=0 & \quad w=1 & \quad w=2 & \quad w=3 & \quad w=4
\end{array}
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Example.

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\epsilon^1 \wedge \epsilon^2, & \quad w=0 \\
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\epsilon^3 \wedge \epsilon^4, & \quad w=4 
\end{align*}
$$

is a basis of $\bigwedge^2 M$: $w = (i_1 - 1) + (i_2 - 2)$ is the weight of $\epsilon^{i_1} \wedge \epsilon^{i_2}$
A Basic Fact

\[ \left( \frac{d^i}{dx^i}(f(j)) \right) = f(i+j) \]
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There is one and only one $A$-algebra homomorphism
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\[ D_t := \sum_{i \geq 0} D_i t^i : \wedge M(p) \rightarrow \wedge M(p)[[t]] \]

\[ (D_i \in \text{End}_A(\wedge M(p)) \]
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There is one and only one $A$-algebra homomorphism

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such that

\[ D_i \epsilon^j = \epsilon^{i+j} \]
Schubert calculus on a Grassmann Algebra.

The explicit way to phrase that $D_t$ is an $A$-algebra homomorphism is

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$$

the fundamental equation of Schubert Calculus (on a Grassmann Algebra).

We are used to such a kind of equations!

If $f, g \in C^\infty(R)$ then:

$$D_t(fg) = D_tf \cdot D_tg$$

where

$$D_t = \sum_{i \geq 0} \frac{1}{i!} d^i dt^i$$

(The Taylor expansion of the product of $f$ and $g$ is the product of the Taylor expansions of $f$ and $g$ respectively.)
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If $f, g \in C^\infty(\mathbb{R})$ then:

$$D_t(fg) = D_t(f)D_t(g) \quad \text{where} \quad D_t = \sum_{i \geq 0} \frac{1}{i!} \frac{d^i}{dx^i} t^i$$
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The fundamental equation of Schubert Calculus
(on a Grassmann Algebra)
SCGA I: Leibniz Rule

The fundamental equation is equivalent to:

\[ D^h(\alpha \wedge \beta) = h_{\wedge i}(D^i\alpha \wedge D^h\beta), \forall \alpha, \beta \in \mathbb{M} \]

which is the \( h \)th order Leibniz rule \((h \geq 0)\). For example:

\[ D^2(\alpha \wedge \beta) = D^2\alpha \wedge \beta + D^1\alpha \wedge D^1\beta + \alpha \wedge D^2\beta \]

One more example:

\[ D^2(\pi_1 \wedge \pi_2) = D^2\pi_1 \wedge \pi_2 + D^1\pi_1 \wedge D^1\pi_2 + \pi_1 \wedge D^2\pi_2 = \pi_3 \wedge \pi_2 + \pi_2 \wedge \pi_3 + \pi_1 \wedge \pi_4 \]

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The fundamental equation is equivalent to:

\[ D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i \alpha \wedge D_{h-i} \beta, \quad \forall \alpha, \beta \in M \]

which is the \( h \)th order Leibniz rule (\( h \geq 0 \)). For example:

\[ D_2(\alpha \wedge \beta) = D_2 \alpha \wedge \beta + D_1 \alpha \wedge D_1 \beta + \alpha \wedge D_2 \beta \]

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\[ D_2(\xi_1 \wedge \xi_2) = D_2 \xi_1 \wedge \xi_2 + D_1 \xi_1 \wedge D_1 \xi_2 + \xi_1 \wedge D_2 \xi_2 = \xi_1 \wedge \xi_4 + \xi_2 \wedge \xi_3 + \xi_1 \wedge \xi_4 = \xi_1 \wedge \xi_4 \]
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which is the \( h^{th} \) order Leibniz rule \( (h \geq 0) \). For example:

\[ D_2(\alpha \wedge \beta) = D_2 \alpha \wedge \beta + D_1 \alpha \wedge D_1 \beta + \alpha \wedge D_2 \beta \]

One more example:

\[ D_2(\epsilon^1 \wedge \epsilon^2) = D_2 \epsilon^1 \wedge \epsilon^2 + D_1 \epsilon^1 \wedge D_1 \epsilon^2 + \epsilon^1 \wedge D_2 \epsilon^2 = \]
\[ = \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4 = \]
The fundamental equation is equivalent to:

\[ D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i \alpha \wedge D_{h-i} \beta, \quad \forall \alpha, \beta \in \bigwedge M \]

which is the \( h^{th} \) order Leibniz rule (\( h \geq 0 \)). For example:

\[ D_2(\alpha \wedge \beta) = D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta \]

One more example:

\[ D_2(\epsilon^1 \wedge \epsilon^2) = D_2\epsilon^1 \wedge \epsilon^2 + D_1\epsilon^1 \wedge D_1\epsilon^2 + \epsilon^1 \wedge D_2\epsilon^2 = \]

\[ = \epsilon^3 \wedge \epsilon^2 + \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4 = \epsilon^1 \wedge \epsilon^4 \]
SCGA II: Integration by Parts \( (\int f \cdot dg = fg - \int df \cdot g) \)
Leibniz’s rule implies
Leibniz’s rule implies

Integration by Parts
Leibniz’s rule implies

**Integration by Parts**

\[ D_h \alpha \wedge e^i = D_h (\alpha \wedge e^i) - D_{h-1} (\alpha \wedge e^{i+1}) \]
Leibniz’s rule implies

Integration by Parts

\[ D_h \alpha \wedge e^i = D_h (\alpha \wedge e^i) - D_{h-1} (\alpha \wedge e^{i+1}) \]

For example:
Leibniz’s rule implies

Integration by Parts

\[ D_h \alpha \wedge \epsilon^i = D_h (\alpha \wedge \epsilon^i) - D_{h-1} (\alpha \wedge \epsilon^{i+1}) \]

For example:

\[ \epsilon^2 \wedge \epsilon^5 = \]
Leibniz’s rule implies

Integration by Parts

\[ D_h \alpha \wedge \epsilon^i = D_h (\alpha \wedge \epsilon^i) - D_{h-1} (\alpha \wedge \epsilon^{i+1}) \]

For example:

\[ \epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1 \epsilon^4 = \]
Leibniz’s rule implies

**Integration by Parts**

\[ D_h \alpha \wedge \epsilon^i = D_h (\alpha \wedge \epsilon^i) - D_{h-1} (\alpha \wedge \epsilon^{i+1}) \]

For example:

\[ \epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1 \epsilon^4 = D_1 (\epsilon^2 \wedge \epsilon^4) \]
Leibniz’s rule implies

**Integration by Parts**

\[ D_h \alpha \wedge \epsilon^i = D_h (\alpha \wedge \epsilon^i) - D_{h-1} (\alpha \wedge \epsilon^{i+1}) \]

For example:

\[ \epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1 \epsilon^4 = D_1 (\epsilon^2 \wedge \epsilon^4) - D_1 \epsilon^2 \wedge \epsilon^4 = \]
SCGA II: Integration by Parts \((\int f \cdot dg = fg - \int df \cdot g)\)

Leibniz’s rule implies

**Integration by Parts**

\[
D_h \alpha \wedge \epsilon^i = D_h (\alpha \wedge \epsilon^i) - D_{h-1} (\alpha \wedge \epsilon^{i+1})
\]

For example:

\[
\epsilon^2 \wedge \epsilon^5 = \epsilon^2 \wedge D_1 \epsilon^4 = D_1 (\epsilon^2 \wedge \epsilon^4) - D_1 \epsilon^2 \wedge \epsilon^4 = D_1 (\epsilon^2 \wedge \epsilon^4) - \epsilon^3 \wedge \epsilon^4
\]
Let $A^*(\bigwedge M(p))$ be the polynomial ring $A[T_1, T_2, \ldots]$. Give degree $i$ to the monomial $T_i$. Let $ev_D: A[T] \to \text{End}_A(\bigwedge M(p))$ be the natural map $T_i \mapsto D_i$ and $A^*(\bigwedge M(p)) := \text{Im}(ev_D) \subseteq \text{End}_A(\bigwedge M(p))$. Letterio Gatto

Schubert Calculus on a Grassmann Algebra
Let $A[T]$ be the polynomial ring $A[T_1, T_2, \ldots]$. 
Let $A[T]$ be the polynomial ring $A[T_1, T_2, \ldots]$. Give degree $i$ to the monomial $T_i$. 

$A^*(\wedge M(p))$
Let $A[T]$ be the polynomial ring $A[T_1, T_2, \ldots]$. Give degree $i$ to the monomial $T_i$.

Let 

$$ev_D : A[T] \rightarrow End_A(\bigwedge M(p))$$

be the natural map $T_i \mapsto D_i$ and
Let $A[T]$ be the polynomial ring $A[T_1, T_2, \ldots]$.

Give degree $i$ to the monomial $T_i$.

Let

$$ev_D : A[T] \longrightarrow \text{End}_A(\bigwedge M(p))$$

be the natural map $T_i \mapsto D_i$ and

$$A^*(\bigwedge M(p)) := \text{Im}(ev_D) \subseteq \text{End}_A(\bigwedge M(p)).$$
Denote by $A^\star(\bigwedge^k M(p))$ the image of the natural restriction map 

$$ \rho_k : A^\star(\bigwedge^k M(p)) \to \text{End} A^\star(\bigwedge^k M(p)) $$

<table>
<thead>
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<tr>
<td>Schubert Calculus on a Grassmann Algebra</td>
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Denote by

\[ A^* (\bigwedge^k M(p)) \]
Denote by

$$A^*(\wedge^k M(p))$$
Denote by

\[ A^*\left( \bigwedge^k M(p) \right) \]

the image of the natural restriction map
Denote by

$$A^*(\bigwedge^k M(p))$$

the image of the natural restriction map

$$\rho_k : A^*(\bigwedge M(p)) \rightarrow \text{End}_A(\bigwedge^k M(p))$$

$$P(D) \quad \mapsto \quad P(D)|_{\bigwedge^k M(p)}$$
Giambelli’s problem has a solution

Theorem. The natural evaluation map:

\[
\text{ev}^1 \wedge \ldots \wedge \text{ev}^k : A^* \left( V^*_M(p) \right) \to V^*_k M(p) \text{ P(D)} \to P(D) \cdot \text{ev}^1 \wedge \ldots \wedge \text{ev}^k
\]

is surjective.

Proof. Enough to prove that for each \( \text{ev}^i_1 \wedge \ldots \wedge \text{ev}^i_k \in V^*_k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\text{ev}^i_1 \wedge \ldots \wedge \text{ev}^i_k = P(D) \cdot \text{ev}^1 \wedge \ldots \wedge \text{ev}^k.
\]

One then concludes using integration by parts. For example:
Giambelli’s problem has a solution

Theorem.

The natural evaluation map:

\[
\begin{cases}
\text{ev}_{\mathbf{1} \wedge \ldots \wedge \mathbf{k}}: A^*(V^*M(p)) \to V_k M(p) \cdot \mathbf{D} \\
\end{cases}
\]

is surjective.

Proof. Enough to prove that for each \( \mathbf{i}_1 \wedge \ldots \wedge \mathbf{i}_k \in \mathbb{k} M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\mathbf{i}_1 \wedge \ldots \wedge \mathbf{i}_k = P(D) \cdot \mathbf{1} \wedge \ldots \wedge \mathbf{k}.
\]

One then concludes using integration by parts. For example:

Letterio Gatto

Schubert Calculus on a Grassmann Algebra
Giambelli’s problem has a solution

**Theorem.** The natural evaluation map:

\[
ev_{i_1} \wedge \ldots \wedge ev_{i_k} : A^*(V(M(p))) \to V^k(M(p)) \shortrightarrow P(D) \cdot ev_{i_1} \wedge \ldots \wedge ev_{i_k}
\]

is surjective.

**Proof.** Enough to prove that for each \(ev_{i_1} \wedge \ldots \wedge ev_{i_k} \in V^k(M(p))\) there exists a polynomial expression \(P(D)\) such that \(ev_{i_1} \wedge \ldots \wedge ev_{i_k} = P(D) \cdot ev_{i_1} \wedge \ldots \wedge ev_{i_k}\).

One then concludes using integration by parts. For example:
Giambelli’s problem has a solution

**Theorem.** The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon^k} : & \quad A^*(\wedge M(p)) \rightarrow \wedge^k M(p) \\
& \quad P(D) \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]
Theorem. The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : \mathcal{A}^*(\wedge M(p)) & \longrightarrow \wedge^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.
Giambelli’s problem has a solution

Theorem. The natural evaluation map:

\[
\begin{cases}
\text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon^k} : \mathcal{A}^*(\wedge M(p)) & \longrightarrow & \wedge^k M(p) \\
P(D) & \longmapsto & P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{cases}
\]

is surjective.

Proof.
Giambelli’s problem has a solution

**Theorem.** The natural evaluation map:

\[
\begin{aligned}
ed_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : \mathcal{A}^*(\wedge M(p)) &\longrightarrow \wedge^k M(p) \\
P(D) &\longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{aligned}
\]

is surjective.

**Proof.** Enough to prove that for each \( \epsilon^i_1 \wedge \ldots \wedge \epsilon^i_k \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k = \text{expression.}
\]
Giambelli’s problem has a solution

**Theorem.** *The natural evaluation map:*

\[
\begin{align*}
\text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon_k} : \mathcal{A}^*(\Lambda M(p)) & \longrightarrow \Lambda^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon_1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

**Proof.** Enough to prove that for each \( \epsilon^i_1 \wedge \ldots \wedge \epsilon^i_k \in \Lambda^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^i_1 \wedge \ldots \wedge \epsilon^i_k = P(D) \cdot \epsilon_1 \wedge \ldots \wedge \epsilon^k.
\]

One then concludes using integration by parts. \( \blacksquare \)
Giambelli’s problem has a solution

**Theorem.** The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon_k} & : \mathcal{A}^*(\wedge M(p)) \rightarrow \wedge^k M(p) \\
\text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon_k} & : P(D) \rightarrow P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

**Proof.** Enough to prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

One then concludes using integration by parts.

For example:
Theorem. The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : A^*(\Lambda M(p)) & \longrightarrow \Lambda^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

Proof. Enough to prove that for each \(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \Lambda^k M(p)\) there exists a polynomial expression \(P(D)\) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

One then concludes using integration by parts. \(\blacksquare\)

For example:

Recall

In a previous slide we saw that, applying Leibniz rule:

\[
D_2(\epsilon^1 \wedge \epsilon^2) = \epsilon^1 \wedge \epsilon^4.
\]
Theorem. The natural evaluation map:

\[
\begin{aligned}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : & \quad \mathcal{A}^*(\wedge M(p)) \rightarrow \wedge^k M(p) \\
& \quad P(D) \quad \mapsto \quad P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{aligned}
\]

is surjective.

Proof. One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts.

For example:
**Theorem.** The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : \mathcal{A}^* (\wedge M(p)) & \longrightarrow \wedge^k M(p) \\
P(D) & \longrightarrow P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

**Proof.** One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts. \( \blacksquare \)

For example:

\[
\epsilon^2 \wedge \epsilon^3 =
\]
Theorem. The natural evaluation map:

\[
\begin{cases}
    \text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon^k} : A^*(\bigwedge M(p)) \rightarrow \bigwedge^k M(p) \\
    P(D) \mapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{cases}
\]

is surjective.

Proof. One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts.

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 =
\]
Theorem. The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} &: A^*(\wedge M(p)) \longrightarrow \wedge^k M(p) \\
\end{align*}
\]

\[
P(D) \quad \longmapsto \quad P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\]

is surjective.

Proof. One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts.

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1(\epsilon^1 \wedge \epsilon^3)
\]
Theorem. *The natural evaluation map:*

\[
\begin{align*}
ev_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} & : A^*(\wedge M(p)) \longrightarrow \wedge^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

**Proof.** One must prove that for each \(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p)\) there exists a polynomial expression \(P(D)\) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts.

---

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 =
\]
**Theorem.** The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : A^*(\wedge M(p)) & \longrightarrow \wedge^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

**Proof.** One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts. \( \blacksquare \)

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 = D_1(D_1(\epsilon^1 \wedge \epsilon^2))
\]
Theorem. The natural evaluation map:

\[
\begin{aligned}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} & : \mathcal{A}^*(\wedge M(p)) \rightarrow \wedge^k M(p) \\
P(D) & \mapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{aligned}
\]

is surjective.

Proof. One must prove that for each \(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p)\) there exists a polynomial expression \(P(D)\) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts. \(\blacksquare\)

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 = D_1(D_1(\epsilon^1 \wedge \epsilon^2)) - D_2(\epsilon^1 \wedge \epsilon^2) =
\]

---

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Schubert Calculus on a Grassmann Algebra
Theorem. The natural evaluation map:

\[
\begin{align*}
\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k} : \mathcal{A}^*(\wedge M(p)) & \longrightarrow \wedge^k M(p) \\
P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{align*}
\]

is surjective.

Proof. One must prove that for each \(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p)\) there exists a polynomial expression \(P(D)\) such that

\[\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.\]

By integration by parts.

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 = D_1(D_1(\epsilon^1 \wedge \epsilon^2)) - D_2(\epsilon^1 \wedge \epsilon^2) =
\]

\[= (D_1^2 - D_2) \cdot \epsilon^1 \wedge \epsilon^2 =
\]
Theorem. The natural evaluation map:

\[
\begin{aligned}
    \text{ev}_{\epsilon_1 \wedge \ldots \wedge \epsilon_k} : & \quad A^*(\wedge M(p)) \longrightarrow \wedge^k M(p) \\

    P(D) & \longmapsto P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{aligned}
\]

is surjective.

Proof. One must prove that for each \( \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \wedge^k M(p) \) there exists a polynomial expression \( P(D) \) such that

\[
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.
\]

By integration by parts. \( \square \)

For example:

\[
\epsilon^2 \wedge \epsilon^3 = D_1 \epsilon^1 \wedge \epsilon^3 = D_1 (\epsilon^1 \wedge \epsilon^3) - \epsilon^4 = D_1 (D_1 (\epsilon^1 \wedge \epsilon^2)) - D_2 (\epsilon^1 \wedge \epsilon^2) = (D_1^2 - D_2) \cdot \epsilon^1 \wedge \epsilon^2 = D_1 D_2 - D_0 D_1 \epsilon^1 \wedge \epsilon^2.
\]
Computing $A^* (\wedge^k M(p))$
Computing $A^*(\wedge^k M(p))$

Hence:
Computing $A^*(\bigwedge^k M(p))$

Hence:

$$A^*(\bigwedge^k M(p)) := \frac{A^*(\bigwedge M(p))}{\ker(\rho_k)} = \frac{A^*(\bigwedge M(p))}{\ker(e_{V}^{1 \wedge \ldots \wedge k})}$$
Computing $\mathcal{A}^*(\bigwedge^k M(p))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(p)) := \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge^k M(p))}{\ker(e_1 \wedge \ldots \wedge e^k)}$$

The map:
Computing $\mathcal{A}^*(\bigwedge^k M(p))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(p)) := \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\ev_{\epsilon^1 \wedge \ldots \wedge \epsilon^k})}$$

The map:

$$\left\{ \begin{array}{l}
\Pi_k : \mathcal{A}^*(\bigwedge^k M(p)) \longrightarrow \bigwedge^k M(p) \\
\rho_k(P(D)) \mapsto P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{array} \right.$$
Computing $\mathcal{A}^*(\bigwedge^k M(p))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(p)) := \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(eV \epsilon^1 \wedge \ldots \wedge \epsilon^k)}$$

The map:

$$\left\{ \begin{array}{ccc} \Pi_k : & \mathcal{A}^*(\bigwedge^k M(p)) & \longrightarrow & \bigwedge^k M(p) \\ \rho_k(P(D)) & \mapsto & P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \end{array} \right\}$$

is said to be *Poincaré Isomorphism*
Computing $\mathcal{A}^*(\bigwedge^k M(p))$

Hence:

$$\mathcal{A}^*(\bigwedge^k M(p)) := \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\rho_k)} = \frac{\mathcal{A}^*(\bigwedge M(p))}{\ker(\text{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k})}$$

The map:

$$\begin{cases}
\Pi_k : \mathcal{A}^*(\bigwedge^k M(p)) \rightarrow \bigwedge^k M(p) \\
\rho_k(P(D)) \mapsto P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k
\end{cases}$$

is said to be \textit{Poincaré Isomorphism}

In a sense:

"$\mathcal{A}^*(\bigwedge^k M(p)) = \bigwedge^k \mathcal{A}^*(M(p))$"

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Schubert Calculus on a Grassmann Algebra
Let \( E \to Y \) be a vector bundle of rank \( n \); furthermore, let:

\[
A := A^\ast(Y)
\]

and

\[
p = X_n - c_1(E) X_{n-1} + \ldots + c_n(E)
\]

Construct \( \mathbb{k} M(p) \). It turns out that (easy):

\[
\mathbb{k} M(p) = M(p) \sim A^\ast(P(E))
\]

which is an \( A \)-module freely generated by \( i := \xi_i - 1 \cap [P(E)] \), where \( \xi = c_1(O_{P(E)}(E)(-1)) \).
Intersection Theory on Grassmann Bundles

Let

$$E \rightarrow Y$$ be a vector bundle of rank $n$;

Furthermore, let:

$$A := A^* (Y)$$ and

$$p = X_{n-1} - c_1(E) X_{n-1} + \ldots + c_n(E)$$

Construct

$$k M(p)$$ before

It turns out that (easy):

$$1 M(p) = M(p) \sim A^* (P(E))$$

which is an $A$-module freely generated by

$$\xi_i := \xi_i - 1 \cap [P(E)]$$, where

$$\xi = c_1(O_P(E)(-1))$$
Let \( E \rightarrow Y \)
Intersection Theory on Grassmann Bundles

Let \( E \rightarrow Y \) be a vector bundle of rank \( n \);

Furthermore, let:

\[
\begin{align*}
A &= A^* (Y) \\
p &= X^{n-1} c_1(E) X^{n-2} + \ldots + c_n(E)
\end{align*}
\]

Construct \( \text{M}^k(p) \) before it turns out that (easy):

\[
\text{M}^1(p) = \text{M}(p) \sim A^* (P(E))
\]

which is an \( A \)-module freely generated by \( \xi_i = \xi_{i-1} \cap [P(E)] \), where

\[
\xi = c_1(O_{P(E)}(-1))
\]
Let $E \to Y$ be a vector bundle of rank $n$;

Furthermore, let:

$$A := A^\ast(Y)$$

It turns out that (easy):

$$\text{ext}^1 M(p) = M(p) = A^\ast(P(E))$$

which is an $A$-module freely generated by

$$i := \xi_i - 1 \cap [P(E)]$$

where

$$\xi = c_1(O_{P(E)}(-1))$$
Let $E \rightarrow Y$ be a vector bundle of rank $n$;

Furthermore, let:

$$A := A^*(Y) \quad \text{and}$$
Let \( E \rightarrow Y \) be a vector bundle of rank \( n \);

Furthermore, let:

\[
A := A^*(Y) \quad \text{and} \quad p = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)
\]
Intersection Theory on Grassmann Bundles

Let \( E \rightarrow Y \) be a vector bundle of rank \( n \);

Furthermore, let:

\[
A := A^*(Y) \quad \text{and} \quad p = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)
\]

Construct \( \bigwedge^k M(p) \) as before
Let $E \to Y$ be a vector bundle of rank $n$;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad p = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\wedge^k M(p)$ as before

$$M := XA[X], \quad M(p) := M/pM$$
Intersection Theory on Grassmann Bundles

Let $E \to Y$ be a vector bundle of rank $n$;

Furthermore, let:

$$A := A^*(Y) \quad \text{and} \quad p = x^n - c_1(E)x^{n-1} + \ldots + c_n(E)$$

Construct $\wedge^k M(p)$ as before
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Let $E \rightarrow Y$ be a vector bundle of rank $n$;

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It turns out that (easy):
Let $E \rightarrow Y$ be a vector bundle of rank $n$;

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It turns out that (easy):

$$\bigwedge^1 M(p) = M(p) \cong A_*(\mathbb{P}(E))$$
Let $E \to Y$ be a vector bundle of rank $n$;

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$$A := A^*(Y) \quad \text{and} \quad p = X^n - c_1(E)X^{n-1} + \ldots + c_n(E)$$

Construct $\bigwedge^k M(p)$ as before

It turns out that (easy):

$$\bigwedge^1 M(p) = M(p) \cong A_*(\mathbb{P}(E))$$

which is an $A$-module freely generated by $e^i := \xi^{i-1} \cap [\mathbb{P}(E)]$, where

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$$
The Main Theorem (Laksov & Thorup, 2006)

The following diagram commutes.

\[
\begin{align*}
\ast & \mid \ast \mid \ast \\
A & \otimes & \ast \\
G(k, E) & \cap & \ast \\
\iota_k & \otimes & \delta_{-1} \\
\mid & \mid & \mid \\
\ast & \mid \ast \\
(\ast \setminus k \mathcal{M}(\mathcal{P})) & \otimes & \ast \\
\mathcal{M}(\mathcal{P}) & \rightarrow & \mathcal{M}(\mathcal{P})
\end{align*}
\]
The Main Theorem (Laksov & Thorup, 2006)

\[ A^* \left( G(k, E) \right) \otimes A^* \left( G(k, E) \right) \cap \rightarrow A^* \left( G(k, E) \right) \]

\[ \iota_k \otimes \delta^{-1} \]

\[ \delta_k \]

\[ A^* \left( \pi \left( k M(p) \right) \right) \otimes \pi \left( k M(p) \right) \rightarrow \pi \left( k M(p) \right) \]

commutes.

\[ \iota_k \left( P(\sigma) \right) = P(D) \]

\[ \delta_k \left( \pi_1 \wedge \ldots \wedge \pi_k \right) = \Delta_{\pi_1, \ldots, \pi_k} \]

\[ \cap \left[ G(k, E) \right] \]
The following diagram

\[
\begin{array}{c}
A^\ast (G(k,E)) \otimes A^\ast (G(k,E)) \cap -\rightarrow A^\ast (G(k,E)) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
A^\ast (\mathbb{M}(p)) \otimes \mathbb{M}(p) \rightarrow \mathbb{M}(p)
\end{array}
\]

where \(\iota_k(P(\sigma)) = P(D)\) and \(\delta_k(\wedge \ldots \wedge \wedge i_1 \ldots i_k) = \Delta_{i_1 \ldots i_k}(\sigma) \cap [G(k,E)]\).
The Main Theorem (Laksov & Thorup, 2006)

The following diagram

\[
\begin{align*}
A^*(G(k, E)) \otimes A^*(G(k, E)) & \longrightarrow A^*(G(k, E)) \\
\iota_k \otimes \delta_k^{-1} & \downarrow \\
A^*(\bigwedge^k M(p)) \otimes \bigwedge^k M(p) & \longrightarrow \bigwedge^k M(p)
\end{align*}
\]
The following diagram

\[
\begin{array}{ccc}
A^\ast(G(k, E)) \otimes A^\ast(G(k, E)) & \xrightarrow{\cap} & A^\ast(G(k, E)) \\
\downarrow \iota_k \otimes \delta_k^{-1} & & \uparrow \delta_k \\
A^\ast(\bigwedge^k M(p)) \otimes \bigwedge^k M(p) & \rightarrow & \bigwedge^k M(p)
\end{array}
\]

commutes.
The following diagram

\[ A^*(G(k, E)) \otimes A^*(G(k, E)) \rightarrow A^*(G(k, E)) \]

\[ \iota_k \otimes \delta_k^{-1} \downarrow \quad \delta_k \uparrow \]

\[ A^*(\bigwedge^k M(p)) \otimes \bigwedge^k M(p) \rightarrow \bigwedge^k M(p) \]

commutes.

(\text{where} \quad \nu_k(P(\sigma)) = P(D)
\text{and} \quad \delta_k(\epsilon_1 \wedge \ldots \wedge \epsilon_k) = \Delta_{i_1, \ldots, i_k}(\sigma) \cap [G(k, E)] )
The Main Theorem (Laksov & Thorup, 2006)

The symmetric structure of $\wedge^k A[X]$

Let $S := A[X_1, \ldots, X_k]^{sym}$. Then $\otimes^k A[X] \rightarrow \wedge^k A[X]$ is $S$-linear

The following diagram

$$
\begin{align*}
A^*(G(k, E)) \otimes A^*(G(k, E)) & \rightarrow A^*(G(k, E)) \\
\delta_k \otimes \delta_k^{-1} & \downarrow \quad \delta_k \uparrow \\
A^*(\wedge^k M(p)) \otimes \wedge^k M(p) & \rightarrow \wedge^k M(p)
\end{align*}
$$

commutes.

where $\nu_k(P(\sigma)) = P(D)$

and $\delta_k(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \Delta_{i_1, \ldots, i_k}(\sigma) \cap [G(k, E)]$
Newton’s Binomial Formulas

What do we gain by our dictionary?

Working on the Exterior Algebra rather than on a single exterior power, we inherit Newton's type binomial formulas!


Letterio Gatto

Schubert Calculus on a Grassmann Algebra
Newton’s Binomial Formulas

What do we gain by our dictionary?
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Schubert Calculus on a Grassmann Algebra
Newton’s Binomial Formulas

What do we gain by our dictionary?

Working on the **Exterior Algebra** rather than on a single exterior power, we inherit

Newton’s type binomial formulas!

see

1st Newton’s formulas \((a + b)^n\)

The first is gotten via a simple induction by iterating

\[
D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta
\]

\[
D_m(\alpha \wedge \beta) = \sum_{j=0}^{m} D_j\alpha \wedge D_{m-j}\beta
\]

holding for each \(\alpha, \beta \in M(p)\) and each \(m \geq 1\).
The first is gotten via a simple induction by iterating

\[ D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta. \]
The first is gotten via a simple induction by iterating

\[ D_1(\alpha \land \beta) = D_1\alpha \land \beta + \alpha \land D_1\beta. \]

\[ D_1^m(\alpha \land \beta) = \sum_{j=0}^{m} \binom{m}{j} D_1^j\alpha \land D_1^{m-j}\beta \quad (1) \]
The first is gotten via a simple induction by iterating

\[ D_1(\alpha \wedge \beta) = D_1 \alpha \wedge \beta + \alpha \wedge D_1 \beta. \]

\[
D_1^m(\alpha \wedge \beta) = \sum_{j=0}^{m} \binom{m}{j} D_1^j \alpha \wedge D_1^{m-j} \beta \quad (1)
\]

holding for each \( \alpha, \beta \in \wedge M(p) \) and each \( m \geq 1 \)
Similarly, iterating
\[ D_h (\alpha \wedge \cdot i) = D_h \alpha \wedge \cdot i + D_h \alpha \wedge \cdot (i + 1), \]
one gets a second Newton's type formula (Cordovez):
\[ D_m h (\alpha \wedge \cdot i) = \sum_{j=0}^m D_j h - 1 (D_m - j h \alpha \wedge \cdot i + j) \] (2)
holding for each \( \alpha \in \mathfrak{M}(p) \) and each \( m \geq 0 \) (when \( h = 1 \) one gets precisely formula (1) for \( \beta = \cdot i \)).

Claim: formula (2) is unspeakable in the classical formulation of Schubert Calculus.

Challenge: disprove the claim!
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, 

Claim: formula (2) is unspeakable in the classical formulation of Schubert Calculus

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Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h \alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,

**Integration by parts**

\[
D_h \alpha \wedge \epsilon^i = D_h(\alpha \wedge \epsilon^i) - D_{h-1}(\alpha \wedge \epsilon^{i+1})
\]
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$,
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton’s type formula (Cordovez):
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton’s type formula (Cordovez):

$$D^m_h(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D^j_{h-1}(D^{m-j}_h \alpha \wedge \epsilon^{i+j})$$ (2)
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton’s type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(D_h^{m-j}\alpha \wedge \epsilon^{i+j}) \quad (2)$$

holding for each $\alpha \in \wedge M(p)$ and each $h, m \geq 0$
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton’s type formula (Cordovez):

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holding for each $\alpha \in \wedge M(p)$ and each $h, m \geq 0$

(when $h = 1$ one gets precisely formula (1) for $\beta = \epsilon^i$)
Similarly, iterating \( D_h(\alpha \wedge \epsilon^i) = D_h\alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1}) \), one gets a second Newton’s type formula (Cordovez):

\[
D^m_h(\alpha \wedge \epsilon^i) = \sum_{j=0}^m \binom{m}{j} D^j_{h-1}(D^{m-j}_h \alpha \wedge \epsilon^{i+j})
\]  

(2)

holding for each \( \alpha \in \bigwedge M(p) \) and each \( h, m \geq 0 \)

(when \( h = 1 \) one gets precisely formula (1) for \( \beta = \epsilon^i \))

**Claim:**

formula (2) is unspeakable in the classical formulation of Schubert Calculus
Similarly, iterating $D_h(\alpha \wedge \epsilon^i) = D_h \alpha \wedge \epsilon^i + D_{h-1}(\alpha \wedge \epsilon^{i+1})$, one gets a second Newton’s type formula (Cordovez):

$$D_h^m(\alpha \wedge \epsilon^i) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(D_h^{m-j} \alpha \wedge \epsilon^{i+j})$$

(2)

holding for each $\alpha \in \wedge M(p)$ and each $h, m \geq 0$

(when $h = 1$ one gets precisely formula (1) for $\beta = \epsilon^i$)

**Claim:**

formula (2) is unspeakable in the classical formulation of Schubert Calculus

**Challenge:**

disprove the claim!
Schubert Calculus on Grassmann Varieties.

From now on let $A = Z$ and $p = X^n$. Then $M_n := M(p^i)$ is the $Z$-module of rank $n$ generated by $(\pi_1, \ldots, \pi_n)$. In this case $\pi_j = 0$ if $j > n$. The weight of $\pi_1 \wedge \ldots \wedge \pi_k$ is $(i_1 - 1) + \ldots + (i_k - k)$. Then:

$k \subseteq M_n \mapsto (k \subseteq M_n)$

The fundamental element $g_k = \pi_1 \wedge \ldots \wedge \pi_k$ is the unique of weight 0 while $\pi_k, n = \pi_n - k + 1 \wedge \ldots \wedge \pi_n$ is the unique of weight $k(n - k)$ (the maximum possible).
From now on let $A = \mathbb{Z}$ and $p = X^n$. 

The weight of $\pi_1 \wedge \ldots \wedge \pi_k$ is $(\pi_1 - 1) + \ldots + (\pi_k - k)$. Then:

$$
\pi_k \in M_n = \pi_k (M_n)
$$

The fundamental element $g_k = \pi_1 \wedge \ldots \wedge \pi_k$ is the unique of weight 0 while $\pi_{n,k} \wedge \ldots \wedge \pi_n$ is the unique of weight $k(n-k)$ (the maximum possible).
From now on let $A = \mathbb{Z}$ and $p = X^n$.

Then $M_n := M(p)$ is a free $\mathbb{Z}$-module of rank $n$ generated by $(\epsilon^1, \ldots, \epsilon^n)$. 
From now on let $A = \mathbb{Z}$ and $p = X^n$.

Then $M_n := M(p)$ is a free $\mathbb{Z}$-module of rank $n$ generated by $(\varepsilon^1, \ldots, \varepsilon^n)$.

In this case $\varepsilon^j = 0$ if $j > n$. 
From now on let $A = \mathbb{Z}$ and $p = X^n$.

Then $M_n := M(p)$ is a free $\mathbb{Z}$-module of rank $n$ generated by $(\epsilon^1, \ldots, \epsilon^n)$. In this case $\epsilon^j = 0$ if $j > n$.

The weight of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ is $(i_1 - 1) + \ldots + (i_k - k)$. Then:

$$\bigwedge^k M_n = \bigoplus (\bigwedge^k M_n)_w$$
From now on let $A = \mathbb{Z}$ and $p = X^n$.

Then $M_n := M(p)$ is a free $\mathbb{Z}$-module of rank $n$ generated by $(\epsilon^1, \ldots, \epsilon^n)$.

In this case $\epsilon^j = 0$ if $j > n$.

The weight of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ is $(i_1 - 1) + \ldots + (i_k - k)$. Then:

$$\bigwedge^k M_n = \bigoplus \left( \bigwedge^k M_n \right)_w$$

The fundamental element

$$g_k = \epsilon^1 \wedge \ldots \wedge \epsilon^k$$

is the unique of weight 0 while

$$\pi_{k,n} = \epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n$$

is the unique of weight $k(n - k)$ (the maximum possible).
One has:

\[ D^m h (k \neq M_n) w \subseteq (k \neq M_n) w + m h \]

i.e., \( D^h \) is homogeneous of degree \( h \).

In particular

\[ D^k (n - k) \frac{1}{1} (k \neq M_n) 0 \sim = (k \neq M_n) k (n - k) \]

i.e.

\[ D^k (n - k) \frac{1}{1} \wedge ... \wedge \frac{1}{k} = d_k, n \cdot \frac{1}{n + 1} = k \wedge ... \wedge \frac{1}{k} \]
One has:

$$D^m_h \subseteq (k \wedge M^n)_{w} + mh$$

i.e., $D_h$ is homogeneous of degree $h$.

In particular

$$D^k_{(n-k)}_{1} \sim (k \wedge M^n)_{k(n-k)}$$

i.e.

$$D^k_{(n-k)}_{1} \wedge ... \wedge k = d^k_{n \cdot n+1-k}$$
One has:

\[ D_h^m (\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh} \]

i.e., \( D_h \) is homogeneous of degree \( h \).
One has:

\[ D_h^m (\bigwedge^k M_n)_w \subseteq (\bigwedge^k M_n)_{w+mh} \]

i.e., \( D_h \) is homogeneous of degree \( h \).

In particular

\[ D_1^{k(n-k)} (\bigwedge^n M_n)_0 \cong (\bigwedge^k M_n)_{k(n-k)} \]
One has:

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i.e., \( D_h \) is homogeneous of degree \( h \).

In particular

\[ D_{k(n-k)}^k (\bigwedge^m M_n)_0 \cong (\bigwedge^k M_n)_{k(n-k)} \]

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One has:

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i.e., \( D_h \) is homogeneous of degree \( h \).

In particular

\[ D_1^{k(n-k)}(\bigwedge M_n)_0 \cong (\bigwedge M_n)_{k(n-k)} \]

i.e.

\[ D_1^{k(n-k)} \epsilon^1 \wedge \ldots \wedge \epsilon^k = d_{k,n} \cdot \epsilon^{n+1-k} \wedge \ldots \wedge \epsilon^n \]
Define: \( i_1 \wedge \cdots \wedge i_k = \begin{cases} 0 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n-k+1, \ldots, n) \\ -1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (n-k+1, \ldots, n) \\ \end{cases} \)

Extend \( \text{by } \mathbb{Z} \)-linearity, getting \( \pi_k : M_n \to \mathbb{Z} \)

The Main Theorem implies that

\[
G(k, n) P(\sigma) \cap \Omega i_1 \wedge \cdots \wedge i_k = \pi_k(D) \text{ if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n-k+1, \ldots, n)
\]

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Schubert Calculus on a Grassmann Algebra
Define:

\[ n \wedge \ldots \wedge i_k = \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n - k + 1, \ldots, n) \\
-1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (n - k + 1, \ldots, n) \\
0 & \text{otherwise}
\end{cases} \]
Define:

\[ \int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \]

\[ = \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n - k + 1, \ldots, n) \\
-1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (n - k + 1, \ldots, n) \\
0 & \text{otherwise}
\end{cases} \]

Extend \( \int \) by \( \mathbb{Z} \)-linearity, getting \( \int : \bigwedge^k M_n \to \mathbb{Z} \)
A piece of Notation

Define:

\[
\int_n^{e_{i_1} \wedge \ldots \wedge e_{i_k} =}
\]

\[
= \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n - k + 1, \ldots, n) \\
-1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (n - k + 1, \ldots, n) \\
0 & \text{otherwise}
\end{cases}
\]

Extend \( \int \) by \( \mathbb{Z} \)-linearity, getting \( \int : \wedge^k M_n \to \mathbb{Z} \)

The Main Theorem
A piece of Notation

Define:
\[ \varepsilon_{i_1} \wedge \ldots \wedge \varepsilon_{i_k} = \]
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0 & \text{otherwise} 
\end{cases} \]

Extend \( \varepsilon \) by \( \mathbb{Z} \)-linearity, getting \( \varepsilon : \bigwedge^k M_n \rightarrow \mathbb{Z} \)

The Main Theorem

\[ A^*(G(k, E)) \otimes A^*(G(k, E)) \xrightarrow{\cap} A^*(G(k, E)) \]
\[ \downarrow \quad \delta_k^{-1} \quad \delta_k \]
\[ A^*(\bigwedge^k M(p)) \otimes \bigwedge^k M(p) \rightarrow \bigwedge^k M(p) \]
A piece of Notation

Define:

\[ \int_n \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \]

\[= \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n - k + 1, \ldots, n) \\
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0 & \text{otherwise}
\end{cases} \]

Extend \( \int \) by \( \mathbb{Z} \)-linearity, getting \( \int : \wedge^k M_n \to \mathbb{Z} \)

The Main Theorem
A piece of Notation

Define:

\[ \int_{n}^{\epsilon_{1} \land \ldots \land \epsilon_{k}} = \]

\[
= \begin{cases} 
1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (n - k + 1, \ldots, n) \\
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0 & \text{otherwise}
\end{cases}
\]

Extend \( \int \) by \( \mathbb{Z} \)-linearity, getting \( \int : \wedge^k M_n \to \mathbb{Z} \)

The Main Theorem implies that

\[ \int_{G(k,n)} P(\sigma) \cap \Omega_{i_1 \ldots i_k} = \int P(D) \epsilon_{i_1} \land \ldots \land \epsilon_{i_k} \]
Let us compute the degree of $G(2, 2+n)$ in the Plücker embedding. It is

$$G(2, n+2) \sigma_2 n_1 \cap [G(2, n+2)] = n+2 D_2 n_1$$

It is the coefficient of $\pi_2, n+2 := n+1 \wedge \pi_2 n+2$ in the expansion of $D_2 n_1 \pi_1 \wedge \pi_2$.

Notice that:

$\text{wt}(\pi_1 + n \wedge \pi_2 + n) = 2n, \text{deg}(D_2 n_1) = 2n,$

$\text{wt}(\pi_1 \wedge \pi_2) = 0$ and

$\text{rk}_Z(\pi_2 M n + 2) = 2n_1.$
Let us compute the degree of $G(2, 2 + n)$ in the Plücker embedding.
Let us compute the degree of \( G(2, 2 + n) \) in the Plücker embedding. It is

\[
\int_{G(2,n+2)} \sigma_1^{2n} \cap [G(2, n + 2)] = \int_{n+2} D_1^{2n} \epsilon^1 \wedge \epsilon^2
\]
Let us compute the degree of $G(2, 2 + n)$ in the Plücker embedding. It is

$$
\int_{G(2,n+2)} \sigma_1^{2n} \cap [G(2, n + 2)] = \int_{n+2} D_1^{2n} \epsilon^1 \wedge \epsilon^2
$$

It is the coefficient of $\pi_{2,n+2} := \epsilon^{n+1} \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n} \epsilon^1 \wedge \epsilon^2$.
Let us compute the degree of $G(2, 2 + n)$ in the Plücker embedding. It is

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It is the coefficient of $\pi_{2,n+2} := \epsilon^{n+1} \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n} \epsilon^1 \wedge \epsilon^2$

Notice that:

$$\text{wt}(\epsilon^{1+n} \wedge \epsilon^{2+n}) = 2n, \quad \text{deg}(D_1^{2n}) = 2n, \quad \text{wt}(\epsilon^1 \wedge \epsilon^2) = 0$$

and

$$\text{rk}_\mathbb{Z}(\wedge^2 M_{n+2})_{2n} = 1.$$
Using the 1st Newton formula:
\[ D^2 n^1 \wedge n^2 = \sum_{j=0}^{n^j} D^2 n^j \]

In the sum we get contributions only from \( j = n \) and \( j = n + 1 \):
\[ D^2 n^1 \wedge n^2 = \sum_{j=0}^{n^j} 1 \]
\[ + \sum_{j=0}^{n^j} 2 + 2n - j \]
\[ = \sum_{j=0}^{n^j} \frac{n!}{n!(n+1)!} \]

Letterio Gatto
Schubert Calculus on a Grassmann Algebra
Using the 1st Newton formula:
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\[
D_1^m(\alpha \wedge \beta) = \sum_{j=0}^{m} \binom{m}{j} D_1^j \alpha \wedge D_1^{m-j} \beta
\]
Using the 1st Newton formula:
Using the 1st Newton formula:

\[ D_{1}^{2n} \epsilon^1 \wedge \epsilon^2 = \]
Using the 1st Newton formula:

\[ D_1^{2n} \epsilon^1 \land \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} D_1^j \epsilon^1 \land D_1^{2n-j} \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} \epsilon^{1+j} \land \epsilon^{2+2n-j} \]
Using the 1st Newton formula:

\[ D_1^{2n} \epsilon^1 \wedge \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} D_1^j \epsilon^1 \wedge D_1^{2n-j} \epsilon^2 = \sum_{j=0}^{2n} \binom{2n}{j} \epsilon^{1+j} \wedge \epsilon^{2+2n-j} \]

In the sum we get contributions only from \( j = n \) and \( j = n + 1 \):

\[ D_1^{2n} \epsilon^1 \wedge \epsilon^2 = \binom{2n}{n} \epsilon^{1+n} \wedge \epsilon^{2+n} + \binom{2n}{n+1} \epsilon^{2+n} \wedge \epsilon^{1+n} \]
Using the 1st Newton formula:

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\[ = \left[ \binom{2n}{n} - \binom{2n}{n+1} \right] \epsilon^{1+n} \wedge \epsilon^{2+n} = \frac{2n!}{n!(n+1)!} \epsilon^{1+n} \wedge \epsilon^{2+n} \]
Exactly by the same technique one finds:

\[ n + k \neq 1 \wedge \ldots \wedge \neq i_k = \omega_{i_1, \ldots, i_k} = (n + k - i_1) \ldots (n + k - i_k)! \cdot \ldots \cdot (n + k - i_k)! (3) \]

where \( w = \sum_{j=1}^{k} (i_j - j) \)

It is the degree of the Schubert variety \( \Omega_{i_1, \ldots, i_k}(F) \), \( F \) a given complete flag of \( \mathbb{C}^n \).
Exactly by the same technique one finds:

\[
\int_{n+k} D_1^{kn} e^1 \wedge \ldots \wedge e^k = \omega_{i_1, \ldots, i_k} = \frac{(kn - w)! \prod_{j<k} (i_k - i_j)}{(n + k - i_1)! \cdot \ldots \cdot (n + k - i_k)!} \tag{3}
\]
Exactly by the same technique one finds:

\[
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\]

where \( w = \sum_{j=1}^{k} (i_j - j) \)

It is the degree of the Schubert variety

\[\Omega_{i_1 \ldots i_k}(F^\bullet),\]

\( F^\bullet \) a given complete flag of \( \mathbb{C}^n \).
Rational Space curves having flexes at prescribed points.

Question (Ranestad): Find a list of the number of rational curves in $\mathbb{P}^3$ of degree $n + 3$ having inflectional tangent at $2n$ general marked points. Any such a curve can be gotten by projecting a rational normal curve in $\mathbb{P}^{n+3}$ from a $\mathbb{P}^{n-1}$ which intersects the osculating plane at the marked points. Therefore the sought for number is that of the $\mathbb{P}^{n-1}$'s having such a behaviour. This amounts to compute the integral.

$$\int G(n, n+4) \sigma_2 n^2 \cap [G(n, n+4)] = \int G(4, n+4) \sigma_2 n^2 \cap [G(4, n+4)]$$
Rational Space curves having flexes at prescribed points.

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$$\int_{G(n,n+4)} \sigma_2^{2n} \cap [G(n, n + 4)] = \int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4, n + 4)]$$
Summary

One usually computes $\sigma^2_n$ via iteration of Pieri's formula.

You may ask Schubert (1) doing it for you, but... when $n = 12$ you get the following message

Execution stopped: Stack limit reached. (Vainsencher, Økland – private communication).

However, we have a formula:

\[ \text{(1) S. Katz and S. A. Strømme, “Schubert”, a Maple } \]
\[ \text{package for intersection theory and enumerative geometry, http://math.uib.no/schubert/} \]
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However, we have a formula:

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G(4, n+4) \quad \sigma^2 \quad n^2 \quad \cap \quad [G(4, n+4)] = [4]

X_l_1 + l_2 + l_3 + l_4 = 2n\quad l_1, l_2, l_3, l_4 \geq 0

0 \leq m_2 \leq l_2 - m_2 \quad m_3 \leq l_3 + l_2 - m_2 \quad \cdot \omega(I(l_1, l_2, l_3, l_4, m_1, m_2, m_3))

where

I(l_1, l_2, l_3, l_4, m_1, m_2, m_3) is the ordered 4-tuple of positive integers.

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Schubert Calculus on a Grassmann Algebra
\[ \int_{G(4,n+4)} \sigma_2^{2n} \cap [G(4, n + 4)] = \] (4)
\[ \int_{G(4, n+4)} \sigma_2^{2n} \cap [G(4, n + 4)] = \]

\[ \sum_{l_1 + l_2 + l_3 + l_4 = 2n} \begin{pmatrix} 2n \\ l_1, l_2, l_3, l_4 \end{pmatrix} \begin{pmatrix} l_2 \\ m_2 \end{pmatrix} \begin{pmatrix} l_3 + l_2 - m_2 \\ m_3 \end{pmatrix} \cdot \omega I(l_1, l_2, l_3, l_4, m_1, m_2, m_3) \]
\[
\int_{G(4, n+4)} \sigma_2^{2n} \cap [G(4, n+4)] = (4)
\]

\[
\sum \binom{2n}{l_1, l_2, l_3, l_4} \binom{l_2}{m_2} \binom{l_3 + l_2 - m_2}{m_3} \cdot \omega(I(l_1, l_2, l_3, l_4, m_1, m_2, m_3))
\]

where

\[I(l_1, l_2, l_3, l_4, m_1, m_2, m_3)\]

is the ordered 4-tuple of positive integers

\[(1 + l_1, 2 + l_2 + m_2, 3 + l_3 + m_3, 4 + 2l_4 + (l_2 - m_2) + (l_3 - m_3)).\]
Summary

A very important question:
WHO CARE?

Letterio Gatto
Schubert Calculus on a Grassmann Algebra
A very important Question:
A very important Question: WHO CARE?
A very important Question:

**WHO CARE?**

We don’t, we may produce many similar formulas!
We got formula (4) by applying 2nd Newton formula to $D^2 n_2 (\bar{\pi}_1 \wedge \bar{\pi}_2 \wedge \bar{\pi}_3 \wedge \bar{\pi}_4) = D^2 n_2 (\bar{\pi}_1 \wedge \bar{\pi}_2 \wedge \bar{\pi}_3)^2 = 0$.

Then we apply the same formula to $D^2 n - l_1 n_2 (\bar{\pi}_2 \wedge \bar{\pi}_3 \wedge \bar{\pi}_4) = D^2 n - l_1 n_2 (\bar{\pi}_2 \wedge \bar{\pi}_3)^2$ and then once again, and so on...
We got formula (4) by applying 2nd Newton formula
We got formula (4) by applying 2nd Newton formula

\[ D_h^m(\epsilon^i \wedge \alpha) = \sum_{j=0}^{m} \binom{m}{j} D_{h-1}^j(\epsilon^{i+j} \wedge D_h^{m-j} \alpha) \]
We got formula (4) by applying 2nd Newton formula
We got formula (4) by applying 2nd Newton formula to

\[ D_2^{2n}(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4) = D_2^{2n}((\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3) \wedge \epsilon^4) \]

\[ \sum_{l_1=0}^{2n} \binom{2n}{l_1} D_1^{l_1}(\epsilon^{1+l_1} \wedge D_2^{2n-l_1}(\epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4)). \]
We got formula (4) by applying 2nd Newton formula

to

\[ D_2^{2n}(\epsilon^1 \land \epsilon^2 \land \epsilon^3 \land \epsilon^4) = D_2^{2n}((\epsilon^1 \land \epsilon^2 \land \epsilon^3) \land \epsilon^4) \]

\[ \sum_{l_1=0}^{2n} \binom{2n}{l_1} D_1^{l_1}(\epsilon^{1+l_1} \land D_2^{2n-l_1}(\epsilon^2 \land \epsilon^3 \land \epsilon^4)) \].

Then we apply the same formula to

\[ D_2^{2n-l_1}(\epsilon^2 \land \epsilon^3 \land \epsilon^4) = D_2^{2n-l_1}(\epsilon^2 \land (\epsilon^3 \land \epsilon^4)) \]

and then once again, and then...
Summary

we wrote a trivial CoCoA (version 4.7) code to compute the list varying $n$, and with an Apple iBook G4, 1.2GHz, RAM, 768Mb ←− my iBook in a couple of hours we got the following list:

Letterio Gatto

Schubert Calculus on a Grassmann Algebra
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← my iBook
... we wrote a trivial CoCoA (version 4.7) code to compute the list varying $n$, and

with an Apple iBook G4, 1.2GHz, RAM, 768Mb

← my iBook

in a couple of hours we got the following list:
<table>
<thead>
<tr>
<th>n</th>
<th>#(summands)</th>
<th>( R_n := \int \sigma_2^{2n} \cap [G(4, 4+n)] )</th>
<th>execution time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.082s</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>0</td>
<td>0.242s</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
<td>1</td>
<td>0.614s</td>
</tr>
<tr>
<td>3</td>
<td>142</td>
<td>5</td>
<td>1.449s</td>
</tr>
<tr>
<td>4</td>
<td>331</td>
<td>126</td>
<td>3.340s</td>
</tr>
<tr>
<td>5</td>
<td>641</td>
<td>3396</td>
<td>6.434s</td>
</tr>
<tr>
<td>6</td>
<td>1191</td>
<td>114675</td>
<td>12.081s</td>
</tr>
<tr>
<td>7</td>
<td>1981</td>
<td>4430712</td>
<td>20.053s</td>
</tr>
<tr>
<td>8</td>
<td>3221</td>
<td>190720530</td>
<td>32.755s</td>
</tr>
<tr>
<td>9</td>
<td>4866</td>
<td>8942188632</td>
<td>50.085s</td>
</tr>
<tr>
<td>10</td>
<td>7256</td>
<td>449551230102</td>
<td>1m 20s</td>
</tr>
<tr>
<td>11</td>
<td>10268</td>
<td>23948593282950</td>
<td>2m 55s</td>
</tr>
<tr>
<td>12</td>
<td>14418</td>
<td>1339757254689348</td>
<td>2m 44s</td>
</tr>
<tr>
<td>13</td>
<td>19466</td>
<td>78153481093195800</td>
<td>4m 02s</td>
</tr>
<tr>
<td>14</td>
<td>26156</td>
<td>4727142898098368085</td>
<td>5m 2s</td>
</tr>
<tr>
<td>15</td>
<td>34086</td>
<td>295116442188446065635</td>
<td>9m 9s</td>
</tr>
<tr>
<td>16</td>
<td>44286</td>
<td>18945322608397492982250</td>
<td>10m 46s</td>
</tr>
<tr>
<td>17</td>
<td>56141</td>
<td>1246718376589846006057200</td>
<td>11m 52s</td>
</tr>
<tr>
<td>18</td>
<td>71031</td>
<td>83878801924226511500933250</td>
<td>16m 37s</td>
</tr>
<tr>
<td>19</td>
<td>88071</td>
<td>57568600111979383129907915050</td>
<td>19m 55s</td>
</tr>
<tr>
<td>20</td>
<td>109061</td>
<td>402290757162008042628235950300</td>
<td>25m 53s</td>
</tr>
<tr>
<td>21</td>
<td>132783</td>
<td>28575935656515287427874861725000</td>
<td>34m 37s</td>
</tr>
<tr>
<td>22</td>
<td>161533</td>
<td>2060372706082551084572192852992530</td>
<td>01h 07m</td>
</tr>
</tbody>
</table>
Two days ago, Jan Magnus Økland sent me the list up to \( n = 30 \). He got it in a couple of hours... using Schubert2 for Macaulay (Grayson, Daniel and Stillmann) on a machine with processor speed 2.2 GHz and 16Gb Ram.

Then, yesterday morning I tried with CoCoA \( n = 23 \) and \( n = 24 \). In a couple of hours I got

\[
R_{23} = 150602793256105806699840616089824880
\]

and

\[
R_{24} = 11147684597786902087815929474416203276
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Summary

Thank You!

(Letterio Gatto)

http://calvino.полит.ит/~gatto/

Schubert Calculus on a Grassmann Algebra
Thank You!
Thank You!
(Grazie)
Thank You!
(Grazie)
Thank You!
(Grazie)
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(Grazie)

letterio

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(Grazie)

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