

Recurrence sequences for beginners

Fabio Fagnani

Dipartimento di Matematica

Politecnico di Torino

Email: fabio.fagnani@polito.it

<http://calvino.polito.it/~fagnani/>

1 First order recurrence sequences

Let D be a subset of \mathbb{R} . Suppose we have a function $f : D \rightarrow D$ and a number $\alpha \in D$. We can construct a sequence by considering the iterative scheme

$$a_{n+1} = f(a_n), \quad a_0 = \alpha. \quad (1)$$

Notice that

$$a_0 = \alpha, \quad a_1 = f(\alpha), \quad a_2 = f(f(\alpha)), \dots$$

A sequence a_n constructed in this way is called a (*first order*) *recurrence sequence*; the term first order is to indicate that each term of the sequence is evaluated starting from the previous one. We will see later that more complex iterative schemes can indeed be considered. We will refer to $a_{n+1} = f(a_n)$ as to the *recurrence scheme*. Notice that, by itself, it does not determine any sequence unless we specify the initial value α . Notice that it is fundamental that the image of f is contained inside its domain D : otherwise, the various compositions of f with itself, might not be well defined.

Examples of recurrence schemes we will consider are the following

$$a_{n+1} = \frac{1 - a_n}{2}, \quad (2)$$

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right), \quad (3)$$

$$a_{n+1} = 2a_n + 1. \quad (4)$$

Recurrence schemes come up in many fields of pure and applied mathematics, of theoretical engineering and of theoretical computer science. They can describe iterative algorithms: (2) for instance can be used to trisect a paper in three equal parts while (3) gives a possible way to approximate $\sqrt{2}$. Recurrence schemes can also be used to describe the discrete-time evolution of some physical or biological system, n in this case plays the role of a discrete time. Other times they can be the complexity measure of some combinatorial problem which depends on a natural parameter n : the recurrence scheme (4) comes up, for instance, when we try to compute the number of moves needed to solve a famous combinatorial game, the so-called Hanoi's tower problem.

Which are the main questions regarding recurrence sequences?

- *Find a closed way to express the recurrence sequence*: notice that if we want to compute a_{1000} of a recurrence sequence as (1), we need to apply 1000 times the function f . It would be nice if we could find a closed analytic expression for a_n as a function of the index n .
- *Study the asymptotic behavior of the recurrence sequence*: in those cases when a closed formula can not be found, we would still like to analyze the behavior of the sequence when n grows: in particular, we would like to establish if the sequence is increasing, decreasing, if it admits limit and possibly the value of the limit.

1.1 Asymptotic analysis: introductory examples

We will first focus on the second question which is of fundamental importance in various branches of pure and applied mathematics. A crucial fact is that many properties of the sequence a_n can easily be read out of the iterating function f . The following are a couple of immediate remarks:

- (P1) If α is such that $f(\alpha) = \alpha$, it follows that the sequence a_n defined in (1) is constantly equal to α .
- (P2) If $f(x) \leq x$ for all x , it follows that $a_{n+1} = f(a_n) \leq a_n$, namely a_n is a decreasing sequence. Similarly, if $f(x) \geq x$ for all x , a_n is an increasing sequence. More generally, if the sign of $f(x) - x$ does not remain constant, we will still have that a_n is decreasing (increasing) as long as a_n belongs to those intervals where $f(x) \geq x$ (respectively, $f(x) \leq x$).
- (P3) If $a_n \rightarrow l$ for $n \rightarrow +\infty$ and f is continuous in l , then it must hold that $f(l) = l$. Indeed, since f is continuous in l , it follows that $f(a_n) \rightarrow f(l)$. On the other hand, $f(a_n) = a_{n+1} \rightarrow l$. By the uniqueness of the limit, it follows that $f(l) = l$.

We analyze our first example, making use of the above remarks.

Example 1: Consider the recurrence sequence

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right), \quad a_0 = \alpha > 0. \quad (5)$$

In this case the function $f :]0, +\infty[\rightarrow]0, +\infty[$ is given by $f(x) = (x + 2/x)/2$: see Figure 1.

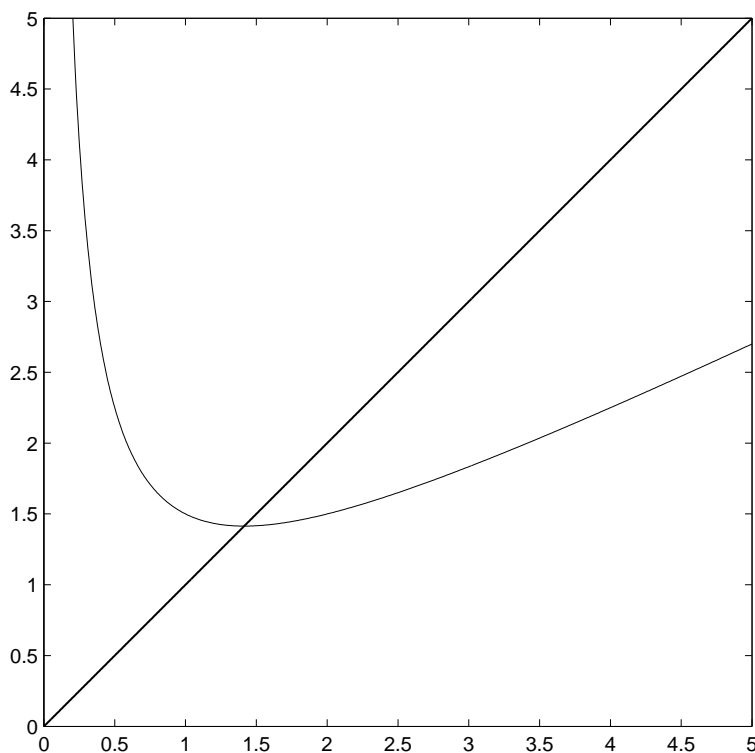


Figure 1:

Notice that $f(x) = x$ is equivalent to $x^2 = 2$, namely in our domain $x = \sqrt{2}$. Moreover, it is immediate to check that, $f(x) \leq x$ if and only if $x \geq \sqrt{2}$. Moreover, it can be shown that f is decreasing in $]0, \sqrt{2}]$ and increasing in $[\sqrt{2}, +\infty[$.

Suppose first that $a_0 = \alpha > \sqrt{2}$. Since f is increasing in $[\sqrt{2}, +\infty[$, it follows that

$$a_n \geq \sqrt{2} \Rightarrow a_{n+1} = f(a_n) \geq f(\sqrt{2}) = \sqrt{2}$$

In other words, this means that if $a_0 \geq \sqrt{2}$, then $a_n \geq \sqrt{2}$ for every n . Actually, this last deduction, apparently obvious, uses inside the so-called induction principle (which can be found in any textbook in basic analysis and algebra): from the fact that $a_0 \geq \sqrt{2}$ and the fact that if $a_n \geq \sqrt{2}$ for some n , then also $a_{n+1} \geq \sqrt{2}$ the induction principle allows to conclude that $a_n \geq \sqrt{2}$ for all n . Often in the sequel we will use the induction principle to prove properties of recurrence sequences. Going back to a_n , knowing that $a_n \geq \sqrt{2}$ for all n , it then follows (property (P2)) that a_n is a decreasing sequence. Hence, it must admit a limit l which is finite since $a_n \geq \sqrt{2}$ for all n . Since f is continuous, by (P3), l must satisfy the conditions $l \geq \sqrt{2}$ and $f(l) = l$. Clearly, the only possibility is $l = \sqrt{2}$. What happens if we start instead from an initial value $a_0 = \alpha \in]0, \sqrt{2}[$? Notice that $a_1 = f(a_0) \geq f(\sqrt{2}) = \sqrt{2}$. But then, from the index 1 on, we are back in the case considered before: we will have that $a_n \geq \sqrt{2}$ for all $n \geq 1$ and $a_n \rightarrow \sqrt{2}$ for $n \rightarrow +\infty$. Hence, we have proven that for any possible choice of $\alpha \in]0, +\infty[$,

$$\lim_{n \rightarrow +\infty} a_n = \sqrt{2}.$$

We leave as a simple exercise to study the case when $\alpha < 0$.

The technique used in the above example, based on the monotonicity of $f(x)$, can be used in many situations. It is convenient to give it a general formulation:

(P4) Suppose that $f : D \rightarrow D$ is such that $f(l) = l$ for some $l \in D$. Then, exactly as in the previous example, it holds

$$\begin{aligned} f \text{ increasing in } D \cap]-\infty, l] \text{ and } a_0 \leq l &\Rightarrow a_n \leq l \quad \forall n \in \mathbb{N} \\ f \text{ increasing in } D \cap [l, +\infty[\text{ and } a_0 \geq l &\Rightarrow a_n \geq l \quad \forall n \in \mathbb{N} \end{aligned}$$

Example 2: Consider the recurrence sequence

$$a_{n+1} = 2a_n + 1, \quad a_0 = \alpha. \quad (6)$$

In this case the function f is given by $f(x) = 2x + 1$. We have $f(x) = x$ if and only if $x = -1$. Moreover, $f(x) \leq x$ if and only if $x \leq -1$. Since f is an increasing function, we can apply property (P4) which says that if $a_0 = \alpha \geq -1$, then $a_n \geq -1$ for all n , and a_n is thus increasing. Hence it must admit a limit l . If l was finite, the only possibility would be $l = -1$. This is what happens if $\alpha = -1$. However if $\alpha > -1$, it follows that $a_n \geq \alpha > -1$. Hence l can not be equal to -1 . The only possibility is therefore $l = +\infty$. We have thus proven that in this case $a_n \rightarrow +\infty$ for $n \rightarrow +\infty$. A similar argument shows that $a_n \rightarrow -\infty$ if $a_0 = \alpha < -1$.

Example 3: Consider the recurrence sequence

$$a_{n+1} = \frac{1 - a_n}{2}, \quad a_0 = \alpha.$$

In this case the function f is given by $f(x) = (1 - x)/2$ ad $D = \mathbb{R}$. Notice that $f(x) = x$ if and only if $x = 1/3$. Hence, if $a_0 = \alpha = 1/3$, then $a_n = 1/3$ for every $n \in \mathbb{N}$. We now prove that indeed $a_n \rightarrow 1/3$ always. Notice that f is decreasing on its domain \mathbb{R} . This implies that if for instance $a_0 = \alpha < 1/3$, then $a_1 = f(a_0) \geq f(1/3) = 1/3$. Continuing, $a_2 = f(a_1) \leq 1/3$ and so on: the sequence a_n has an oscillatory behavior around the point $1/3$ and is not a priori clear if it converges or not. It is very useful in these circumstances to consider the two-steps recursive map $f^2 := f \circ f$:

$$f^2(x) = \frac{1 - \frac{1-x}{2}}{2} = \frac{1+x}{4}.$$

Notice that through f^2 we can describe, separately, the evolution of a_n for even and odd n :

$$a_{2n+2} = f^2(a_{2n}), \quad a_{2n+1} = f^2(a_{2n-1}).$$

It is immediate to see that f^2 is now strictly increasing on \mathbb{R} (not unexpected since we have composed two strictly decreasing maps!) Also it is obvious that $f^2(x) = x$ if and only if $x = 1/3$ and $f^2(x) \leq x$

if and only if $x \geq 1/3$. Suppose we start from $a_0 = \alpha \leq 1/3$ (the case $a_0 = \alpha \geq 1/3$ is completely analogous). Then, because of the properties of f^2 , using property (P2), (P3), and (P4), it follows that $a_{2n} \leq 1/3$ for every n , is increasing and converges to $1/3$. Moreover, notice that $a_1 \geq 1/3$. Similarly, it follows that $a_{2n+1} \geq 1/3$ for every n , is decreasing and converges to $1/3$. Putting the two facts together, we can now conclude, that

$$\lim_{n \rightarrow +\infty} a_n = \frac{1}{3}.$$

1.2 Asymptotic analysis: other cases

In all the examples analyzed sofar, the sequence a_n admits a limit finite or infinite. Of course, it is easy to construct examples where this does not happen. In general, it is the presence of intersection points between the graph of f and the line $y = x$ where f is decreasing which may cause a lot of pathologies in the asymptotic behavior of the sequence a_n . Lack of limit and also 'chaotic behavior' can show up in these cases. Here we limit ourselves to present a simple academic example

Example 4: Consider the recurrence sequence

$$a_{n+1} = \frac{1}{a_n}, \quad a_0 = \alpha. \quad (7)$$

In this case the function $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$ is given by $f(x) = 1/x$. Clearly, $f(x) = x$ if and only if $x = \pm 1$. Of course if we start from $a_0 = \pm 1$, a_n is constant. If, on the other hand, $a_0 \neq \pm 1$, then clearly, $a_1 \neq a_0$ and $a_{2n} = a_0$ for every n , while $a_{2n+1} = a_1$ for every n . This shows that in this case a_n is periodic of period 2 and does not converge.

There are other first order recurrence sequences which do not fit in the scheme (1). An example is the following

Example 6: Consider the recurrence sequence

$$a_{n+1} = \frac{1}{n}a_n + 1, \quad a_1 = \alpha \geq 0. \quad (8)$$

The scheme depends explicitly on n ; none of the techniques learned sofar can be used in this case. We first prove that a_n is always a bounded sequence. Notice that clearly we always have $a_n \geq 0$. To obtain an upper bound on a_n , assume first that $\alpha \geq 1$ and let us prove that $a_n \leq 1 + \alpha$ for every n . Of course this is true for $n = 1$ and for $n = 2$. To prove it for all others n we use a technique known as 'induction principle' (details can be found in any textbook in basic analysis or algebra). We prove that, given any $n \geq 2$, if we assume that $a_n \leq 1 + \alpha$, then it must also hold $a_{n+1} \leq 1 + \alpha$. This together with the fact that $a_2 \leq 1 + \alpha$ will allow us to establish the claim. So, assume that for a given $n \geq 2$, $a_n \leq 1 + \alpha$. Now

$$a_{n+1} = \frac{1}{n}a_n + 1 \leq \frac{1}{n}(1 + \alpha) + 1 \leq \frac{1}{2}(1 + \alpha) + 1 \leq 1 + \alpha,$$

(last inequality holds true specifically because $\alpha \geq 1$). What about if instead $\alpha \in [0, 1[$? Well, this case can easily be treated with the help of previous case in the following way: let (a_n) be the sequence defined by (8) with an $\alpha \in [0, 1[$ and let (b_n) be the sequence, satisfying the same recursive scheme but with $\alpha = 1$. We know from previous considerations that $b_n \leq \alpha + 1 = 2$ for every n . On the other hand, it should be clear that $a_n \leq b_n$ for every n (try to prove this fact formally using again the induction principle). Hence we also have that $a_n \leq 2$ for every n . We have thus proven that the sequence (a_n) is bounded for any initial condition $\alpha \geq 0$. We are now ready to study the limit behavior. Notice indeed that, since a_n is bounded, it follows that $a_n/n \rightarrow 0$ for $n \rightarrow +\infty$. But then, from the recursive relation we obtain that

$$\lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} \left(\frac{1}{n}a_n + 1 \right) = 0 + 1 = 1.$$

Hence, we have obtained that for any initial condition $\alpha \geq 0$,

$$\lim_{n \rightarrow +\infty} a_n = 1.$$

As an exercise study the case when $\alpha < 0$.

1.3 Closed formulas

We now try to address the other problem we had posed: how to find exact analytic expressions for recurrence sequences. We have to say first of all, that this problem can be solved only in very special cases. Most of recurrence sequences do not admit such a representation.

In certain cases a closed formula is very easy to be found. For instance if we consider

$$a_{n+1} = ra_n, \quad a_0 = \alpha \tag{9}$$

where $r, \alpha \in \mathbb{R}$ we immediately obtain

$$a_n = r^n \alpha$$

This example can be generalized a little bit. Consider indeed,

$$a_{n+1} = ra_n + s, \quad a_0 = \alpha$$

where $r, s, \alpha \in \mathbb{R}$. For a moment forget about the initial condition and concentrate on the recurrence scheme itself:

$$a_{n+1} = ra_n + s. \tag{10}$$

Of course there will be many different sequences compatible with it, one for every initial condition; these sequences will be said to satisfy the scheme (10). Suppose that a'_n and a''_n are two sequences satisfying (10), namely,

$$a'_{n+1} = ra'_n + s, \quad a''_{n+1} = ra''_n + s.$$

Consider now the difference sequence

$$b_n = a''_n - a'_n.$$

It satisfies the recurrence scheme

$$b_{n+1} = a''_{n+1} - a'_{n+1} = ra''_n + s - (ra'_n + s) = r(a''_n - a'_n) = rb_n.$$

This implies that $b_n = r^n \beta$ for some $\beta > 0$. Therefore, $a''_n = a'_n + r^n \beta$. This argument can be easily inverted to prove that if we start from a sequence a'_n satisfying the scheme (10), then the sequence $a''_n = a'_n + r^n \beta$ also satisfies (10) for any possible choice of β . This says that if we can find just one sequence satisfying (10), then all the others can be constructed simply adding the term $r^n \beta$. On the other hand, it is quite simple to find a particular sequence satisfying (10). Indeed, consider $a_n = a$ constant: the recurrence relation forced on this sequence yields $a = ra + s$. If $r \neq 1$ this has solution $a = (1 - r)^{-1}s$. The constant sequence $a_n = (1 - r)^{-1}s$ satisfies the recurrence scheme (10). Hence, the general sequence satisfying (10) can be written in the form

$$a_n = (1 - r)^{-1}s + r^n \beta.$$

Going back to (9), imposing the initial condition $a_0 = \alpha$ yields $\beta = \alpha - (1 - r)^{-1}s$. We thus obtain the closed formula

$$a_n = (1 - r)^{-1}s + r^n(\alpha - (1 - r)^{-1}s), \quad n \in \mathbb{N}. \tag{11}$$

Notice that if $|r| < 1$, this yields

$$\lim_{n \rightarrow +\infty} a_n = (1 - r)^{-1}s,$$

in accord with the analysis of Example 1 done above ($r = -1/2$ and $s = 1/2$ yielding $(1 - r)^{-1}s = 1/3$). Instead, in those cases when $r > 1$, the limit will always diverge except when we choose $\alpha = (1 - r)^{-1}s$, in which case the sequence is constant.

If $r = 1$, there is no constant sequence satisfying (10). Notice however that in this case we have $a_{n+1} = a_n + s$ from which it is immediate to derive the formula $a_n = ns + a_0$. Hence in this case a closed formula for the sequence defined by (9) is given by

$$a_n = ns + \alpha, \quad n \in \mathbb{N}. \tag{12}$$

1.4 Speed of convergence

Once we have established that a sequence converges to a limit l , it is also important to be able to say something about the velocity of this convergence. This is particularly important when the sequence describes the behavior of an algorithm or the evolution of some physical system. In those cases when a close formula is available, the speed of convergence can, in many cases, be computed directly. For example, in the case treated above $a_{n+1} = ra_n + s$ with $|r| < 1$, it is clear from (11) that the convergence is of geometric type: $a_n - (1-r)^{-1}s$ goes to 0 'as r^n '. In Landau symbols terms, we can write, if $\alpha \neq (1-r)^{-1}s$,

$$a_n - (1-r)^{-1}s \asymp r^n, \quad \text{for } n \rightarrow +\infty.$$

Instead if $r > 1$ we have that

$$a_n \asymp r^n, \quad \text{for } n \rightarrow +\infty.$$

Can we say anything about the speed of convergence in case where a closed formula is not available? As usual, we work with an example, specifically with Example 1 treated above.

Example 1 (revisited): Suppose we choose $a_0 = \alpha \geq \sqrt{2}$. We now that $a_n \geq \sqrt{2}$ for every n . It follows that

$$\frac{a_n - \sqrt{2}}{a_n} \leq 1, \quad \forall n \in \mathbb{N} \quad (13)$$

We now have the following chain of equalities

$$\begin{aligned} a_{n+1} - \sqrt{2} &= \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) - \sqrt{2} \\ &= \frac{1}{2} \frac{a_n^2 + 2 - 2\sqrt{2}a_n}{a_n} \\ &= \frac{1}{2} \frac{(a_n - \sqrt{2})}{a_n} (a_n - \sqrt{2}). \end{aligned} \quad (14)$$

From (14) and (13) we now obtain

$$a_{n+1} - \sqrt{2} \leq \frac{1}{2} (a_n - \sqrt{2}).$$

Hence we obtain, iterating these inequalities,

$$a_n - \sqrt{2} \leq \frac{1}{2^n} (a_0 - \sqrt{2}).$$

In other terms, the distance from the limit decreases to 0 at least as $(1/2)^n$. In Landau symbols terms:

$$a_n - \sqrt{2} = O((1/2)^n), \quad \text{for } n \rightarrow +\infty.$$

1.5 Higher order recurrence sequences

Recurrence sequences can also be defined through recurrence schemes where a term of a sequence a_{n+1} is determined by not just the previous one a_n , but possibly by other past values of the sequence a_{n-1}, a_{n-2}, \dots

The simplest case are the second order schemes where a_{n+1} depends on a_n and a_{n-1} .

An example is the celebrated Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 1, F_1 = 1. \quad (15)$$

Notice that two initial conditions in this case are needed to define the sequence. Here are the first few terms of the Fibonacci sequence:

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, \dots$$

To prove that F_n diverges to $+\infty$ is quite easy: notice that clearly, $F_n > 0$ for every n and from (15) $F_{n+2} \geq F_{n+1}$ for every n so that the sequence is increasing. Hence it must admit a limit $F_n \rightarrow l$ for $n \rightarrow +\infty$. If l was finite, we would obtain

$$l = \lim_{n \rightarrow +\infty} F_{n+2} = \lim_{n \rightarrow +\infty} (F_{n+1} + F_n) = l + l.$$

hence $l = 0$ but this is impossible since $F_n \geq 1$ for every n . Hence the only possibility is that $F_n \rightarrow +\infty$. We would like to understand however 'how fast' the Fibonacci sequence grows to $+\infty$. Notice that since, F_n is increasing, we have that $F_{n+1} \geq F_n$ and therefore, $F_{n+2} \leq 2F_{n+1}$. This is saying that the Fibonacci sequence should grow less than a geometric sequence of type 2^n .

In the sequel we will actually determine a closed formula for this sequence from which also the growth behavior will be completely clear. In order to find such a formula, we need to forget about the initial conditions and consider only the recurrence scheme

$$a_{n+2} = a_{n+1} + a_n. \quad (16)$$

We had noticed before that we expect growths lower than the geometric sequence 2^n . We could try to see if there are sequences of type $a_n = \lambda^n$ with $\lambda \neq 0$ compatible with the recurrence scheme (16). We obtain

$$\lambda^{n+2} = \lambda^{n+1} + \lambda^n, \forall n \in \mathbb{N}.$$

Dividing by the non-zero term λ^n we see that the condition above is equivalent to the simple equation

$$\lambda^2 - \lambda - 1 = 0,$$

which has solutions

$$\lambda_1 = \frac{1 - \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 + \sqrt{5}}{2}.$$

This means that the two sequences $a'_n = \lambda_1^n$ and $a''_n = \lambda_2^n$ are both compatible with our recurrence scheme (16). If we now consider any linear combination of these two sequences

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n, \quad (17)$$

a straightforward verification shows that this is still compatible with the recurrence scheme (16) (this is due to the fact that the relation among a_{n+2} , a_{n+1} and a_n in the recurrence scheme is linear). In this way we have found a family of sequences all compatible with the recurrence scheme (16): the family is parameterized by the two constants C_1 and C_2 . Suppose now we want to impose a pair of initial conditions: $a_0 = \alpha$ and $a_1 = \beta$. We get the system of equations:

$$\begin{cases} C_1 + C_2 = \alpha \\ C_1 \lambda_1 + C_2 \lambda_2 = \beta \end{cases}$$

It is immediate to see that, since $\lambda_1 \neq \lambda_2$ this system has a (unique) solution C_1, C_2 given by

$$C_1 = \frac{\alpha \lambda_2 - \beta}{\lambda_2 - \lambda_1}, \quad C_2 = \frac{\beta - \alpha \lambda_1}{\lambda_2 - \lambda_1}$$

We have thus proven that the recurrence sequence given by

$$a_{n+2} = a_{n+1} + a_n, \quad a_0 = \alpha, \quad a_1 = \beta. \quad (18)$$

can be represented in the form

$$a_n = \frac{\alpha \lambda_2 - \beta}{\lambda_2 - \lambda_1} \lambda_1^n + \frac{\beta - \alpha \lambda_1}{\lambda_2 - \lambda_1} \lambda_2^n \quad (19)$$

We have found a closed formula for any possible initial conditions; notice that this argument indirectly also shows that the sequences in (17) are all the possible sequences compatible with the recurrence scheme (16). In all cases there are two geometrical behaviors showing up in the solution: λ_1^n and λ_2^n . Notice however that $|\lambda_1| < 1$ so that $\lambda_1^n \rightarrow 0$ for $n \rightarrow +\infty$. On the other hand $\lambda_2 > 1$. The important term, for large n is thus given by the one with λ_2^n : it is the one responsible for the growth rate of the

Fibonacci sequence. λ_2 is the so called golden ratio and is an ubiquitous number in many fields of mathematics.

In the special case of the initial conditions connected with the Fibonacci sequence $\alpha = \beta = 1$, using the simple fact that $\lambda_2 + \lambda_1 = 1$, we can rewrite the solution in a more compact form:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right] \quad (20)$$

This is a closed formula for the Fibonacci sequence: in spite of its look, this formula always gives natural numbers! The asymptotic growth is the following:

$$F_n \asymp \left(\frac{1 + \sqrt{5}}{2} \right)^n, \quad \text{for } n \rightarrow +\infty.$$

Fibonacci sequences show up in many contexts. Below we recall one of these. Consider the following subset of binary sequences:

$$\Omega_n = \{\omega \in \{0, 1\}^n \mid \omega_i \omega_{i+1} \neq 00 \ \forall i = 1, \dots, n-1\}.$$

Let $a_n = |\Omega_n|$. We want to find an explicit expression for the sequence a_n . We proceed in the following way: consider the following subsets of Ω_n

$$\Omega_n^0 = \{\omega \in \Omega_n \mid \omega_n = 0\}, \quad \Omega_n^1 = \{\omega \in \Omega_n \mid \omega_n = 1\}$$

and let $a_n^0 = |\Omega_n^0|$ and $a_n^1 = |\Omega_n^1|$. Notice that we have the following recursive relations

$$\begin{aligned} a_{n+1}^0 &= a_n^0 + a_n^1 \\ a_{n+1}^1 &= a_n^0 \end{aligned}$$

from which it follows that

$$a_{n+1} = a_{n+1}^0 + a_{n+1}^1 = (a_n^0 + a_n^1) + a_n^0 = (a_n^0 + a_n^1) + (a_{n-1}^0 + a_{n-1}^1) = a_n + a_{n-1}.$$

Shifting indices, we thus obtain

$$a_{n+2} = a_{n+1} + a_n$$

the same recurrence scheme defining the Fibonacci sequence! Which are the initial conditions? Well, it is immediate to check that $a_1 = 2$ and $a_2 = 3$. We can always start from 0 and put $a_0 = 1$. This shows that a_n coincides with the sequence of Fibonacci numbers with the index shifted of one unit: $a_n = F_{n+1}$. We thus obtain the closed formula

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+2} \right] \quad (21)$$