Stability Analysis and Synthesis for Scalar Linear Systems With a Quantized Feedback

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***Abstract*—It is well known that a linear system controlled by a quantized feedback may exhibit the wild dynamic behavior which is typical of a nonlinear system. In the classical literature devoted to control with quantized feedback, the flow of information in the feedback was not considered as a critical parameter. Consequently, in that case, it was natural in the control synthesis to simply choose the quantized feedback approximating the one provided by the classical methods, and to model the quantization error as an additive white noise. On the other hand, if the flow of information has to be limited, for instance, because of the use of a transmission channel with limited capacity, some specific considerations are in order. The aim of this paper is to obtain a detailed analysis of linear scalar systems with a stabilizing quantized feedback control. First, a general framework based on a sort of Lyapunov approach encompassing known stabilization techniques is proposed. In this case, a rather complete analysis can be obtained through a nice geometric characterization of asymptotically stable closed-loop maps. In particular, a general tradeoff relation between the number of quantization intervals, quantifying the information flow, and the convergence time is established. Then, an alternative stabilization method, based on the chaotic behavior of piecewise affine maps is proposed. Finally, the performances of all these methods are compared.**

***Index Terms*—Chaotic control, communication constraints, quantized feedback, stability, stabilization.**

I. INTRODUCTION

THE analysis of dynamical systems in which the feedback is a quantized function has been a central issue in control theory, since the advantages of digital control with respect to analog control have been recognized. The original approach used to deal with quantization in feedback systems consisted in modeling the difference between a signal and its quantized version simply as an additive white noise. This method was quite successful both for its simplicity and because it provided an acceptable accuracy of the analysis when the quantization was not too coarse.

For a more accurate analysis, it is convenient to model the digital control as the interaction between a continuous dynamical system and a computer, which is a finite state machine. This combination can be considered as an instance of a hybrid system. This alternative point of view yields really useful contributions in the study of digital control systems only if the difference between the signals and their quantizations becomes remarkable so that the additive white noise model becomes meaningless. This is the case, for instance, when many remotely positioned plants have to be controlled in a centralized way by the transmission of the control signals through communication channels. The fact that these channels, besides being digital, have finite capacity introduces into the control project a new parameter, that is the amount of information flow that we allow in the feedback loops. In other words, the amount of information exchange required by the control has to be considered as an additional cost which enters in the design. The contributions provided by [1]–[10] are in this direction.

The stabilization problem by quantized time invariant state feedback was first considered in a certain detail in [11]. In that paper, the closed-loop system was studied as a general nonlinear system and mathematical instruments such as ergodic theory were first proposed for the analysis of the asymptotic behavior of the system. These mathematical methods allowed the author of [11] to obtain more insight on the problem, even though their complexity limited the applicability essentially to scalar systems. In this set up only uniform quantizers approximating linear feedback functions were considered. In [10], the stabilization problem was considered without restricting the feedback function to be a uniform quantizer. By means of quadratic Lyapunov functions it was shown that stabilization can be obtained by using a logarithmic quantizer, so that the quantized feedback can be coarse when the state is far from the equilibrium, while it has to be finer and finer as the state approaches the equilibrium. In this way, it is possible to save in quantization precision and to stabilize with less information flow in the feedback loop. The quantized feedback schemes studied in [11] and [10] are time-invariant and memoryless (except in the first part of [11]). Also, [4] is essentially devoted to memoryless quantized control schemes where however the transmission protocol between the plant and the controller is modeled in a different way: sampling time is here not uniform and variable length coding is used to transmit information bits. Other time-varying schemes have been proposed in [2], [5]–[7], and [9].

In this paper, we will work out a detailed analysis of the stabilization problem for the special case of scalar linear systems by means of quantized time-invariant memoryless feedbacks. We will propose new synthesis schemes and we will prove fundamental performance limitations of these quantized feedback schemes. As in [10], and differently from [11], we will not stick to a particular quantizer, but rather we will leave it as a possible design parameter. The two methods proposed in [11] and [10] will be put in a more general perspective and compared, on the basis of their performances, with another stabilization technique which is based on the intrinsic chaotic behavior of this

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class of nonlinear systems. The use of chaos for control purposes was first proposed in the classical paper [12] and a nice survey on the use of this method in the context of the control of mechanical systems is presented in [13]. The idea is first to find a quantized feedback which makes a subset I of the state space invariant and such that inside I the dynamics is chaotic. Then, since the state will move chaotically inside I, generically it will visit any subset J of I. Therefore, if we now modify the feedback making also J invariant, we obtain that all initial states inside I will eventually fall into the smaller set J. The analysis of such feedbacks makes a crucial use of the ergodic theory of one-dimensional (1–D) piecewise affine maps.

In order to compare these three stabilization methods, we refer to two parameters which measure their performances. The first parameter N is the number of quantization intervals used by the stabilizing feedback which is a quantitative description of the information flow between the system and the control. The second parameter T is related with the time needed for a state to reach the target set. Both the parameters will be, in fact, functions of the contraction rate C which is the ratio between the size of the bigger set I, where the initial state is supposed to start from, and the smaller target set J, where the state is designed to be attracted. We will establish a number of general inequalities linking these three parameter N, T and C suggesting that the proposed schemes can not be improved.

In this paper, as in [11], we chose to limit our analysis to scalar linear systems. If the classical stabilization problem for scalar linear systems is trivial, it is true that quantization already poses serious mathematical difficulties in this case. If we want to understand in depth the finer aspects of the performance evaluation and of the fundamental limitations of these control schemes, in our opinion, the scalar case is the right framework where to start from. It is clear that in the applications we are mainly interested in the multidimensional case, but we believe that the scalar case will be instrumental for a possible extension in that direction. On the one hand, we believe that some of the results proposed in this paper can have a first direct application to some classes of multidimensional systems with special structures [1] (diagonal systems or systems with only one unstable eigenvalue). On the other hand, some of the results obtained here for the scalar case suggest the right strategy to follow for possible extensions to the multidimensional case. For instance, the chaotic scheme, we believe, can be extended using the recent ergodic theory of multidimensional piecewise affine maps. Also, some of the fundamental limitations results should be extendable if one replace the 1-D geometric techniques used in this paper with the more abstract symbolic dynamics formalism. We are currently working in these directions. We do not expect of course to be able to have a picture as complete and as clear as the one we are obtaining in the scalar case. However, we think that these investigations on the scalar case will play the role of referring methods for the future research.

Finally, here are a few words regarding the choice of sticking to time-invariant memoryless feedbacks. A general principle in control theory is to try to reach a desirable performance trying to use a control as simple as possible. It is well known that linear control systems with state observation can be stabilized with time-invariant memoryless feedbacks and that the same is true if we add quantization. This is a good reason to start from these controllers. Of course, the use of more general feedbacks could in principle change the relations among the performance parameters T, N and C; for instance, since quantization reflects in a partial state observation, it is true that dynamics in the feedback might allow in certain situations to refine the state enstimates [14]. It is, however, true that in order to use this more refined knowledge of the state we need to have more freedom in the input choice and so to allow more input levels. In this case, it will naturally come up with other two parameters, the cardinalities of the controller output space and of the controller state–space. These will possibly link with the others to describe more general fundamental limitations. These questions, which to our knowledge are not explicitely treated in the literature, will be the subject of our future investigations. Finally, we would like to remark that the use of dynamical feedbacks poses also new problems in the stability analysis, since in this case the overall system is really a hybrid system and it is just in this set up that its stability has to be analyzed [15].

A. Problem Statement

Consider the following discrete-time, 1-D linear model:

$$x_{t+1} = ax_t + u_t \tag{1}$$

where $a \in \mathbb{R}$. Let $k : \mathbb{R} \to \mathbb{R}$ be a piecewise constant function with only a finite or countable number of discontinuities. If we use k as a static feedback in (1), we obtain the closed-loop system

$$x_{t+1} = \Gamma(x_t) \tag{2}$$

where $\Gamma(x) := ax + k(x)$ is a piecewise affine map. Differently from what happens when Γ is affine, autonomous systems like (2) in which Γ is piecewise affine can exhibit a very wild behavior. Their dynamical properties have been extensively studied in the past [16], [17].

The type of system theoretic issues which can be addressed in this setting are of a double nature. On the one hand, it is interesting to analyze the behavior of such closed-loop systems as in [11]. On the other hand, it is important to develop synthesis techniques for achieving specific control objectives. In this paper we will concentrate on stability issues [10].

It is obvious that, if we restrict to quantized feedbacks k in which all quantization intervals have size bigger than a positive real (this is true in particular when we have finite quantization intervals), it is not possible to obtain stability or asymptotic stability to an equilibrium point (see [11]), as it can be done by continuous feedback maps. Only the so called “practical stability” can be achieved, which means that the state converges, in some sense, to a certain interval J, which can be seen as playing the role of an equilibrium point in this set up. Therefore, the first thing we can ask is whether (2) admits invariant intervals, namely, intervals I such that $\Gamma(I) \subseteq I$ so that, if $x_0 \in I$, then $x_t \in I$ for every $t \geq 0$. Inside I the dynamics of Γ can be of various nature: x_t may wander for ever, describing a dense orbit in Ior, rather, it may converge to some smaller subset $J \subseteq I$. The following definition captures two different possible ways in which this convergence can occur.

Definition: Given two intervals $J \subseteq I$, which are invariant with respect to Γ, we say that Γ is (I,J)*-stable* if for every $x_0 \in I$, there exists an integer $t_0 \geq 0$ such that $x_t \in J$ for every $t \geq t_0$. We say that Γ is *almost* (I,J)*-stable* if the convergence to J as defined previously occur for almost all $x_0 \in I$, with respect to the Lebesgue measure.

It is clear that the previous definitions of stability and almost stability only depend on $\Gamma|_I$. We could, thus, assume that Γ is defined only on I.

Remark: Our definition of stability is essentially the same to the one considered in [10] and [11]. In [1] and [4], the authors instead consider stability without requiring convergence, which is more similar to our concept of invariance of an interval. Finally, in [2], [6], [7], and [9], classical asymptotic stability is studied. This is made possible thanks to the use of time-varying quantized control schemes.

In the definition of stability and almost stability shown previously, we have not enforced any requirement on the transient behavior of the system, namely on the behavior of the trajectory before falling into J. Indeed the transient behavior is an important system theoretic issue and will be considered in this paper. An important quantitative figure connected with the transient is the first entrance-time function

$$T_J : I \to \mathbb{N} \cup \{+\infty\}$$

defined by

$$T_J(x) = \inf\{t \in \mathbb{N} \,|\, \Gamma^t x \in J\}. \tag{3}$$

We define $T_J(x) = +\infty$ if $\Gamma^t x \notin J$ for all t. The map T_J is always finite exactly when we have stability, while it is almost surely finite when we have almost stability. Quantitative figures connected with T_J, and which will be considered later on, are its infinity norm

$$\|T_J\|_\infty := \sup\{T_J(x) \,|\, x \in I\}$$

or its average value with respect to the normalized Lebesgue measure $\mathbf{E}(T_J)$.

In the sequel, we will concentrate on the following issues.

1) Find conditions which allow to concretely check (I,J)-stability and almost (I,J)-stability for piecewise affine maps as (2).
2) Given a system as (1) and intervals $J \subseteq I$, find a piecewise constant feedback $k : \mathbb{R} \to \mathbb{R}$ such that the obtained closed-loop system (2) is (I,J)-stable or almost (I,J)-stable.
3) Give estimations of the entrance time T_J for closed-loop stable systems. The entrance time T_J and the number of quantization intervals N of the feedback are the two variables which one would like to minimize in the stabilization problems described in 2). However, the two goals are evidently competitive to each other and it will be interesting to investigate the possible tradeoff between these two objectives. A third important figure in the stabilization process is the ratio C between the lengths of the two intervals I and J, called the *contraction rate*. This parameter will also play a role in describing the relation between T_J and N.

We conclude the section by providing a short outline of the content of this paper. In Section II, we introduce all basic definitions and notations. We then present some results on the structure of the set of invariant intervals for piecewise affine maps. Theorems 1 and 2 show how the existence of a suitable continuous family of invariant intervals suffice to guarantee almost stability or stability: it can be interpreted as a sort of Lyapunov stability. Theorems 3 and 4 provide practical tools for the synthesis of quantized feedbacks yielding this Lyapunov stability and which generalize the schemes proposed in [10] and [11]. These tools are used in Section III, where some general stabilization techniques are analyzed in full detail. We analyze the parameters N and $\|T_J\|_\infty$ for these examples and we present some results, Theorems 6 and 7, and Corollary 1, bounding the performance achievable through this type of stabilization technique. These results are completely new, at our knowledge, and can be seen as extensions of the performance limitation already obtained in [1] and [4] and here recalled in Theorem 5. In Section IV, we propose a new stabilization technique which makes use of the chaotic dynamical properties of piecewise affine maps. In this case we only achieve almost stability, but with a number of quantization intervals N which is independent of the contraction C. Section V is devoted to find an estimate of the mean entrance time for the almost stable system obtained in Section IV: the main result is Corollary 2. Finally, in Section VI, we present some conclusive remark and some indication for future research.

II. PIECEWISE AFFINE MAPS AND INVARIANT INTERVALS

We start the section with the precise definition of a quantized feedback and of a piecewise affine map. A map $k : \mathbb{R} \to \mathbb{R}$ is said to be a *quantized* map if there exists a finite or countable family of disjoint open intervals I_h (called *quantization intervals*) whose union is dense in $\mathbb{R}$ and of reals u_h such that $k(x) = u_h$ for all $x \in I_h$. For now, we prefer not to specify the values of k on the boundaries of the intervals I_h.

Given a quantized map k, the closed-loop map $\Gamma(x) = ax + k(x)$ is affine on each quantization interval of k with slope equal to a. In this paper, a *piecewise affine* map will always be a map constructed in this way. If a is such that $|a| > 1$, then the piecewise affine map will be called *expanding*.

On the basis of our definition the value of the piecewise affine map Γ is not specified at the discontinuity points. This is usually what is done in dynamical systems literature, however it creates some problems in our context, especially with our definition of stability. The following remark explains how to overcome this difficulty.

Remark: The map Γ has two natural extensions in each discontinuity point. If $x \in \partial I_h$, the boundary of I_h, define $\Gamma_h(x)$ as the continuity extension of $\Gamma_{|I_h}$ to x. Fix now a closed interval I and let $x \in I$ be a discontinuity point. If x is on the border of I, then there is no ambiguity in the choice of the valued of $\Gamma(x)$ and this will be defined as $\Gamma_h(x)$, where h is such that $I_h \cap I \neq \emptyset$ and $x \in \partial I_h$. If x is an interior point of I, then we will define $\Gamma(x) := \{\Gamma_{h_1}(x), \Gamma_{h_2}(x)\}$, where h_1, h_2 are such that $x \in \partial I_{h_1}$ and $x \in \partial I_{h_2}$. In this way Γ formally becomes a multivalued map. However, we will not insist on this point and for the rest of the paper we will use the notation as Γ were a usual one-valued map with the implicit understanding that, when an assertion about $\Gamma(x)$ is made, it has to be intended to hold for all its possible values. Given $x_0 \in I$, the orbit denoted by $x_t = \Gamma^t(x_0)$ now may consist in many different sequences. However, except for an at most countable number of x_0, the orbit consists of a unique sequence. With this enlarged definition of Γ and of its orbits, we can now better clarify the concept of invariant interval and of stability. We will say that I is invariant by Γ if for every $x_0 \in I$ the set $\Gamma(x_0)$ is contained in I. Moreover, if $J \subseteq I$ is a subinterval, we will say that Γ is (almost) (I, J)-stable if both I and J are invariant and for (almost) every $x_0 \in I$ every sequence of the orbit $\Gamma^t(x_0)$ is definetely inside J.

A quantized map $k : \mathbb{R} \to \mathbb{R}$ is said to be *uniform* if for all $h \in \mathbb{Z}$ we have that

$$\begin{aligned} I_h &= (\alpha + h\Delta, \alpha + (h+1)\Delta) = \alpha + h\Delta + (0, \Delta) \\ u_h &= \beta + h\Lambda \end{aligned}$$

where $\alpha, \beta, \Delta, \Lambda \in \mathbb{R}$ and $\Delta > 0$. The corresponding piecewise affine map Γ is also called *uniform*. The map Γ is completely determined by the quintuple of parameters $(a, \alpha, \beta, \Delta, \Lambda)$ and it satisfies the following two properties:

1) *Quasiperiodicity*

$$\Gamma(x + h\Delta) = \Gamma(x) + h\Lambda.$$

2) *Linear boundness*

$$mx + q_0 \le \Gamma(x) \le mx + q_1 \tag{4}$$

where we let

$$\begin{aligned} m &:= \frac{\Lambda}{\Delta} + a \\ q_0 &:= \beta - \frac{\Lambda}{\Delta}\alpha - \Lambda \quad q_1 := \beta - \frac{\Lambda}{\Delta}\alpha \quad \text{if } \Lambda > 0 \\ q_0 &:= \beta - \frac{\Lambda}{\Delta}\alpha \quad q_1 := \beta - \frac{\Lambda}{\Delta}\alpha - \Lambda \quad \text{if } \Lambda < 0. \end{aligned} \tag{5}$$

Notice that, when $\Lambda/\Delta = -a$, the closed-loop map Γ is bounded. More precisely, in this case Γ satisfies the inequalities $q_0 \le \Gamma(x) \le q_1$ and so the state of the system in one step gets into the interval

$$J := [q_0, q_1]. \tag{6}$$

Uniform quantized feedbacks k satisfying the above condition $\Lambda/\Delta = -a$ and the corresponding closed-loop maps Γ will be both called *regular*. A regular piecewise affine map is characterized by a quadruple of parameters $(a, \alpha, \beta, \Delta)$.

Regular piecewise affine maps actually possess a large family of invariant intervals: in the notation above, notice indeed that any interval $I \supseteq J$ is clearly invariant. The interval J is the smallest element of the family. This situation does not occur only when the closed-loop map is regular, as the following results will show.

We will say that the two invariant intervals I and J are Γ-*connectable* if we can find a family of invariant intervals $I_\rho = [r(\rho), s(\rho)], \rho \in [0,1]$, such that the extremes $r(\rho), s(\rho)$ are continuous functions in ρ and such that $I_0 = J$ and $I_1 = I$. The family of intervals $I_\rho = [r(\rho), s(\rho)], \rho \in [0,1]$ is called a continuous arc of invariant intervals. If I and J are Γ-connectable and $J \subseteq I$, it is easy to see that we can find a continuous arc I_ρ of invariant intervals such that $I_0 = J, I_1 = I$ and $I_{\rho_1} \subsetneq I_{\rho_2}$ if $\rho_1 < \rho_2$. Such an arc will be called *strictly increasing*.

Connectability is equivalent to the existence of a Lyapunov function for the system. From the family of invariant intervals $I_\rho, \rho \in [0,1]$, satisfying the aforementioned definition, it is possible to define the following function:

$$V(x) := \inf\{\rho \,|\, x \in I_\rho\} \quad \forall x \in I \tag{7}$$

which plays the role of a Lyapunov function for the system, since it can be easily shown that

$$\Delta V(x) := V(\Gamma(x)) - V(x) \le 0 \quad \forall x \in I. \tag{8}$$

It can be shown that $V(x)$ is in general not continuous, but only lower semicontinuous. Conversely, the existence of a lower semicontinuous Lyapunov function satisfying (8) implies that $J \subseteq I$ are Γ connectable.

The following result, which links connectability to almost stability, is therefore not so surprising.

Theorem 1: Let Γ be an expanding piecewise affine map. Let $J \subseteq I$ be two Γ-connectable intervals. Then Γ is almost (I, J)-stable.

Proof: Let $(r(\rho), s(\rho)), \rho \in [0,1]$, be a strictly increasing continuous arc of invariant intervals such that $J = [r(0), s(0)]$ and $I = [r(1), s(1)]$. Let $x_0 \in I$ and consider the sequence $\{x_t\}_{t=0}^{\infty}$, where $x_t := \Gamma^t(x_0)$. Define the subsets

$$\begin{aligned} \mathcal{U} &:= \{t \in \mathbb{N} \,|\, x_t \in [r(1), r(0)[\} \\ \mathcal{V} &:= \{t \in \mathbb{N} \,|\, x_t \in]s(0), s(1)]\}. \end{aligned}$$

From the existence of the family $[r(\rho), s(\rho)]$ of decreasing invariant intervals it is easy to argue that the sequence $\{x_t\}_{t \in \mathcal{U}}$ is increasing, meaning that $t_1, t_2 \in \mathcal{U}$ and $t_1 < t_2$ implies that $x_{t_1} \le x_{t_2}$. In a similar way, we can argue that the sequence $\{x_t\}_{t \in \mathcal{V}}$ is decreasing. This implies, in particular, that there cannot exist periodic orbits of minimal period greater than two belonging to $I \backslash J$.

Define now the set

$$\mathcal{X} := \{x \in I \setminus J \,|\, \Gamma^2(x) = x\}$$

which coincides with the set of equilibrium points and of period two orbits belonging to $I \setminus J$. Moreover, define

$$\mathcal{X}_\infty := \bigcup_{t=1}^{\infty} \Gamma^{-t}(\mathcal{X})$$

which coincides with the set of all initial states which gets in finitely many steps into $\mathcal{X}$. Observe that $\mathcal{X}$ is the set of equilibrium points of Γ^2 and, since Γ^2 is still an expanding piecewise affine map, we can argue that $\mathcal{X}$ is finite and that $\mathcal{X}_\infty$ is countable.

We now show that, if the initial state $x_0 \in I \backslash \mathcal{X}_\infty$, there exists an integer $t_0 \ge 0$ such that $x_t \in J$ for every $t \ge t_0$.

Assume by contradiction that this is not the case. The fact that $x_0 \notin \mathcal{X}_\infty$ implies that the set $\{x_t \,|\, t = 0, 1, 2, \ldots\}$ is included in

$[r(1), r(0)[\cup]s(0), s(1)]$ and it contains infinitely many points. This implies that at least one of the subsets $\mathcal{U}$ and $\mathcal{V}$ contains infinitely many indexes. Assume that both the sets contain infinitely many indexes (the other cases are simpler and can be handled in a similar way). Consider, therefore, the subsequences $\{x_i\}_{i\in\mathcal{U}}$ and $\{x_i\}_{i\in\mathcal{V}}$. As already observed $\{x_i\}_{i\in\mathcal{U}}$ is strictly increasing and $\{x_i\}_{i\in\mathcal{V}}$ is strictly decreasing. Therefore, they both converge. Call $y_{\mathcal{U}}$ and $y_{\mathcal{V}}$ their respective limits. Since Γ is piecewise continuous and $\{x_i\}_{i\in\mathcal{U}}$ is monotone, the subsequence $\{\Gamma(x_i)\}_{i\in\mathcal{U}}$ of $\{x_t\}_{t=1}^{\infty}$ will also converge. Its limit may be either $y_{\mathcal{U}}$ or $y_{\mathcal{V}}$. We distinguish three cases.

1) Assume that $\lim_{t\in\mathcal{U}}\Gamma(x_t) = y_{\mathcal{U}}$. In this case, necessarily, $\Gamma(x_t) \in [r(1), r(0)[$ for $t \in \mathcal{U}$ except at most finitely many indexes. Therefore, there exists $\bar{t}$ such that for all $t \geq \bar{t}$ we have that $x_t \in [r(1), r(0)[$ which contradicts the assumption that both the subsets $\mathcal{U}$ and $\mathcal{V}$ contain infinitely many indexes.
2) The same occurs if we assume that $\lim_{t\in\mathcal{V}}\Gamma(x_t) = y_{\mathcal{V}}$.
3) Finally consider the case when $\lim_{t\in\mathcal{U}}\Gamma(x_t) = y_{\mathcal{V}}$ and $\lim_{t\in\mathcal{V}}\Gamma(x_t) = y_{\mathcal{U}}$. In this case we can argue that $\Gamma(x_t) \in]s(0), s(1)]$ for $t \in \mathcal{U}$ except at most finitely many indexes, while $\Gamma(x_t) \in [r(1), r(0)[$ for $t \in \mathcal{V}$ except at most finitely many indexes. This implies that there exists $\bar{t}$ such that for all $t \geq \bar{t}$ we have that

$$x_t \in [r(1), r(0)[\Longrightarrow x_{t+1} \in]s(0), s(1)]$$
$$x_t \in]s(0), s(1)] \Longrightarrow x_{t+1} \in [r(1), r(0)[.$$

This implies, in particular, that the sequence $x_{2\tau}, 2\tau \geq \bar{t}$ is contained in one of the two intervals $[r(1), r(0)[$ or $]s(0), s(1)]$ and converges monotonically to $y_{\mathcal{U}}$ or $y_{\mathcal{V}}$. Since Γ^2 is piecewise continuous and expanding, it is easy to see that this can happen if and only if $x_{2\tau}$ becomes constant, and this is in contradiction with the fact that $x_0 \notin \mathcal{X}_\infty$.

This proves the theorem. ■

Connectability does not, in general, imply stability. However, the following reinforcement will. Two intervals I and J are said to be *strongly* Γ*-connectable* if there exists a strictly increasing continuous arc I_ρ of invariant intervals such that $I_0 = J$ and $I_1 = I$ with the property (which we will call Γ*-contractivity*) that for every $\rho > 0$, there exists $\rho' < \rho$ such that $\Gamma(I_\rho) \subseteq I_{\rho'}$. We have the following extension of Theorem 1.

Theorem 2: Let Γ be an expanding piecewise affine map. Let $J \subseteq I$ be two strongly Γ-connectable intervals. Then, Γ is (I, J)-stable.

Proof: It is immediate to realize that the set $\mathcal{X}$, defined in the proof of Theorem 1, consisting of orbits of period 2, is empty. The proof of Theorem 1 then shows that Γ is (I, J)-stable. ■

The fact that $J \subseteq I$ are strongly Γ-connectable means, in terms of the Lyapunov function defined in (7), that $\Delta V(x) < 0$ for all $x \in I \subseteq J$. Therefore, the proof of Theorem 2 could done also following the same arguments used in the proof of the classical Lyapunov theorem [18].

Theorems 1 and 2 give sufficient conditions for stability and almost stability as the existence of a suitable continuous family of invariant intervals. If we want to stabilize a system by enforcing the existence of such a family of intervals, we need to have results which allow us to impose this property by simple requests on Γ. The following result will serve the purpose by providing a connection between continuous arcs of invariant intervals and the geometric structure of the graph of Γ. This connection follows by extending the condition according which a map Γ is invariant with respect to an interval I if and only if the graph of $\Gamma_{|I}$ (the restriction of Γ to I) is contained in the square $I \times I$.

Theorem 3: Let Γ be a piecewise affine map and let $I_\rho = (r(\rho), s(\rho)), \rho \in [0, 1]$, be a strictly increasing continuous arc of intervals. Define the following functions:

$$S(x) := \inf\{s(\rho) \mid r(\rho) = x\} \tag{9}$$
$$R(x) := \sup\{r(\rho) \mid s(\rho) = x\}. \tag{10}$$

Then, I_ρ are invariant for all $\rho \in [0, 1]$ if and only if the following inequalities hold true:

$$x \leq \Gamma(x) \leq S(x) \qquad \forall x \in [r(1), r(0)[\tag{11}$$
$$R(x) \leq \Gamma(x) \leq x \qquad \forall x \in]s(0), s(1)] \tag{12}$$
$$r(0) \leq \Gamma(x) \leq s(0) \qquad \forall x \in [r(0), s(0)]. \tag{13}$$

Proof: Assume that the three inequalities hold. We have to show that, fixed $\rho \in [0, 1]$, for every $x \in [r(\rho), s(\rho)]$, we have that $\Gamma(x) \in [r(\rho), s(\rho)]$. We distinguish three cases. Suppose first that $x \in [r(1), r(0)[$. This implies that there exists $\bar{\rho} \in]0, \rho]$ such that $x = r(\bar{\rho})$. It follows from (11) that

$$r(\rho) \leq r(\bar{\rho}) \leq \Gamma(r(\bar{\rho})) = \Gamma(x) \leq S(r(\bar{\rho})) \leq s(\bar{\rho}) \leq s(\rho).$$

If instead $x \in]s(0), s(1)]$, then there exists $\bar{\rho} \in]0, t]$ such that $x = s(\bar{\rho})$. It follows from (12) that

$$r(\rho) \leq r(\bar{\rho}) \leq R(s(\bar{\rho})) \leq \Gamma(s(\bar{\rho})) = \Gamma(x) \leq s(\bar{\rho}) \leq s(\rho).$$

Finally, if $x \in [r(0), s(0)]$, we are in the situation described by (13) which again ensures that $r(\rho) \leq \Gamma(x) \leq s(\rho)$.

Assume conversely that $(r(\rho), s(\rho))$ are invariant for all $\rho \in [0, 1]$. Let $x \in [r(1), r(0)[$ and let $\rho \in [0, 1]$ be such that $x = r(\rho)$. Then, $x \leq \Gamma(x) \leq s(\rho)$ and, hence, $x \leq \Gamma(x) \leq S(x)$. Condition (12) is proven analogously, while (13) is obvious. ■

The inequalities characterizing the continuous arcs of invariant intervals provided by the previous theorem are illustrated in Fig. 1.

It is easy to realize that equalities in (11) or (12) can only be achieved at some discontinuity point. If this never occurs, it can be shown that the arc in Theorem 3 is also Γ contractive. Under the hypothesis that the number of quantization intervals inside I is finite, it can be shown moreover that contractivity implies that the entrance time T_J is a bounded function of the initial state. The following theorem provides an effective and easily computable bound. With no loss of generality we will assume that 0 is the mid point of the interval J.

Theorem 4: Let Γ be a piecewise affine map and let $(r(\rho), s(\rho)), \rho \in [0, 1]$, be a strictly increasing continuous arc of invariant intervals. Let $I := [r(1), s(1)]$ and $J := [r(0), s(0)]$ and assume that $r(0) = -s(0)$. Let $S(x)$ and

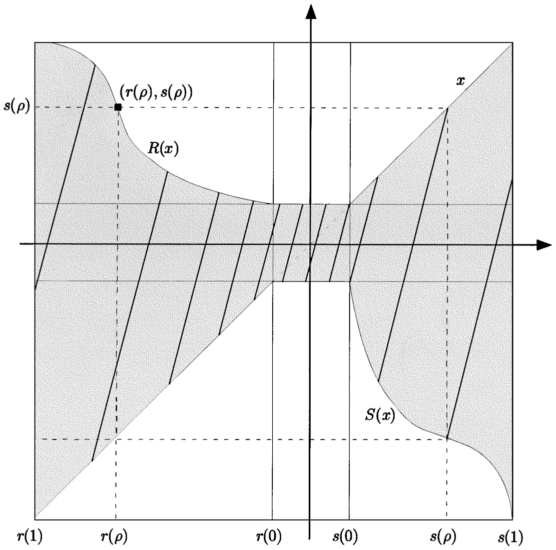


Fig. 1. Graph of a piecewise affine map Γ satisfying the inequalities in Theorem 3.

$R(x)$ be the functions defined in Theorem 3 and let $0 < \delta < 1$. If the following inequalities hold true:

$$\delta x \le \Gamma(x) \le \delta S(x) \qquad \forall x \in [r(1), r(0)[\qquad (14)$$

$$\delta R(x) \le \Gamma(x) \le \delta x \qquad \forall x \in]s(0), s(1)] \qquad (15)$$

$$r(0) \le \Gamma(x) \le s(0) \qquad \forall x \in [r(0), s(0)] \qquad (16)$$

then, the arc is Γ-contractive and the first entrance time T_J satisfies the following inequality

$$T_J \le \left\lceil \frac{2 \log C}{\log(\delta^{-1})} \right\rceil$$

where C denotes the contraction rate.

Proof: Contractivity is clear from the proof of Theorem 3. We now prove the bound on the first entrance time. It follows from the assumptions that $s(0) = |J|/2$ and $r(0) = -|J|/2$. Let $x_0 \in I$ and consider $x_t := \Gamma^t(x_0)$. Assume that $x_t \notin J$. Then, there exists $\rho_t \in [0,1]$ such that either $x_t = r(\rho_t)$ or $x_t = s(\rho_t)$. This means, in particular, that $x_t \in [r(\rho_t), s(\rho_t)]$. The same arguments used in the proof of Theorem 3 now show that $\delta r(\rho_t) \le x_{t+1} \le \delta s(\rho_t)$. Assume that $x_{t+1} \notin J$. In this case, either $|s(\rho_{t+1})| \le \delta |s(\rho_t)|$ or $|r(\rho_{t+1})| \le \delta |r(\rho_t)|$. In any case we have that $|r(\rho_{t+1})||s(\rho_{t+1})| \le \delta |r(\rho_t)||s(\rho_t)|$. This shows that

$$x_s \notin J \quad \forall s \le t \Rightarrow |r(\rho_t)||s(\rho_t)| \le \delta^t |r(\rho_0)||s(\rho_0)| \le \delta^t \frac{|I|^2}{4}$$

where the last inequality follows from the fact that $|r(\rho_0)| + |s(\rho_0)| \le |I|$ implies $|r(\rho_0)||s(\rho_0)| \le |I|^2/4$. These arguments imply that for all $t \in \mathbb{N}$ such that

$$\delta^t \frac{|I|^2}{4} \le \frac{|J|^2}{4}$$

we have that $x_t \in J$. This fact implies that

$$T_J \le \left\lceil \frac{\log(|I|^2/|J|^2)}{\log(\delta^{-1})} \right\rceil = \left\lceil \frac{2 \log C}{\log(\delta^{-1})} \right\rceil$$

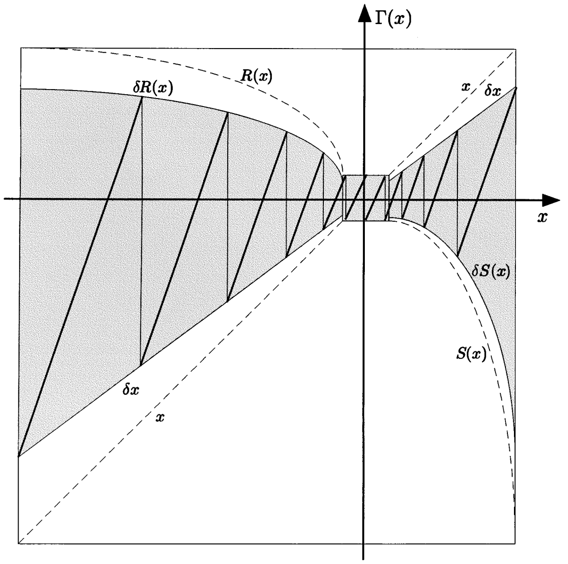


Fig. 2. Graph of a piecewise affine map Γ satisfying the inequalities in Theorem 4.

where $\lceil z \rceil$ means the smallest integer greater than or equal to z. ■

The inequalities in Theorem 4 are illustrated in Fig. 2.

III. QUANTIZED FEEDBACKS YIELDING STABILITY

In this section, we show how stabilization techniques, based on the existence of a continuous arc of invariant intervals, can be obtained by using the results presented in previous section. The problem we will consider is the following.

Given the scalar system (1) and intervals $J \subseteq I$, find a quantized feedback k such that for the corresponding closed-loop map Γ the pair J, I becomes connectable or strongly connectable. Or, in other terms, find a piecewise affine map Γ with slope a such that J and I are connectable or strongly connectable. Moreover, we want to understand which is the minimum number of quantization intervals N among all the possible quantized feedback solving the above problem. Finally, we want to find the relations existing between the number of quantization intervals N and the entrance time T_J. Clearly, we look for constructive answers to these questions.

A. Quantized Feedbacks Yielding Invariant Intervals

We start from the following preliminary result already appeared in a slight different form in [1] and [4]. It provides a complete answer to the previous questions in the case when $I = J$, which simply corresponds to the problem of making invariant a given interval I.

Theorem 5: The minimum number of quantization intervals inside an interval I needed by a piecewise affine map Γ of slope a to maintain I invariant is $N = \lceil |a| \rceil$, where $\lceil |a| \rceil$ means the smallest integer greater than or equal to $|a|$.

Proof: A straightforward computation shows that, if Γ is such that I is invariant, then the number of quantization intervals inside I can not be smaller than $|a|$. On the other hand, it is clear from (6) that, $N = \lceil |a| \rceil$ quantization intervals suffice. ■

Remark: In [2], [6], and [7], the authors present a condition for stability which, using our notation, corresponds to the inequality $N > |a|$. In the above papers, this condition insures asymptotic stability, while in Theorem 5 we are only considering the existence of an invariant interval. The reason for this discrepancy is in the fact that in [2], [6], and [7], the authors use time-varying schemes.

Fix now $N = \lceil |a| \rceil$. In general, given the interval $I = [r, s]$, there are many different possible piecewise affine maps Γ with slope a, keeping I invariant, and having N quantization intervals inside I. Particularly relevant for their simplicity are regular piecewise affine maps which can be characterized using conditions (4) and (5). These yield the following inequalities (the second one should actually be considered modulo Δ):

$$\begin{cases} \frac{s-r}{\lceil |a| \rceil} \le \Delta \le \frac{s-r}{|a|} \\ 0 \le r - \alpha \le \lceil |a| \rceil \Delta - (s - r) \\ r \le q_0 \le s - |a|\Delta. \end{cases} \tag{17}$$

The first inequality provides the parameter Δ and makes the second inequality solvable. The second inequality yields the parameter α, while the third gives β. Finally, notice that regularity implies that $\Lambda = -a\Delta$.

We now illustrate two examples.

Example 1: Choose $\Delta = |a|^{-1}(s - r)$. It follows from (17) that there are infinitely many regular piecewise affine maps having Δ as above and keeping I invariant depending on α and β satisfying the second and the third condition in (17). Notice that the third inequality in (17) is, in this case, equivalent to $q_0 = r$ showing that β is completely determined by the choice of α.

There are two interesting cases corresponding to $\alpha = ((r + s)/2) - (N\Delta/2)$ and $\alpha = r$. The first choice yields a closed-loop Γ which is symmetric with respect to the vertical axis passing for the midpoint of I, as shown in Fig. 3. The dynamical systems corresponding to the second choice with $a > 1$ are generally known as *β-expansions* and have extensively been studied in the 1950s by Parry [19] and Renyi [20].

Example 2: Another possible choice is to take $\Delta = \lceil |a| \rceil^{-1}(s - r)$. In this way, $\alpha = r$ and the closed-loop maps vary only with the possible choices of β satisfying the third condition in (17).

All the constructions shown previously coincide in the special case when a is an integer so that $N = |a|$. Observe that the construction in Example 1 is universal in the class of regular piecewise affine maps in the following sense: if Γ is any regular piecewise affine map keeping I invariant, then $\Gamma(I)$ is contained inside the unique minimum invariant interval I' of Γ which is specified in (6). Moreover, the restriction of Γ to I' corresponds to the case presented in Example 1.

B. Some Stabilization Techniques

We now come to the stabilization problem. Given two intervals $J \subsetneq I$, Theorems 1–4 give the tools for designing quantized feedbacks ensuring almost (I, J)-stability and (I, J) stability. We illustrate how these results can be used by some general examples.

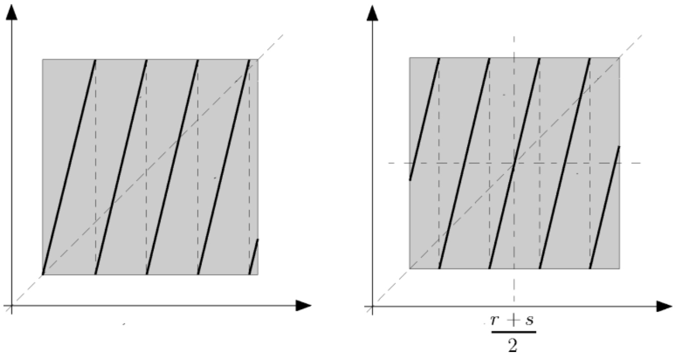


Fig. 3. Graph of the piecewise affine maps Γ presented in Example 1 when $\alpha = r$ and $\alpha = ((r + s)/2) - (N\Delta/2)$.

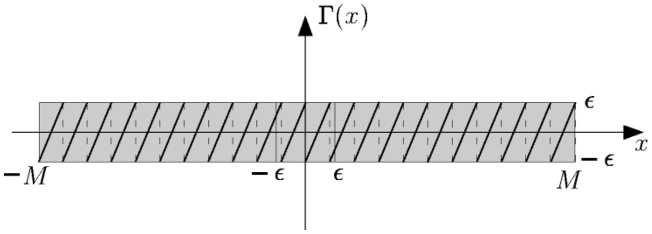


Fig. 4. Graph of $\Gamma(x)$ providing a one step dead-beat quantized feedback.

Consider again the system

$$x_{t+1} = ax_t + u_t$$

and suppose that we want to obtain (I, J)-stability, where $I = [-M, M]$, $J = [-\epsilon, \epsilon]$ and $\epsilon < M$.

1) One Step Dead-Beat Quantized Feedback: The first possibility is to choose a uniform quantized feedback yielding a regular closed-loop map which keeps the interval J invariant, by using conditions (5) and (6). The quantized feedback obtained in this way can be interpreted as the uniform quantized feedback which approximates the linear feedback yielding the dead-beat control (see [11]). The regular closed-loop map, which is shown in Fig. 4, is characterized by the first entrance time $T_J = 1$: just one step is sufficient to get into the target set J from any initial state in I, independently of how large is the contraction rate $C = |I|/|J| = M/\epsilon$. This good performance with respect to T_J, is paid by the fact that this strategy is very demanding in terms of the number N of quantization intervals that are needed. In fact, it is easy to show that

$$N = 2 \left\lceil \frac{|a|C}{2} \right\rceil$$

which is a function depending linearly on the contraction rate C.

2) Two Steps Dead-Beat Quantized Feedback: For obtaining a quantized feedback yielding a closed-loop system which converges from I to J within two steps it is sufficient to impose that in the first step the closed-loop map Γ reduces the interval I to another intermediate interval $I_1 = [-r, r]$, and in the second step it contracts I_1 into J. This can be done if Γ satisfies the inequalities

$$\begin{cases} |\Gamma(x)| \le \epsilon & \text{if } |x| \le r \\ |\Gamma(x)| \le r & \text{if } r < |x| \le M \end{cases}$$

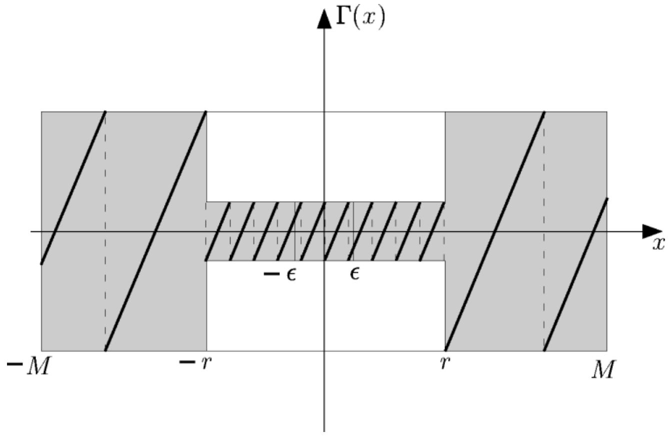


Fig. 5. Graph of $\Gamma(x)$ providing a two steps dead-beat quantized feedback.

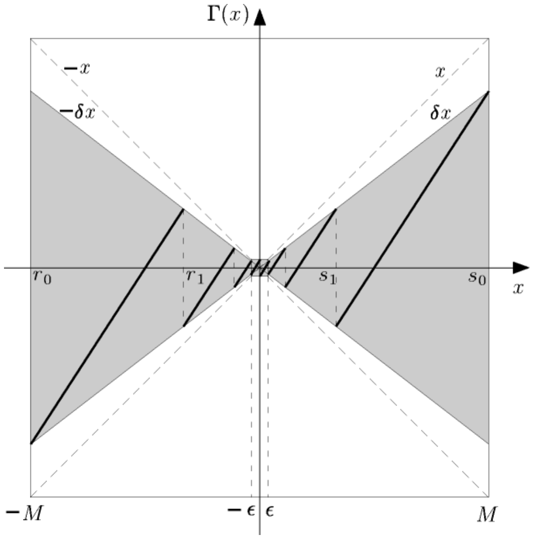


Fig. 6. Graph of $\Gamma(x)$ satisfying (19).

As illustrated in Fig. 5, it can be seen that the previous inequalities can be satisfied using a quantized feedback with a number of quantization intervals

$$N = 2\left\lceil |a|\frac{r}{2\epsilon}\right\rceil + 2\left\lceil |a|\frac{M-r}{2r}\right\rceil.$$

Choosing r which minimizes N, it can be shown that N depends on C according to the following approximate formula:

$$N \simeq |a|(2\sqrt{C}-1)$$

which shows that in this case we have a square root dependence of N on C.

3) Logarithmic Quantized Feedback: The two strategies proposed above both allowed us to obtain strong Γ-connectability from I to J. We now present an alternative stabilization strategy which yields the same property, but with a remarkably smaller number of quantization intervals. We will show in the sequel two different ways to obtain this goal, depending on the choice of the increasing continuous arc of intervals between J and I imposed on the closed-loop map.

In the first case, we consider the following family of intervals:

$$[-\rho, \rho], \qquad \rho \in [\epsilon, M].$$

Theorem 3 ensures that these intervals are invariant for the closed-loop map Γ if and only if Γ satisfies the inequalities

$$\begin{cases} |\Gamma(x)| \le |x| & \text{if } \epsilon \le |x| \le M \\ |\Gamma(x)| \le \epsilon & \text{if } |x| \le \epsilon. \end{cases} \tag{18}$$

By Theorem 1, this choice produces a closed-loop system which is almost (I,J)-stable. If we require that

$$\begin{cases} |\Gamma(x)| \le \delta|x| & \text{if } \epsilon \le |x| \le M \\ |\Gamma(x)| \le \epsilon & \text{if } |x| \le \epsilon \end{cases} \tag{19}$$

where $0 < \delta < 1$, then, by Theorems 2 and 4, we obtain (I,J)-stability and, clearly, we have

$$|x_{t-1}| > \epsilon \Rightarrow |x_t| \le \delta^t M$$

from which it is easy to verify the following bound on the first entrance time:

$$T_J \le \frac{\log C}{\log(\delta^{-1})}.$$

This logarithmic bound on the entrance time as a function of the contraction rate $C = |I|/|J| = M/\epsilon$ is in agreement with the general bound provided by Theorem 4.

The most efficient way to obtain a closed-loop map Γ satisfying (19) is illustrated in Fig. 6. An easy computation shows that the points s_k and r_k providing the quantization intervals in Fig. 6 are

$$\begin{aligned} r_k &= -\left(\frac{a-\delta}{a+\delta}\right)^k M \\ s_k &= \left(\frac{a-\delta}{a+\delta}\right)^k M, \qquad k = 0, 1, 2, \ldots. \end{aligned}$$

This shows that, using this symmetric feedback function $k(x)$, it is possible to satisfy condition (19) with a number of quantization intervals

$$N = 2\left\lceil \frac{\log C}{\log(a+\delta) - \log(a-\delta)} \right\rceil + \lceil |a| \rceil \tag{20}$$

which is again a logarithmic function of the contraction rate C. The technique follows essentially the method proposed in [10], restricted to the scalar case. Observe that while the case $\delta < 1$, which yields stability, ensures also a certain degree of robustness of the result with respect to small variations of the parameter a determining the system and the parameters of the quantized feedback, this is not the case when $\delta = 1$ corresponding to the closed-loop system which is only almost stable.

Taking a different family of invariant intervals we can obtain different results. Choose the family of intervals

$$\begin{aligned} &[-\rho, M], \qquad \rho \in [\epsilon, M] \\ &[-\epsilon, \rho], \qquad \rho \in [\epsilon, M]. \end{aligned}$$

By Theorem 3, these form a family of invariant intervals for the closed-loop map Γ if and only if Γ satisfies the inequalities

$$\begin{cases} x \leq \Gamma(x) \leq M & \text{if } -M \leq x \leq -\epsilon \\ 0 \leq \Gamma(x) \leq x & \text{if } \epsilon \leq x \leq M \\ |\Gamma(x)| \leq \epsilon & \text{if } |x| \leq \epsilon. \end{cases} \tag{21}$$

For simplicity, we consider only the case $a > 1$ and the problem of obtaining almost (I, J) stability. The most efficient way of achieving this goal is illustrated in Fig. 7. An easy computation shows that the points s_k providing the quantization intervals in Fig. 7 are

$$s_k = \left(\frac{a-1}{a}\right)^k M, \qquad k = 0, 1, 2, \ldots.$$

This shows that the number of quantization intervals which is needed in this case is

$$\left\lceil \frac{\log C}{\log a - \log(a-1)} \right\rceil + \frac{3}{2}\lceil a \rceil \leq N \leq \left\lceil \frac{\log C}{\log a - \log(a-1)} \right\rceil + 2\lceil a \rceil$$

and so N is again a logarithmic function of the contraction rate C which grows more slowly than the function (20), when letting $\delta = 1$.

C. Some General Performance Bounds

Previous examples suggest some general considerations. In the dead-beat quantized feedback examples we had that $T_J = 1$ or $T_J = 2$, both independent of the contraction C, and that the number of quantization intervals was $N \sim C$ in the first case and $N \sim \sqrt{C}$ in the second. In the logarithmic quantized feedback example, instead, we had $T_J \sim \log C$ and $N \sim \log C$. This shows that in all cases the number N of quantization intervals grows at least logarithmically with C and that it grows even faster if we impose some restrictions on T_J as a function of C. Indeed, it is reasonable to expect the existence of a tradeoff between the N and T_J as functions of C. We will show in the sequel that the fact that N grows at least logarithmically with C is an intrinsic consequence of the requirement that $J \subseteq I$ are connectable. We will propose moreover a bound involving the parameters N, T_J and C from which it will be possible to deduce a general link existing between the asymptotic behavior of N and T_J as functions of C.

In the sequel, we will make the following assumptions. Consider an expanding piecewise affine map Γ and invariant interval I. We will assume that the number of quantization intervals N of Γ intersecting I is finite. This implies that for any subset $\bar{I}$ of I, the number of quantization intervals intersecting $\bar{I}$, denoted by $N(\bar{I})$, is finite. With this notation we have that $N = N(I)$. This first result shows that connectability implies that N grows at least logarithmically with C.

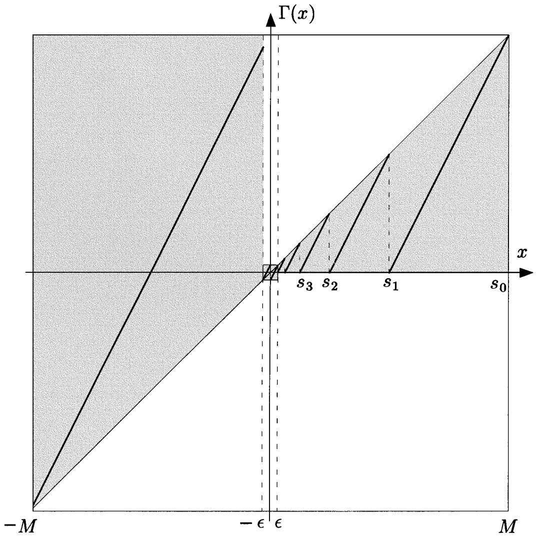


Fig. 7. Graph of $\Gamma(x)$ satisfying (21).

Theorem 6: Let Γ be an expanding piecewise affine map with slope a, and let $J \subseteq I$ be two invariant intervals which are Γ connectable. Moreover, let N be the number of quantization intervals of Γ inside I and let $C = |I|/|J|$ where $|I|$ and $|J|$ denote the length of I and J, respectively. Then

$$N \geq (|a| - 1) + \frac{\log C}{\log(|a|+1) - \log(|a|-1)}. \tag{22}$$

Proof: We start by proving the following preparatory fact: if $I_1 = [r_1, s_1]$ and $I_2 = [r_2, s_2]$ are two invariant intervals for Γ such that $I_1 \subsetneq I_2$, then

$$|I_2| > \frac{|a|+1}{|a|-1}|I_1| \Rightarrow N(I_2) \geq N(I_1) + 1. \tag{23}$$

Assume by contradiction that $N(I_2) = N(I_1)$. Assume also that $r_1 - r_2 \geq s_2 - s_1$, the other case being completely analogous. Necessarily, $r_2 < r_1$ and, therefore, r_1 is not a discontinuity of Γ, which is affine on $[r_2, r_1 + \epsilon]$, for some $\epsilon > 0$. If $a > 1$, this implies that $\Gamma([r_2, r_1]) \subseteq \Gamma([r_2, r_1 + \epsilon]) \subseteq [r_2, s_1]$. Thus, we obtain

$$(r_1 - r_2)|a| \leq s_1 - r_2 = |I_1| + (r_1 - r_2)$$

which implies that $(r_1 - r_2) \leq |I_1|/(|a| - 1)$ and so

$$|I_2| \leq 2(r_1 - r_2) + |I_1| \leq \frac{|a|+1}{|a|-1}|I_1|. \tag{24}$$

This contradicts the left hand side of (23). If instead $a < -1$, we have that $\Gamma([r_1, r_2]) \subseteq [r_1, s_2]$. Thus, we obtain

$$(r_1 - r_2)|a| \leq s_2 - r_1 \leq |I_1| + (r_1 - r_2)$$

which, as before, leads to (24). Therefore (23) is proved.

Fix now any number $K > (|a| + 1)/(|a| - 1)$ and choose a sequence of invariant intervals

$$J = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\nu \subseteq I$$

where

$$|I_j| = K|I_{j-1}|.$$

Notice that such a sequence surely exists, as long as $K^\nu \leq C$, because of the assumption of concatenability. An iterative application of (23) shows that

$$N(I) \geq N(I_\nu) \geq N(J) + \nu.$$

If we choose

$$\nu \geq \frac{\log C}{\log K} - 1$$

and we recall that $N(J) \geq |a|$, by Theorem 5, we obtain that

$$N(I) \geq (|a| - 1) + \frac{\log C}{\log K}$$

and this, by the way K was chosen, clearly implies the result. ■

We now consider the relation between the first entrance time and the number of quantization intervals. Let $J \subseteq I$ be two invariant intervals for Γ and assume that $\Gamma^T(I) \subseteq J$. Then, it is easy to see that

$$N_\Gamma^T \geq N_{\Gamma^T} \geq |a|^T C$$

where N_Γ and N_{Γ^T} denote the number of quantization intervals inside I of Γ and Γ^T, respectively. This implies that

$$N \geq |a| C^{1/T} \tag{25}$$

where $C = |I|/|J|$. Inequality (25) already gives some information. Indeed, if we impose that the first entrance time T_J is constantly equal to 1, then (25) shows that number of quantization intervals N has to increase at least linearly in C. More generally, if T_J is constant in C, N will increase at least as C^{1/T_J}. This shows that the performances obtained in the dead-beat quantized feedbacks shown previously cannot be improved. However, if we allow $\|T_J\|_\infty$ to grow with C, inequality (25) becomes less useful. In order to clarify this point and to better introduce the next results, we need to be more precise.

Assume that we have a sequence of closed-loop piecewise affine maps Γ_i (with fixed slope a), a sequence of pair of intervals $J_i \subseteq I_i$ which are, respectively, strongly Γ_i-connectable. Denote by N_i the number of quantization intervals of Γ_i inside I_i and by T_i the infinity norm of the first entrance time inside J_i relative to Γ_i. Denote, moreover, $C_i = |I_i|/|J_i|$ and assume that $C_i \to \infty$. According to the growth we impose on T_i, we expect a different growth on N_i. It follows from Theorem 6 that N_i has to grow at least as $\log C_i$. Moreover, since we presented examples in Section III in which both N_i and T_i had the same order of $\log C_i$, then we can argue that the only interesting regimes which deserve to be investigated are those in which T_i grows with C_i less than logarithmically, i.e.,

$$\lim_{i\to\infty} \frac{T_i}{\log C_i} = 0.$$

In this case from (25) we can only argue that $N_i \to \infty$, fact which already follows from Theorem 6. A more refined use of concatenability will allow us to obtain better results.

Basically, we need a sharper inequality than (25) and to obtain this we need to find a better estimate for the number of intervals for the iterated maps Γ^k. This will be done in a number of steps. Let $J \subseteq I$ be two Γ-connectable invariant intervals and let $I_\rho, \rho \in [0, 1]$, be an strictly increasing continuous arc of invariant intervals such that $I_0 = J$ and $I_1 = I$. We now consider a sequence of invariant intervals constructed in the following way. Define

$$\begin{cases} \rho_0 = 0 \\ \rho_j = \sup\{\rho \geq \rho_{j-1} \,|\, N(I_\rho \setminus I_{\rho_{j-1}}) \leq 2\}. \end{cases} \tag{26}$$

Denote $H_j = I_{\rho_j}$. It is clear that there exists $\nu \in \mathbb{N}$ such that $H_{\nu-1} \neq I$ and $H_\nu = I$. Moreover, from the definition of H_j, it follows that for all $j < \nu$ one or both the extremes of H_j coincide with discontinuity points for Γ. This implies that

$$N(J) + j - 1 \leq N(H_j) \leq N(J) + 2j \quad 0 \leq j \leq \nu. \tag{27}$$

In particular, we have that

$$N(J) + \nu - 1 \leq N(I) \leq N(J) + 2\nu. \tag{28}$$

We now use the incapsulated intervals H_j's to give an estimate of the number of quantization intervals in I for the iterated map Γ^k.

Lemma 1: Denote by $N^k(H_j)$ the number of quantization intervals of Γ^k inside H_j. Then

$$N^k(H_j) \leq \sum_{h=0}^{k} \binom{j+h-1}{h} 2^h N(J)^{k-h}. \tag{29}$$

Proof: Observe preliminary that, if $J \subseteq I$ are two invariant intervals, $N^2(I) \leq N(I \setminus J)N(I) + N^2(J)$ and, more in general, $N^k(I) \leq N(I \setminus J)N^{k-1}(I) + N^k(J)$. We will use this fact for proving (29) by induction on k. For $k = 1$, (29) is equivalent to the right-hand side of (27). Assume now that (29) holds true for $k - 1$ and any $j \leq \nu$ and let us prove it for k. Iterating the previous observation we get

$$N^k(H_j) \leq \sum_{i=1}^{j} N(H_i \setminus H_{i-1}) N^{k-1}(H_i) + N(J)^k.$$

By definition of H_i we have $N(H_i \setminus H_{i-1}) \leq 2$ and so, using induction, it follows that

$$\begin{aligned} N^k(H_j) &\leq 2\sum_{i=1}^{j} N^{k-1}(H_i) + N(J)^k \\ &\leq 2\sum_{i=1}^{j}\sum_{h=0}^{k-1} \binom{i+h-1}{h} 2^h N(J)^{k-1-h} + N(J)^k \\ &= N(J)^k + \sum_{h=0}^{k-1} 2^{h+1} N(J)^{k-1-h} \sum_{i=1}^{j} \binom{i+h-1}{h} \\ &= \sum_{h=0}^{k} \binom{j+h-1}{h} 2^h N(J)^{k-h} \end{aligned} \tag{30}$$

where in the last equation we used the identity [21, p. 822]

$$\sum_{i=1}^{j} \binom{i+h-1}{h} = \binom{h+j}{h+1}. \tag{31}$$

■

By taking $k = T$ and $j = \nu$ in formula (29), we obtain the bound

$$N^T(I) \le N(J)^T \sum_{h=0}^{T} \binom{\nu + h - 1}{h} \left[\frac{2}{N(J)}\right]^h. \qquad (32)$$

We recall from previous considerations that we are interested in analyzing the situation when $T/\log C$ is small. We have the following result.

Theorem 7: Let $a \in \mathbb{R}$ be such that $|a| > 1$. Then, there exist positive constants K and H, depending only on a, such that, for any piecewise affine map Γ of slope a making the intervals $J \subseteq I$ strongly Γ connectable, we have that

$$T \le H \log C \Longrightarrow N \ge KTC^{\frac{1}{T}} \qquad (33)$$

where $C = |I|/|J|$ is the contraction rate, N is the number of quantization intervals in I, and $T = \|T_J\|_\infty$ is the infinity norm of the first entrance time map in J.

Proof: Let $b := N(J)$. Observe that, from the proof of Theorem 6, we can argue that

$$N(I) \ge b - 1 + \frac{\log C}{[\log(|a|+1) - \log(|a|-1)]}$$

and so, using (28), we have that

$$\nu \ge \frac{\log C}{2[\log(|a|+1) - \log(|a|-1)]} - 1.$$

If we fix

$$H = \frac{1}{[b+2][\log(|a|+1) - \log(|a|-1)]}$$

then $1 \le T \le H \log C$ implies

$$\nu \ge \frac{b+2}{2} H \log C - 1 \ge \frac{b+2}{2} T - 1 \ge \frac{b}{2} T. \qquad (34)$$

Since $b \ge 2$

$$\begin{aligned} N^T(I) &\le b^T \sum_{h=0}^{T} \binom{\nu + h - 1}{h} \left[\frac{2}{b}\right]^h \\ &\le b^T \sum_{h=0}^{T} \binom{\nu + h - 1}{h} = \binom{T + \nu}{T} b^T \end{aligned}$$

where we used the identity (31). Using (25) and the fact that $\binom{T+\nu}{T} \le [(T+\nu)eT^{-1}]^T$, we can argue that

$$|a|^T C \le \binom{T + \nu}{T} b^T \le \left[eb\frac{T+\nu}{T}\right]^T.$$

Notice now that (34) implies that $T + \nu \le (1 + 2/b)\nu \le 2\nu$ which yields

$$\frac{N}{2} \ge \frac{\nu}{2} \ge \frac{T+\nu}{2} \ge \frac{|a|}{2eb} TC^{\frac{1}{T}}.$$

■

A nice consequence of Theorem 7 is the following result.

Corollary 1: Let Γ_i be a sequence of expanding piecewise affine maps with fixed slope a, making the intervals $J_i \subseteq I_i$ strongly Γ_i-connectable. For each Γ_i denote by $C_i = |I_i|/|J_i|$ its contraction rate, by N_i its number of quantization intervals in I_i and by T_i the infinity norm of its first entrance time map in J_i. Then

$$\frac{T_i}{\log C_i} \to 0 \Longrightarrow \frac{N_i}{\log C_i} \to +\infty. \qquad (35)$$

Proof: Write $T_i = \log C_i \alpha_i$ where α_i is infinitesimal for $i \to +\infty$. Then, it follows from Theorem 7 that for i sufficiently large we have that

$$N_i \ge KT_iC_i^{\frac{1}{T_i}} = K \log C_i \alpha_i e^{\frac{1}{\alpha_i}} \qquad (36)$$

and the result immediately follows from the fact that $\alpha_i e^{1/\alpha_i} \to +\infty$. ■

IV. CHAOTIC QUANTIZED FEEDBACKS YIELDING ALMOST STABILITY

In this section, we will propose a completely different technique to implement a quantized feedback controller. This method yields only almost stability, but it requires less quantization intervals than what is needed to obtain stability. The key idea will be to use, in a fundamental way, the ergodic properties of expanding piecewise affine maps. We will obtain in this way quantized feedbacks yielding almost stability with a number of quantization intervals not depending on the contraction rate $C = |I|/|J|$.

We start with a notation. If f and g are two real functions and J is an interval, define

$$(f \wedge_J g)(x) := \begin{cases} f(x) & \text{if } x \in J^c \\ g(x) & \text{if } x \in J. \end{cases}$$

Let $J \subseteq I$ be two intervals. Let k_I, k_J be quantized feedbacks making the intervals I and J invariant and let Γ_I, Γ_J be the corresponding closed-loop maps. Consider the new quantized feedback $k_{(I,J)} := k_I \wedge_J k_J$ and let $\Gamma_{(I,J)}$ be the corresponding closed-loop map. Clearly, $\Gamma_{(I,J)}$ leaves both I and J invariant. Since $\Gamma_{(I,J)} = \Gamma_I \wedge_J \Gamma_J$, it is clear that all $\Gamma_{(I,J)}$ are almost (I,J)-stable for any subinterval J of I if and only if the map $\Gamma = \Gamma_I$ possesses the following property.

P) Given any interval $J \subseteq I$, for almost every $x \in I$ there exists an integer $t \ge 0$ such that $\Gamma^t(x) \in J$.

The following result states that this property can always be fulfilled.

Theorem 8: Given a system (1) with $|a| > 1$ and any interval I, there always exists a uniform quantized feedback with $N = \lceil |a| \rceil$ quantization intervals such that

1) the interval I is invariant with respect to the closed-loop map;
2) the closed-loop map possesses property P).

Consequently, there exists a quantized feedback with $N = 2\lceil |a| \rceil$ quantization intervals which makes the closed-loop system almost (I,J) stable.

Remark: As already recalled in the Remark after Theorem 5, the quantized schemes proposed in [6], [7], [2] reach asymptotic stability with $N > |a|$. Our scheme instead uses $N = 2\lceil |a| \rceil$ quantization intervals and only reaches almost stability.

However, while our method uses time-invariant feedbacks, the schemes in [2], [6], and [7] are time-varying.

The rest of this section is devoted to prove Theorem 8. Notice first that the second part of this theorem is direct consequence of the first part and of previous discussion. In order to prove the first part of Theorem 8 we will need to use some basic ergodic properties of piecewise affine maps which will be now briefly recalled, general references for this subject being [16], [22], and [23].

Let $\Gamma : I \to I$ be an expansive piecewise affine map. A probability measure μ on I (equipped with the Borel σ-algebra) is said to be invariant by Γ, if $\mu(\Gamma^{-1}(A)) = \mu(A)$ for every Borel subset $A \subseteq I$. Moreover, μ is said to be ergodic, if the following happens: for every pair of Borel subsets $A, B \subseteq I$ such that $\mu(A) > 0$ and $\mu(B) > 0$, there exists $t_0 \in \mathbb{N}$ such that $\mu(A \cap \Gamma^{-t_0} B) > 0$. We denote by λ the Lebesgue measure on I normalized to 1. Notice that if $J \subseteq I$ is a subinterval, $\lambda(J) = |I|/|J|$. We say that μ is absolutely continuous with respect to λ, if $\lambda(A) = 0$ implies that $\mu(A) = 0$ for every Borel subset $A \subseteq I$. Finally, if both μ is absolutely continuous with respect to λ and λ is absolutely continuous with respect to μ, then we say that the measures μ and λ are equivalent.

In general, the Lebesgue measure λ is not invariant by Γ. However, as shown in [23], there always exists an invariant measure which is ergodic and absolutely continuous with respect to λ. By the Radon–Nikodym theorem it follows that μ admits a density with respect to λ, namely, there exists a nonnegative integrable function $\phi \in L^1(\lambda)$ such that $\mu(A) = \int_A \phi(x) d\lambda(x)$ for every Borel subset A. The invariant measure μ found in [23] has also the property that ϕ is a bounded variation function, namely

$$\bigvee_I \phi := \sup \left\{ \sum_{i=1}^{n-1} |\phi(x_{i+1}) - \phi(x_i)| \right\} < \infty$$

where the previous sup is over all possible finite families of points $x_1 < x_2 < \cdots < x_n$ in I. The following standard result illustrates the behavior of Γ inside the support of μ.

Proposition 1: Let $J \subseteq I$ be a subinterval such that $\mu(J) > 0$. Then, for μ-almost every $x \in I$, there exists an integer $t \geq 0$ such that $\Gamma^t(x) \in J$.

Proof: Consider the subset

$$W := \bigcap_{t=0}^{+\infty} \Gamma^{-t}(J^c) = \{x \in I \,|\, T^t x \notin J \ \forall t \geq 0\}.$$

We need to prove that $\mu(W) = 0$. If by contradiction $\mu(W) > 0$, then ergodicity would imply that $\mu(W \cap \Gamma^{-t_0} J) > 0$ for some $t_0 \in \mathbb{N}$, which is absurd by the way in which W has been defined. ■

If the measure μ were equivalent to λ, then Proposition 1 would clearly imply property P). There exists a purely topological condition on Γ which guarantees the equivalence of μ and λ. A map $\Gamma : I \to I$ is said to be *covering*, if for every open interval $U \subseteq I$, there exists $t \in \mathbb{N}$ such that $\Gamma^t(U) = I$. It is shown in [16] that, if Γ is covering, then there exists a $\delta > 0$ such that the density ϕ of the invariant measure μ is such that $\phi(x) \geq \delta$ for every $x \in I$. This fact clearly yields that μ is equivalent to λ and, actually, also that μ is the unique absolutely continuous invariant measure for Γ (see [22] for details). Property P) is, therefore, implied by covering.

In the remaning part of the section we will investigate under which conditions a piecewise affine map is covering. We start with a simple preparatory result which is commonly used in the literature on piecewise affine maps (it can be found, for instance, in [16]). Anyhow, we prefer to present also a short proof since this result will be used many times.

Lemma 2: Let $\Gamma : I \to I$ be an expanding piecewise affine map with slope a and consider any open interval $U \subseteq I$. The following properties hold true.

a) If $|a| > 1$, then there exists $t \in \mathbb{N}$ such that $\Gamma^t(U)$ includes an open subinterval which contains a point of discontinuity of Γ.

b) If $|a| > 2$, then there exists $t \in \mathbb{N}$ such that $\Gamma^t(U)$ includes an open subinterval which contains two points of discontinuity of Γ.

Proof: Define a sequence of subintervals $U_t \subseteq \Gamma^t(U) \subseteq I$ in the following iterative way: let $U_0 = U$ and, given U_{t-1}, define U_t to be the largest of the intervals $\Gamma(U_{t-1}) \cap I_h$ as I_h varies among the all quantization intervals of Γ. Observe that when $\Gamma(U_{t-1})$ contains no points of discontinuity, then $\lambda(U_t) = |a|\lambda(U_{t-1})$. Since $|a| > 1$, this can not occur for all t. This proves a). Observe now that when $\Gamma(U_{t-1})$ contains no more than one point of discontinuity, then $\lambda(U_t) \geq (|a|/2)\lambda(U_{t-1})$. This can not occur for all t if $|a| > 2$. This proves b). ■

As a direct consequence of item b) of the previous lemma and of the definition of regularity, we have the following proposition.

Proposition 2: Let Γ be a regular piecewise affine map with slope a and let I be the interval which is the range of Γ. If $|a| > 2$, then $\Gamma : I \to I$ is covering.

The case $|a| \leq 2$ is more delicate. In general it is no longer true that any regular piecewise affine map is covering. There exist three specific quantized feedback which provide a covering closed-loop map in three different cases. In the sequel we will present these feedback maps without giving the proof of the fact that the closed-loop maps are indeed covering.

Case 1) It can be seen from [19] and [20] that any quantized feedback providing a piecewise affine map Γ corresponding to a β-expansion (see Example 1) is covering if $a > 1$.

Case 2) It can be shown that any quantized feedback providing a piecewise affine map Γ which is symmetric with respect to the vertical axis passing through the mid-point of I (see Example 1) is covering if $|a| \geq \sqrt{2}$

Case 3) Assume with no loss of generality that in this case $I = [0, 1]$. Then it can be shown that the quantized feedback providing the piecewise affine map Γ defined as follows:

$$\Gamma(x) = \begin{cases} -ax + 1 & 0 \leq x \leq 1/a \\ -a(x-1) + b & 1/a \leq x \leq 1 \end{cases} \tag{37}$$

where $b = (1+a)^{-1}$, is covering if $-\sqrt{2} < a < -1$.

An alternative way to solve the difficulties arising when $|a| \leq 2$ is to proceed as follows. Instead of applying the control u_t in the system (1) at each time t, apply it only every n steps so that the problem is now to stabilize the sampled system

$$x_{(t+1)n} = a^n x_{tn} + u_{tn}.$$

If $|a| > 1$, by choosing n big enough, we obtain that $|a^n| = |a|^n > 2$.

V. STATISTICS OF ENTRANCE TIMES FOR COVERING MAPS

The strategy proposed in the previous section looks simpler and more efficient than the one based on a continuous family of invariant intervals. The feedback proposed there requires a number of quantization intervals which is independent of the size of the two intervals I and J and thus, in general, remarkably smaller than the number needed to have connectability. Of course, there is a price to pay. Indeed with this strategy we do not obtain stability, but even more we do not have any *a priori* information on the way the convergence process from I to J take place, while in the stable case a sort of Lyapunov function guarantees a monotonicity of the convergence. It is therefore of fundamental importance to investigate in deeper detail the convergence in this case.

This section is devoted to obtain an estimate of the mean entrance time needed for a state to enter in the invariant interval J for an almost (I, J)-stable map of the type proposed in Section IV. To solve this problem we will use the tools proposed in [16] for the estimation of the decay of correlation in piecewise expanding maps together with an idea in [24].

We start with some general considerations. Let $\Gamma : I \to I$ be any piecewise affine map and let $J \subseteq I$ be a subinterval. Consider the first entrance time map $T_J : I \to \mathbb{N} \cup \{+\infty\}$ as defined in (3). The map T_J can be seen as a discrete random variable with respect to the probability space I equipped with the Borel σ-algebra and with the uniform probability λ. Property P) can, therefore, be equivalently expressed by saying that the map T_J is finite almost surely or that $\lambda(T_J = \infty) = 0$. The statistics of T_J is completely determined by the discrete distribution $\lambda(T_J = n)$. In particular, the mean first entrance time can be expressed as

$$\mathbf{E}(T_J) = \sum_{n \in \mathbb{N} \cup \{\infty\}} n \lambda(T_J = n).$$

Observe that mean first entrance time may be infinite even if P) is satisfied. In the sequel, we will show in particular that for covering maps this mean time is always finite.

Let $\Gamma : I \to I$ be a covering piecewise affine map and let $\Gamma_J : J \to J$ be another piecewise affine map. Consider $\Gamma \wedge_J \Gamma_J$. It is clear that the first entrance time into J of the two maps Γ and $\Gamma \wedge_J \Gamma_J$ coincide. For this reason we will need to focus only on the estimation of the mean first entrance time in J for the covering piecewise affine map Γ.

We need to recall now a fundamental fact about covering expansive piecewise affine maps: the exponential decay of correlations. If $\Gamma : I \to I$ is a covering piecewise affine map, then, as observed above, there exists a unique invariant probability measure which is absolutely continuous with respect to λ. Denote this probability measure by μ and the corresponding density by ϕ. We have the following result (see [16] and[22]).

There exist positive constants K_1, K_2 and $\theta \in [0, 1[$ such that for every absolutely integrable function f defined over I, for every function g defined over I of bounded variation, and for any $n \in \mathbb{N}$, we have that

$$\left| \int_I f(\Gamma^n x) g(x)\, dx - \int_I f(x)\, d\mu(x) \int_I g(x)\, dx \right| \leq K_1 \int_I |f(x)|\, dx \int_I |g(x)|\, dx (1 + K_2 \bigvee_I g) \theta^n \quad (38)$$

where $\bigvee_I g$ denotes the total variation of the function g on I.

Take $f(x) = \chi_B(x)$ and $g(x) = \phi(x)\chi_A(x)$, where A is a union of q disjoint intervals, B is any measurable subset and where χ_B is the characteristic function of the set B which is defined as follows:

$$\chi_B(x) := \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

It is clear that g is of bounded variation and that

$$\bigvee_I g \leq \bigvee_I \phi + 2q\|\phi\|_\infty. \quad (39)$$

Substituting in (38), we obtain that

$$\begin{aligned} &\mu(A \cap \Gamma^{-n} B) - \mu(A)\mu(B) \\ &\quad \leq \left| \mu(A \cap \Gamma^{-n} B) - \mu(A)\mu(B) \right| \\ &\quad \leq K_1 \mu(A) \lambda(B) \left(1 + K_2 \bigvee_I g \right) \theta^n \end{aligned} \quad (40)$$

for all $n \in \mathbb{N}$. Using the fact that $\phi(x) \geq \delta$ for a suitable $\delta > 0$ we can argue that

$$\lambda(B) \leq \delta^{-1} \mu(B). \quad (41)$$

This fact and (40) imply that

$$\mu(A \cap \Gamma^{-n} B) \leq \mu(A)\mu(B)(1 + K\theta^n) \quad \forall n \in \mathbb{N} \quad (42)$$

where

$$K := \delta^{-1} K_1 \left(1 + K_2 \left(\bigvee_I \phi + 2q\|\phi\|_\infty \right) \right) \quad (43)$$

is a constant not depending on the particular sets A and B, but only on the number q of the disjoint intervals constituting A.

We are now in position to present the following result.

Proposition 3: Let $\Gamma : I \to I$ be a covering expanding piecewise affine map. Then, there exist positive constants H_0 and H_1 such that for every subinterval J of I we have that

$$\mathbf{E}(T_J) \leq \left(H_0 + H_1 \log \frac{1}{\lambda(J)} \right) \frac{1}{\lambda(J)} \quad (44)$$

where λ denotes the uniform probability measure on I.

Proof: The proof is based on an idea of [24]. Notice first that

$$\{T_J \geq N + 1\} = \bigcap_{j=0}^{N} \Gamma^{-j} J^c.$$

If we fix $n \in \mathbb{N}$ arbitrarily and for any $k \in \mathbb{N}$ we define

$$B_k := \bigcap_{j=0}^{k} \Gamma^{-jn} J^c$$

we then have that

$$\{T_J \geq N+1\} \subseteq B_{\lfloor \frac{N}{n} \rfloor} \tag{45}$$

where the symbol $\lfloor N/n \rfloor$ means the greatest integer smaller than or equal to N/n. Letting $A := J^c$, which implies that $q = 2$, and using (42), we obtain that

$$\mu(B_k) = \mu(A \cap \Gamma^{-n} B_{k-1}) \leq \mu(A)\mu(B_{k-1})(1+K\theta^n). \tag{46}$$

By an iterative use of (46), and using (41) and (45), we obtain the following:

$$\begin{aligned}\lambda(T_J \geq N+1) &\leq \delta^{-1}\mu(T_J \geq N+1) \leq \delta^{-1}\mu\left(B_{\lfloor \frac{N}{n} \rfloor}\right) \\ &\leq \delta^{-1}\mu(J^c)^{\lfloor \frac{N}{n} \rfloor + 1}(1+K\theta^n)^{\lfloor \frac{N}{n} \rfloor}.\end{aligned} \tag{47}$$

This yields the following estimate:

$$\begin{aligned}\mathbf{E}(T_J) &= \sum_{N \geq 0} \lambda(T_J \geq N+1) \\ &\leq \delta^{-1}\mu(J^c) \sum_{N \geq 0} [\mu(J^c)(1+K\theta^n)]^{\lfloor \frac{N}{n} \rfloor}.\end{aligned} \tag{48}$$

Notice that the previous estimate holds for any $n \in \mathbb{N}$. It is always possible to choose n in such a way that $\mu(J^c)(1+K\theta^n) < 1$ and this choice allows us to argue from (48) that

$$\mathbf{E}(T_J) \leq \frac{\delta^{-1}\mu(J^c)n}{1-\mu(J^c)(1+K\theta^n)}. \tag{49}$$

In order to get a more useful estimate we fix n in such a way that

$$\mu(J^c)(1+K\theta^n) \leq 1 - \frac{1}{2}\mu(J).$$

This holds true for any n such that

$$n \geq (\log\theta^{-1})^{-1} \log \frac{2K(1-\mu(J))}{\mu(J)}.$$

If we choose

$$n = \left\lfloor (\log\theta^{-1})^{-1} \log \frac{2K(1-\mu(J))}{\mu(J)} + 1 \right\rfloor$$

we then obtain

$$\begin{aligned}\mathbf{E}(T_J) &\leq \frac{1-\mu(J)}{\delta\mu(J)/2}\left(\frac{1}{\log\theta^{-1}}\log\frac{2K(1-\mu(J))}{\mu(J)} + 1\right) \\ &\leq \frac{2}{\delta\mu(J)}\left(\frac{1}{\log\theta^{-1}}\log\frac{2K}{\mu(J)} + 1\right) \\ &\leq \frac{2}{\delta^2\lambda(J)}\left(\frac{1}{\log\theta^{-1}}\log\frac{2K}{\delta\lambda(J)} + 1\right).\end{aligned}$$

■

Remark: Notice that the two constants H_0 and H_1 depend on the parameters K_1, K_2, and θ which appear in the decay of correlation formula (38), and on $\delta, \bigvee_I \phi, ||\phi||_\infty$ all linked to the density ϕ of the invariant probability measure μ. Therefore, they a priori depend both on the map Γ and on the interval I on which it is defined. It is important to point out, however, that H_0 and H_1 do not change when we perform an affine transformation of coordinates, since this type of transformation does not modify any of the above parameters. Consider indeed an affine invertible γ defining a bi-injection between I and another interval $\tilde{I}$ and define $\tilde{\Gamma} = \gamma^{-1} \circ \Gamma \circ \gamma$ on $\tilde{I}$. The density of the invariant probability measure for $\tilde{\Gamma}$ which is absolutely continuous with respect to Lebesgue is given by $\tilde{\phi} = \phi \circ \gamma^{-1}$. Clearly, $\tilde{\phi}$ has the same total variation and infinity norm of ϕ, and also the same parameter δ. A straightforward verification shows also that (38) holds true both for Γ and for $\tilde{\Gamma}$ with the same parameters K_1, K_2, and θ.

We can now state the following final result which is a consequence of the results of this section and of Theorem 8.

Corollary 2: Given a system (1) with $|a| > 1$, there exist positive constants H_0 and H_1, only depending on a, such that for any pair of intervals $J \subseteq I$, there exists a uniform quantized feedback making the closed-loop system Γ almost (I, J)-stable with the following characteristics:

1) number of quantization intervals $N = 2\lceil |a| \rceil$;
2) mean first entrance time

$$\mathbf{E}(T_J) \leq (H_0 + H_1 \log C)C \tag{50}$$

where $C = |I|/|J|$ denotes the contraction rate.

Of course, any practical use of (50) needs explicit estimates of the constants H_0 and H_1. A careful reading of [16] indeed offers some tools to estimate these quantities. The following is the specialization of certain estimations done in [16] to the case of uniform piecewise affine maps.

Let $\Gamma : I \to I$ be a uniform piecewise affine map with parameters $(a, \alpha, \beta, \Delta, \Lambda)$. We assume that $|a| > 2$, which is a very mild assumption, since it can easily be removed by considering appropriate powers of Γ. Let $\mathcal{P}$ be the partition of I into quantization intervals and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ consisting of those subintervals of length equal to Δ. For any interval $K \subseteq I$, we define

$$\begin{aligned}N(\Gamma, K) &:= \inf\{N \in \mathbb{N} \,|\, \Gamma^N K = I\} \\ N_0(\Gamma, K) &:= \inf\{N \in \mathbb{N} \,|\, \Gamma^N K \supseteq U, \exists U \in \mathcal{P}_0\}\end{aligned}$$

and for any $\epsilon > 0$ define moreover

$$\begin{aligned}N(\Gamma, \epsilon) &:= \sup\{N(\Gamma, K) \,|\, |K| \geq \epsilon\} \\ N_0(\Gamma, \epsilon) &:= \sup\{N_0(\Gamma, K) \,|\, |K| \geq \epsilon\}\end{aligned}$$

where $|K|$ denotes the length of the interval K. In order to be able to get estimates of this quantity in terms of a, it is useful to consider also

$$\bar{N}(\Gamma) := \sup\{N(\Gamma, K) \,|\, K \in \mathcal{P}_0\}.$$

Notice that, if Γ is regular, then $\bar{N}(\Gamma) = 1$. In general, we have that

$$N(\Gamma, K) \leq N_0(\Gamma, K) + \bar{N}(\Gamma)$$

which implies that

$$N(\Gamma, \epsilon) \leq N_0(\Gamma, \epsilon) + \bar{N}(\Gamma). \tag{51}$$

It follows from the same arguments used in the proof of Lemma 2 that

$$N_0(\Gamma,\epsilon) \leq \frac{\log(2\Delta/\epsilon)}{\log(|a|/2)} + 1. \tag{52}$$

From this, we can obtain an estimate of $N(\Gamma,\epsilon)$. Define now

$$A := \frac{2}{|a|}\max_{U\in\mathcal{P}}\frac{1}{\lambda(U)} \qquad B := \max\left\{1, \frac{2A|a|}{|a|-2}\right\}$$

and

$$N^* := \max\left\{N\left(\Gamma, \frac{1}{2B}\right), \frac{\log|a|}{\log(|a|/2)}\right\}$$

which are quantities that can be computed or estimated from the parameters a, α, β, Δ, and Λ of the map Γ. In [16], the following estimates on the density ϕ of the invariant measure μ has been obtained

$$\begin{gathered}\frac{1}{2}|a|^{-N(\Gamma,\frac{1}{2B})} \leq \phi(x) \leq |I|^{-1} + B \qquad \forall x \in I\\ \bigvee_I \phi \leq B.\end{gathered} \tag{53}$$

This, in particular, implies that we can choose

$$\delta = \frac{1}{2}|a|^{-N(\Gamma,\frac{1}{2B})}. \tag{54}$$

Finally, again from [16], we have that K_1, K_2, and θ in (38) can be chosen as follows:

$$\theta = \left(\frac{1-3|a|^{N^*}+2^{N^*}}{1+3|a|^{N^*}+2^{N^*}}\right)^{\frac{1}{N^*}} \tag{55}$$

and

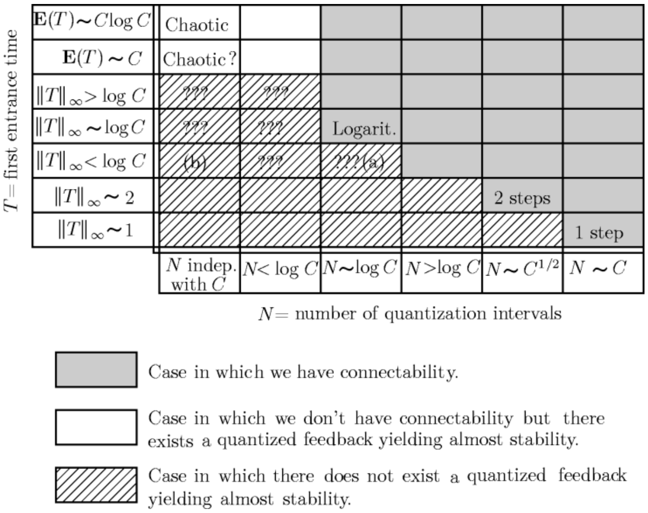
$$\begin{aligned}K_1 &= 2(1+B)e^{\theta^{-N^*}}\theta^{-N^*}\log(3|a|^{N^*}+2^{N^*})\\ K_2 &= \frac{|a|-2}{A|a|}.\end{aligned} \tag{56}$$

From (53), (55), and (56), we can find estimates of the parameters H_0 and H_1 which appear in (50). Aside from the question of the sharpness of these estimates, they show a continuity of the bound provided by (50) with respect to small perturbations of a, as long as $|a| > 2$ and the extremes of I are not discontinuity points for Γ.

VI. CONCLUSION

We have presented some stabilization methods for scalar linear systems by means of a static quantized feedback control. The different strategies have been analyzed in detail and compared to each other on the basis of the amount of information flow they require in the feedback-loop, and on the basis of their performances expressed in terms of the convergence time. It is worth noticing that it is possible to develop stabilization strategies using different methods at the same time. For instance, if $I_1 \subseteq I_2 \subseteq I_3$ are three nested intervals, we can use the logarithmic quantized feedback to obtain (I_3, I_2)-stability and a chaotic quantized feedback to obtain almost (I_2, I_1)-stability.

The results in Sections II and III concerning the stabilization through the existence of a continuous family of invariant intervals can be considered quite complete and this seems to be the case for the construction of the chaotic quantized feedback as well. On the other hand, the analysis of the chaotic quantized feedback done in Section V has margin for improvement. The bound of the mean entrance time (50) can not be considered tight. Indeed, simulation results seems to suggest that the mean entrance time grows only linearly with the contraction rate C. Moreover, it is not clear whether the proposed chaotic quantized feedback can be improved, namely whether there exists or not a quantized feedback requiring the same amount of information flow, but which yields a mean entrance time which grows more slowly with C.



$T=$ first entrance time

	N indep. with C	$N<\log C$	$N\sim\log C$	$N>\log C$	$N\sim C^{1/2}$	$N\sim C$
$\mathbf{E}(T)\sim C\log C$	□ Chaotic	□	▒	▒	▒	▒
$\mathbf{E}(T)\sim C$	□ Chaotic?	□	▒	▒	▒	▒
$\|T\|_\infty > \log C$	▨ ???	▨ ???	▒	▒	▒	▒
$\|T\|_\infty \sim \log C$	▨ ???	▨ ???	▒ Logarit.	▒	▒	▒
$\|T\|_\infty < \log C$	▨ (b)	▨ ???	▨ ???(a)	▒	▒	▒
$\|T\|_\infty \sim 2$	▨	▨	▨	▨	▒ 2 steps	▒
$\|T\|_\infty \sim 1$	▨	▨	▨	▨	▨	▒ 1 step

$N=$ number of quantization intervals

▒ Case in which we have connectability.

□ Case in which we don't have connectability but there exists a quantized feedback yielding almost stability.

▨ Case in which there does not exist a quantized feedback yielding almost stability.

Fig. 8. Cases corresponding to the different asymptotic behavior of the first entrance time and the number of quantization intervals.

Another issue which is worth to be investigated is whether it is possible to generalize the tradeoff results connecting the asymptotic behavior of the number of quantization levels and the mean entrance time of the type obtained in Section III to the case in which the existence of a continuous family of invariant intervals is not assumed.

In Fig. 8, the various cases corresponding to the different asymptotic behavior of the first entrance time and the number of quantization intervals as functions of the contraction rate are compared. Question marks point out the cases in which we can only propose a conjecture. Notice only that case a) correspond to the situation described by Corollary 1. This shows that in this case connectability is not possible. We think that also almost stability can not be obtained. In case b), a straightforward application of the bound (25) shows that almost stability is not possible.

Another important problem which has not been considered in this paper concernes the robustness of the chaotic stabilization technique. It is possible to show that in the chaotic stabilization the mean entrance time is a continuous function of the slope a. We conjecture that, more in general, the mean entrance time is a continuous function of Γ, if we endow the space of piecewise affine maps with the pointwise convergence topology.

We conclude with some considerations on the possibility of extending the results presented in this paper to the multidimensional case. In fact, we expect that some results will be probably easier to extend, while others will be harder. It is our opinion that the results presented in the first part of the paper on connectability have more chances to be extended using instruments offered by the theory of Lyapunov stability. Theorem 7 is instead a deeper result and we expect that other instruments have to be used to extend it to the multidimensional case: the best candidates are the tools offered by symbolic dynamics and the theory of Markov chains with countable state space. These methods seem very promising since they naturally lead to the language of coding and of automata which is probably the most appropriate for a deeper analysis of interconnections between continuous systems and digital controllers. As far as the chaotic quantized feedback scheme is concerned, we believe that the recent ergodic theory of multidimensional piecewise affine maps is the most suitable framework for extending our results to the multidimensional case.

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