A Recursive Deconvolution Approach to Disturbance Reduction

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***Abstract*—Active noise control (ANC) uses an estimate of the noise affecting a system in order to remove its effect from the output. In some applications, it is possible to directly measure the noise and a feedforward compensator can be used. Other applications require instead that the noise be estimated from its effect on the system. This results in an *adaptive feedback ANC*, see a previous paper by Gan and Kuo. The identification of the disturbance from the output of the linear system is a deconvolution problem. In this paper, we study a deconvolution technique for the active reduction of the influence of the disturbance on the output of a linear control system. The regulator that we present does not affect the behavior of the system in the absence of disturbances.**

***Index Terms*—Disturbance reduction, inverse problems, iterative deconvolution.**

I. Introduction

IN THIS paper, we consider a linear, finite-dimensional system, described by

$$\begin{cases} x' = Ax + D(v-u) + Df_0, \; y = Cx, \quad t \geq 0 \\ x(0) = x_0. \end{cases} \tag{1}$$

Here, x and y are vectors of respective dimensions q, p while u, v, and f_0 are m-vectors and the matrices have consistent dimensions and are constant. The input v represents a disturbance whose effect on the output y must be reduced "as much as possible" by using the control u. In other terms, we want the output y to be driven by the input f_0 in spite of the presence of the disturbance v. The disturbance v can be an exogenous signal but it can be as well an endogenous signal due to unmodeled dynamics or nonlinearities. This problem has been studied since the beginning of systems theory. The input f_0 represents the known signal that drives the system. It may be a feedback control, or an exogenous signal.

The usual solution to this problem consists in the addition of a (static or dynamic) feedback loop to (1) which cancels, if possible, or attenuates with respect to a suitable "cost functional" the effect of the disturbance. For example, if f_0 is a low frequency signal, it is possible to attenuate high frequency disturbances with a filter. It is clear that this method cannot be used if the disturbance v and the input f_0 have energy in close frequencies. The H^∞ problem is a further instance: The controller u is still a feedback which reduces the sensitivity, i.e., the norm of the transformation from v to y, below a given tolerance. Quite often a high gain control is used, whose relation with the method we propose is discussed later on, see Section IV-D. The approach in [26], mostly a theoretical interest, led to the geometric theory of linear systems and finally to the Morse form, used in Section V.

The action of the feedbacks constructed by these methods changes the dynamic of the free evolution. In particular, even if v is not present, the system obtained after the interconnection with the compensator is different from the nominal system

$$x' = Ax + Df_0 \quad x(0) = x_0 \quad y = Cx. \tag{2}$$

Instances in which this cannot be permitted are shown in the examples discussed in Section II. The solution of (2) is denoted x_{f_0}. *Our goal is the construction of an input* u *such that the output* y *of system (1) approximates* $y_{f_0} = Cx_{f_0}$.

The idea of this paper is as follows. If by some fortunate case we could know in advance the disturbance v, then it would be possible to cancel it: put $u = v$. Of course, this is a most unusual case, but it may well be possible that we are able to recursively construct an approximation $\hat{v}$ of the *unknown* disturbance v just looking at the output y. If this is the case then we can use $u = \hat{v}$ to reduce the effect of v. See [25] for an analogous idea.

We use a recursive deconvolution method in order to identify the input v and then to compensate its effect. In this way we obtain a feedback regulator which adapts itself to the actual input v; in particular, it is zero if the disturbance is zero. Our approach presents another nice quality: it can also be used if the output y is sampled at discrete time instants and we shall give explicit estimates showing that the compensator proposed here is robust with respect to the sample times and observation errors. We combine for this a technique of recursive deconvolution (based on penalization) as presented in [17] and the results in [8]. The deconvolution problem is fundamental in applied sciences and it has been studied in a lot of papers. See, in particular, [5], [7], [19], [23], and the references therein for the deconvolution problem for causal systems.

The idea of using a deconvolution technique in order to identify and then to cancel a disturbance is currently used in particular applications, especially in active noise reduction, see [10], [11], [14], [21], [24], and [27]. See [4] for an application of this idea to laser control. However, these papers are application oriented and they do not present a general analysis of the algorithm.

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For example, often the system is represented as a discrete time system from the outset.

We sum up: We are presenting a precise mathematical analysis of an adaptive methodology for disturbance cancellation *which does not affect the free dynamics of the given system.* We feel this mathematical analysis is now needed due to the great number of papers which study the application of this idea, mostly on a heuristic basis; to the fact that some particular applications (especially to active noise reduction) have already been used in industrial production; and that more involved applications are already being tested in laboratory experiments, see [4]. Just to stress the usefulness of this approach, we quote from [27, p. 242]: "Active noise control (ANC) system cancels the unwanted noise based on the principle of superposition; specifically, an anti-noise of equal amplitude and opposite phase is generated and combined with the primary noise, thus resulting in the cancellation of both noises. The ANC system efficiently attenuates low frequency noise where passive methods are either ineffective or tend to be very expensive or bulky." Another important feature of the method studied here is that the cancellation is produced by a second "model" system, which is added to the original one, possibly a mechanical and expensive system. In some cases the addition of a model, which may be produced by a computer simulation, can be more convenient than the modification of an already implemented system. This can also be used to alleviate a new problem that has cropped up.

As an additional feature of this paper, we will assume that the system output is read at discrete time instants τ_k (equispaced for simplicity). This is often unavoidable in particular for digital elaboration of data and it is a common practice in active noise suppression. In fact, quite often a discretized model of the system is used in the belief that it will work like the real continuous time system. In fact, the combination of continuous time and discrete sampling can degrade the performance of the system, as noted for example in [15]. For this reason we make an effort to study precisely this problem and to show the tradeoff between the penalization parameter and the other parameters of the process, in particular the sampling time τ. References to sample data control systems are, for example, [3], [12], and [18]. See, for example, [22] and [28] for the limitations imposed by the sampling on the performance of the system.

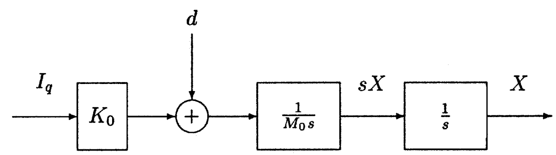
The organization of the paper is as follows: In the next section, we present two guiding examples, taken from the technical literature, which clarify the need for an "external" device which identifies the disturbance, which is then subtracted from the system. In Section III, we present an overview of the method. In Section IV, we will study the case $C = I$, full-state observation. The results of this special case are then adapted to the study of the general case $C \neq I$ in Section V. Of course, our goal of reducing the effect of the disturbance without affecting the free dynamic of the system is not always achievable. The class of systems for which this goal is achievable is characterized in Section V. Simulations on the guiding examples presented in Section II are in Section VI. These simulations, as well as the special applications already appeared in the literature show, from one side, that the method deserves to be (further) tested in real world applications. From the other side, they show that the parameters which appear in the algorithm must be tuned on the basis of the *a priori* information available on the system, on the input f_0, and on the noise. This problem is precisely analyzed in [7].

II. Two Guiding Examples

Our goal is to present a general disturbance reduction scheme based on the following idea: the disturbance is estimated and then subtracted from the input to the system. In order to clarify this point we present two examples taken from the technical literature, where this idea is used. The first example concerns the control of a robot motor against the variations of the transported load. The second example concerns the identification of the fault of an actuator, which reduces its effectiveness, and which is then compensated using a "stronger" control. Features which are common to both the examples (and to a wealth of similar examples) are: 1) under the action of the control f_0, and if there is no disturbance, the performance of the system is satisfactory; 2) there are correlations between the disturbance v and the input signal f_0; and 3) the input signal f_0 and the disturbance v are bounded.

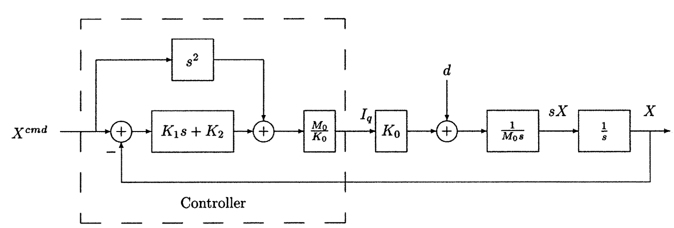
A. Robot Motor

A DD motor can be represented as in the scheme that follows:



where M_0 is the mass of the rotor, K_0 is the thrust constant, X is the angle of the rotor, I_q the thrust current, and d represent disturbances entering into the system. For the sake of simplicity, we use the frequency domain notation.

A typical way to perform path tracking control with this type of motor is to interconnect it with a so-called acceleration controller as displayed in the following figure.



The signal $X^{\rm cmd}$ represents the path to be tracked by the output X. It is fed to the dashed box marked "Controller." This is the acceleration controller, which produces the input fed to the system.

It can easily be shown that

$$X - X^{\rm cmd} = \frac{1}{M_0} \cdot \frac{1}{s^2 + K_1 s + K_2} d.$$

The choice of the weights K_1 and K_2 should ensure asymptotic stability. If d is equal to 0, path tracking is successfully performed: the values of X (and of its first and second derivatives) coincide with the one of $X^{\rm cmd}$ (asymptotically in the presence of fading disturbances). However, in the case when the disturbance d persists, the acceleration controller is unable to filter it out.

As in [15], we shall focus on disturbances due to variations of the transported load. The parameter of the system are set to certain nominal values but, when the load changes, the inertia changes. Of course, a bound on such disturbance is *a priori* known. To attack this problem, [15] proposes to associate an observer to (3), which "estimates the disturbance and parameter variation of the motor" so that the "controller is realized by the feedback of the estimated disturbances by the disturbance observer". Our technique fits into this circle of ideas and actually gives, in this particular setting, a solution similar to the one proposed in [15].

Notice that the input f_0 for this example is $K_0 I_q$, constructed via a feedback loop.

B. Fault Detection Problem

The second guiding example, studied in [29], is taken from the literature on fault detection. It concerns the construction of a fault tolerant aircraft controller. The linearized model considered in [29] is (we adapt the notations)

$$x' = Ax + B\zeta f_0 \quad y = Cx. \tag{4}$$

The scope of the known controller f_0 is to improve the quality of the flight. In concrete applications it can be produced by a feedback loop.

The coefficient ζ is 1 in normal condition while $\zeta \in (0,1)$, constant, in case that an actuator fails.

The nominal values of the parameters (proposed in [29] and which will be used for the simulations in Section VI) are

$$A = \begin{bmatrix} -0.5162 & 26.96 & 178.9 \\ -0.6896 & -1.225 & -30.38 \\ 0 & 0 & -13 \end{bmatrix}$$
$$B = \begin{bmatrix} -175.6 \\ 0 \\ 14 \end{bmatrix}, C = [1 \quad 12.43 \quad 0].$$

We observe that the matrix A is exponentially stable (its eigenvalues are -13 and $-0.8706 \pm 4.2972i$).

In order to put the previous problem in our framework we represent it as

$$x' = Ax + Bf_0 + B[(\zeta f_0 - f_0) - \zeta u] \quad y = Cx$$

where f_0 is the input to be applied if $\zeta = 0$ (i.e., no fault) and $v = (\zeta f_0 - f_0)$ is the disturbance due to the actuator fault (this is clearly bounded). The compensating input is now ζu.

It is clear that if the value by which the actuator is weakened, i.e., ζ, were *exactly known,* the fault would be corrected by $u = f_0 - (1/\zeta)f_0$. However, this value is not known and the authors of [29] propose to *estimate* it using the observation y. The algorithm that we are going to present in general can be adapted to this special problem.

III. Assumptions and an Overview of the Method

Consider the linear finite-dimensional system described by (1) either for t in a bounded interval [0, T] or for $t \geq 0$. The output y is measured at discrete times τ_k, equispaced for simplicity, $\tau_k = k\tau$. Here, τ is a parameter. If the system evolves on a finite time interval [0, T] then we can choose $\tau = T/n$, $0 \leq k \leq n$. Moreover, we assume that the measures taken on the output function are corrupted by errors of known tolerance h, i.e., we assume that at time τ_k we measure a vector ξ_k such that

$$\|y(\tau_k) - \xi_k\| < h. \tag{5}$$

We assume that the input f_0 is approximately known, i.e., we know a function $\tilde{f}_0$ such that

$$\|f_0(t) - \tilde{f}_0(t)\| < \tilde{h}. \tag{6}$$

On the basis of this information we want to construct an estimate $\hat{v}$ of the disturbance which is then used to compensate (1) against v, as done in the references already cited. In order for this to be possible, the estimate of v must be obtained *online*, i.e., at each time t it has to be computed solely on the basis of the measures obtained up to time t. Moreover, if the method has to be of any practical use, the estimation process should be robust with respect to the observation errors and with respect to the sampling time τ. We make precise this concept as follows: the identification algorithm depends on τ, h, $\tilde{h}$ (and a further *penalization parameter* α). When these parameters converge to zero, we want the output of the compensated system to converge to $y_{f_0} = Cx_{f_0}$, the output of (2).

The reason for introducing the parameter α is as follows: The identification of the input v is an *ill posed* problem. This means that small variations in the data can produce large variations in the results, unless special conditions are respected, as in the case of numerical differentiation, see Section IV-D. Robustness of the algorithm is recovered using penalization.

We shall assume that we have an *a priori* information on v and on f_0 namely that both are *bounded*

$$\|v(t)\| < \mathcal{N} \quad \|f_0(t)\| < \mathcal{N}, \qquad t \geq 0. \tag{7}$$

The mere *existence* of the number $\mathcal{N}$, *but not its value,* is used in the algorithm presented in the next section. The actual value of $\mathcal{N}$ should replace the unknown number $\|v\|_\infty$ in the explicit convergence estimates.

The key idea behind the proposed identification algorithm is adapted from [17] and can be illustrated as follows: we associate a "model" to (1), and we use it to test candidate approximants of the input v. We use as a model the following continuous system, which is a copy of the nominal system (in fact, from practical computation a discrete-time model could be more convenient):

$$\begin{cases} \hat{x}' = A\hat{x} + D(\hat{v} - u^{(M)}) + D\tilde{f}_0(t-\tau), \quad z = C\hat{x} \\ \hat{x}(0) = \hat{x}_0. \end{cases} \tag{8}$$

For most of clarity, we introduce the following notations.

- $\hat{v}$ is a signal which mimics the disturbance v.

- $u^{(M)}$ (as shown previously) for the control which acts on the model and $u^{(S)} = u$ for the control which acts on the system.
- Index $_{(k)}$, as in $z_{(k)}$, to denote a function defined on the interval $[\tau_k, \tau_{k+1})$.
- If the argument of a function is negative, then the value of the function is zero. Hence, for example, $\tilde{f}_0(t-\tau) = 0$ for $t < \tau$.

The core of the compensation method is a procedure which, performed at each time τ_k, constructs a function $\hat{v}_{(k)}$, conceivably an estimate of v. In the interval $[\tau_k, \tau_{k+1})$, we will use these functions $\hat{v}_{(r)}$ $(r < k)$ in order to construct the controls $u^{(S)}_{(k)}$ and $u^{(M)}_{(k)}$ which should reduce the action of the disturbance from the outputs, respectively, of the system and of its model.

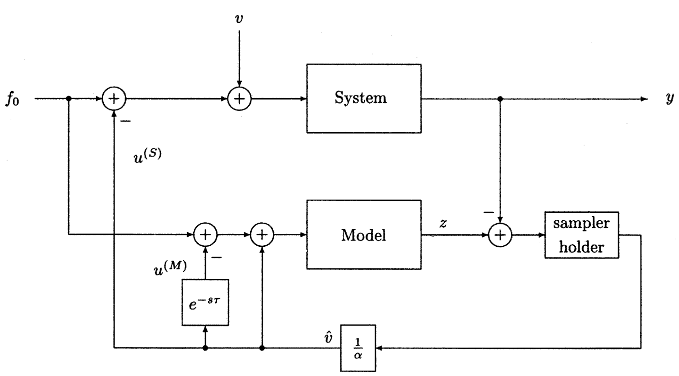
The functions $\hat{v}$, $u^{(S)}$ and $u^{(M)}$ are related by

$$u^{(S)}_{(k)}(t) = \hat{v}_k(t) \qquad u^{(M)}_{(k)}(t) = \hat{v}_{(k-1)}(t-\tau). \tag{9}$$

Hence, $u^{(M)}(t) = 0$ on $[0, \tau)$ and

$$u^{(M)}(t) = u^{(S)}(t-\tau) = \hat{v}(t-\tau), \qquad t \geq \tau. \tag{10}$$

The interconnections of the system and of its model are described by the following scheme, where $e^{-s\tau}$ denotes the delay of one step:



The problem of disturbance reduction makes sense both on a finite interval [0, T] and on $t \geq 0$. In this last case, we must assume exponential stability of the nominal system. In order to keep track of the results which only hold on finite time intervals, we denote by $\mathcal{M}$ a number which depends on the matrices A, C, and D, and possibly on T. It does depend neither on the parameters τ, h, $\tilde{h}$ and α nor on the signals v and f_0. A number that does not even depend on T is denoted by $\mathcal{C}$.

IV. Full State Observation

In this section, we consider (1) with $C = I$, i.e., full state observation. This is the key for the analysis of the general case.

We recall our standing assumptions on the noise (7) and the observations (5) and (6). The initial condition to the model system is put equal to to the first available piece of information, $\hat{x}_0 = \xi_0$. We present an algorithm for the construction of $\hat{v}$ when D is surjective and then we show how the procedure can be modified in the general case. Convergence of $D\hat{v}$ to Dv is then studied in Section IV-C.

A. Case When D is Surjective

We introduce the *error function* (continuous on $t \geq \tau$)

$$e(t) = \hat{x}(t) - x(t-\tau) \quad e_k = e(\tau_k).$$

At this point, a formula for $\hat{v}$ is not yet known. So, let us study the equation of the model with $\hat{v}$ replaced by a general "disturbance" ω.

We introduce

$$g_k = \int_0^\tau e^{Ar} D[\tilde{f}(\tau_k - r) - f(\tau_k - r)]\mathrm{d}r \quad \|g_k\| \leq \mathcal{C}\tau\tilde{h}.$$

For $k \geq 1$, using (10), we compute as follows:

$$\begin{aligned} e_{k+1} &= \hat{x}(\tau_{k+1}) - x(\tau_k) = e^{A\tau}[\hat{x}(\tau_k) - x(\tau_{k-1})] \\ &\quad + \int_0^\tau e^{Ar}\left[-Du^{(M)}(\tau_{k+1} - r) + D\omega(\tau_{k+1} - r)\right]\mathrm{d}r \\ &\quad - \int_0^\tau e^{Ar}\left[-Du^{(S)}(\tau_k - r) + Dv(\tau_k - r)\right]\mathrm{d}r \\ &\quad + \int_0^\tau e^{Ar} D[\tilde{f}(\tau_k - r) - f(\tau_k - r)]\mathrm{d}r \\ &= \left\{e^{A\tau}e_k - \int_0^\tau e^{Ar}Dv(\tau_k - r)\mathrm{d}r + g_k\right\} \\ &\quad + \int_0^\tau e^{Ar}D\omega(\tau_{k+1} - r)\mathrm{d}r. \end{aligned} \tag{11}$$

Now, we define $\hat{v}$ as follows. We put $\hat{v}_{(0)} = 0$. In order to define $\hat{v}_{(k)}$, $k \geq 1$, we look at the system when $t = \tau_k$, $k \geq 1$. At this stage, the function $\hat{v}$ has already been defined and cannot be changed for $t < \tau_k$, while an input ω, still to be specified, acts on $[\tau_k, \tau_{k+1})$. Hence, e_{k+1} is the sum of the contribution on $[0, \tau_k)$ [the brace in (11)] which cannot be changed, and of the contribution of ω on $[\tau_k, \tau_{k+1})$. We consider the sum

$$\|e_{k+1}\|^2 + \alpha \int_0^\tau \|\omega(\tau_{k+1} - r)\|^2 \mathrm{d}r$$

where, as usual in the theory of ill-posed problems, the last addendum is a Tikonov penalization term which is introduce so to have a robust algorithm. We compute

$$\begin{aligned} &\|e_{k+1}\|^2 + \alpha \int_0^\tau \|\omega(\tau_{k+1} - r)\|^2 \mathrm{d}r \\ &= \left\|e^{A\tau}e_k - \int_0^\tau e^{Ar}Dv(\tau_k - r)\mathrm{d}r + g_k\right\|^2 \\ &+ \left\|\int_0^\tau e^{Ar}D\omega(\tau_{k+1} - r)\mathrm{d}r\right\|^2 \qquad (12) \\ &+ 2\langle g_k + e^{A\tau}[\xi_{k-1} - x(\tau_{k-1})] - \int_0^\tau e^{Ar}Dv(\tau_k - r)\mathrm{d}r \\ &\int_0^\tau e^{Ar}D\omega(\tau_{k+1} - r)\mathrm{d}r\rangle \qquad (13) \\ &+ \left\{2\langle e^{A\tau}[\hat{x}(\tau_k) - \xi_{k-1}], \int_0^\tau e^{Ar}D\omega(\tau_{k+1} - r)\mathrm{d}r\rangle\right. \\ &\left. +\alpha \int_0^\tau \|\omega(\tau_{k+1} - r)\|^2 \mathrm{d}r\right\}. \qquad (14) \end{aligned}$$

A common practice in adaptive control defines $\hat{v}$ by minimizing this expression with respect to ω; see [2, Ch. 2]. Unfortunately, the computation of the minimum would require the knowledge of the disturbance (or, alternatively, the use of much more delicate methods, as in [8]). Instead, minimization of the last brace does not involve the disturbance and can be computed solely on the basis of the available data $\hat{x}(\tau_k)$ and ξ_{k-1}; and that (if our algorithm really produces a bounded $\omega = \hat{v}$) the remaining terms which contain $\omega = \hat{v}$ are infinitesimal of higher order when h, $\tilde{h}$, τ and α converge to zero. So, as suggested in [17], we minimize the brace in (14).

Let $\hat{v}_{(k)}$ minimize the brace (14), i.e., let $v_0(s) = 0$ for $s \in [0, \tau)$ and, for $s \in [\tau_k, \tau_{k+1})$, $k \geq 1$

$$\hat{v}_{(k)}(s) = -\frac{1}{\alpha} D^* e^{A^*(\tau_{k+1}-s)} e^{A\tau}[\hat{x}(\tau_k) - \xi_{k-1}]. \qquad (15)$$

We now choose the controls according to (9) and we go on to study the behavior of the system and of the model. We want to show that, under suitable conditions, $\hat{v}$ forces $\hat{x}$ and x to approximate x_{f_0}. We noted that the identification problem is *ill posed.* Therefore, in order to have an algorithm which is robust with respect to the disturbances on the observation and the sampling time, particular consistency conditions must be observed, see for example Theorem 5.

First, of all we will give an estimate for the samples of the error function.

Lemma 1: Let D be surjective. There exist positive constants $\mathcal{C}$ $\mathcal{C}'$ (both soley depend on A and D) such that, if τ, α and τ/α are smaller than $\mathcal{C}'$, then, for every $k \geq 1$ we have

$$\|e_k\| \leq \mathcal{C}\left(h + \alpha\tilde{h} + \tau + \alpha\|v\|_\infty\right). \qquad (16)$$

Proof: We insert (15) into (11) for e_{k+1}. We get, for $k \geq 1$

$$\begin{aligned} e_{k+1} = &-\left[\frac{1}{\alpha}\int_0^\tau e^{Ar}DD^*e^{A^*r}\mathrm{d}r\right]e^{A\tau}[\hat{x}(\tau_k) - \xi_{k-1}] \\ &- \int_0^\tau e^{Ar}Dv(\tau_k - r)\mathrm{d}r + g_k + e^{A\tau}e_k \\ = &\left[I - \left(\frac{1}{\alpha}\int_0^\tau e^{Ar}DD^*e^{A^*r}\mathrm{d}r\right)\right]e^{A\tau}e_k + g_k \\ &- \left[\frac{1}{\alpha}\int_0^\tau e^{Ar}DD^*e^{A^*r}\mathrm{d}r\right]e^{A\tau}[x(\tau_{k-1}) - \xi_{k-1}] \\ &- \int_0^\tau e^{Ar}Dv(\tau_k - r)\mathrm{d}r. \end{aligned}$$

The assumption that D, hence DD^*, is surjective implies that there exists a constant $\mathcal{C}_0 > 0$ such that

$$\left\|I - \frac{1}{\alpha}\int_0^\tau e^{Ar}DD^*e^{A^*r}\mathrm{d}r\right\| \leq 1 - \mathcal{C}_0\frac{\tau}{\alpha}.$$

In fact, DD^* can be diagonalized by an orthogonal transformation in the state space (this does not change the norms) and the diagonal elements are nonzero. Furthermore, e^{As} converges to I for $s \to 0$ so that the integral is of the order of $I\tau$ for $\tau \to 0$.

Let $0 < \mathcal{C}_0(\tau/\alpha) < 1$. We use $\|e^{A\tau}\| < (1 + \mathcal{C}_1\tau)$ and we see

$$\begin{aligned} &\left\|I - \frac{1}{\alpha}\int_0^\tau e^{Ar}DD^*e^{A^*r}\mathrm{d}r\right\| \|e^{A\tau}\| \\ &\qquad \leq 1 - \frac{\tau}{\alpha}(\mathcal{C}_0 + \mathcal{C}_1\alpha - \mathcal{C}_0\mathcal{C}_1\tau) \leq 1 - \frac{\mathcal{C}_0}{2}\frac{\tau}{\alpha} \end{aligned}$$

so that

$$\|e_{k+1}\| \leq \left[1 - \frac{\mathcal{C}_0}{2}\frac{\tau}{\alpha}\right]\|e_k\| + \mathcal{C}_2\left[\frac{\tau h}{\alpha} + \tau\tilde{h} + \tau\|v\|_\infty\right].$$

Using $\|e_1\| \leq \mathcal{C}_3(\tau + h)$, we finally obtain

$$\begin{aligned} \|e_{k+1}\| \leq &\mathcal{C}_2\left(\frac{\tau h}{\alpha} + \tau\tilde{h} + \tau\|v\|_\infty\right)\sum_{j=1}^{k+1}\left[1 - \frac{\mathcal{C}_0}{2}\frac{\tau}{\alpha}\right]^j + \|e_1\| \\ \leq &\frac{\alpha}{\tau}\frac{2}{\mathcal{C}_0}\mathcal{C}_2\left(\frac{\tau h}{\alpha} + \tau\tilde{h} + \tau\|v\|_\infty\right) + \mathcal{C}_3(\tau + h) \\ = &\mathcal{C}\left(h + \alpha\tilde{h} + \tau + \alpha\|v\|_\infty\right). \end{aligned}$$

■

Remark 2: For future use, we explicitly note that Lemma 1 yields the following bound on $\hat{v}$ [see definition (15)]:

$$\|\hat{v}(t)\| \leq \mathcal{C}\left(\|v\|_\infty + \tilde{h} + \frac{h + \tau}{\alpha}\right) \qquad \forall t \geq 0. \qquad (17)$$

We can now obtain a bound of $e(t)$ for every t.

Theorem 3: Let D be surjective and let the interval $[\sigma, +\infty[$ be fixed. There exist positive constants $\mathcal{C}$ $\mathcal{C}'$ such that, if τ, α and τ/α are smaller than $\mathcal{C}'$, the following estimate holds for $t \in [\sigma, +\infty[$:

$$\|e(t)\| \leq \mathcal{C}\left(h + \alpha\tilde{h} + \tau\tilde{h} + \tau(1 + \|v\|_\infty) + \alpha\|v\|_\infty\right). \qquad (18)$$

Proof: Assume that $t > \sigma > \tau$. Let k be the positive integer such that $t \in [k\tau, (k+1)\tau[$. Using the representation

$$e'(t) = Ae(t) + D\hat{v}(t) - Dv(t - \tau) + [\tilde{f}(t - \tau) - f(t - \tau)]$$

and (17), we can estimate $e(t)$ as follows:

$$\begin{aligned} \|e(t)\| \leq e^{\tau\|A\|}\|e_k\| &+ \frac{\tau}{\alpha}e^{2\tau\|A\|}\|D\|^2(\|e_k\| + h) \\ &+ \tau e^{\tau\|A\|}\|D\|\|v\|_\infty + \tau e^{\tau\|A\|}\|D\|\tilde{h}. \end{aligned}$$

Using now suitable upper bounds on τ, τ/α and Lemma 1, we obtain (18). ■

We consider now convergence results when τ, α, h, and $\tilde{h}$ converge to 0 in such a way that the following *consistency condition* holds:

$$\frac{\tau}{\alpha} \to 0 \quad \frac{h}{\alpha} \to 0. \qquad (19)$$

To be formal, we will say that a function of time t and of the four parameters above $f_{\tau,\alpha,h,\tilde{h}}(t)$ converges uniformly to a function $f(t)$ for $t \in I$ for τ, α, h, and $\tilde{h}$ converging to 0 while satisfying the consistency condition (19) if for every $\epsilon > 0$ there exist $\delta > 0$ such that if τ, α, h, $\tilde{h}$, τ/α, and h/α are all below δ, then

$$|f_{\tau,\alpha,h,\tilde{h}}(t) - f(t)| < \epsilon \qquad \forall t \in I.$$

We can now prove a first convergence result.

Theorem 4: Let D be surjective and let the interval $[\sigma, T]$, $0 \leq \sigma < T < +\infty$ be fixed. For τ, α, h, and $\tilde{h}$ converging to 0 in such a way that (19) holds, we have that both $\hat{x}(t)$ and $x(t)$ converge *uniformly* on $[\sigma, T]$ to the solution x_{f_0} of the nominal system (2) (i.e., approximate disturbance cancellation is achieved). Moreover, if A is *exponentially stable* convergence holds on $[\sigma, +\infty)$.

Proof: Consider first $\hat{x}(t)$ which is governed by

$$\hat{x}'(t) = A\hat{x} + D(\hat{v}(t) - u^{(M)}(t)) + D\tilde{f}_0(t-\tau).$$

We have

$$\begin{aligned}\hat{x}(t) = e^{tA}\hat{x}(0) + \int_{t-\tau}^{t} e^{(t-s)A}D\hat{v}(s)ds \\ + \int_0^t e^{(t-s)A}D\tilde{f}_0(s-\tau)ds \\ + (e^{-\tau A} - I)\int_0^{t-\tau} e^{(t-s)A}D\hat{v}(s)ds.\end{aligned}$$

Since $\hat{v}$ is bounded (see Remark 2), $\hat{x}(0)$ converges to $x(0)$ and $\tilde{f}_0$ to f_0, it follows that the last expression converges to

$$x_{f_0}(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}Df_0(s)ds$$

uniformly on compact intervals, as wanted. If A is exponentially stable then we have uniform convergence on $[0, +\infty)$. From Theorem 3, $e(t) \to 0$ uniformly on $[0, +\infty)$, since $x(t) = \hat{x}(t+\tau) - e(t+\tau)$, everything follows. ∎

B. Case When D is Not Surjective

In this section, we study the case when D is not surjective (and $C = I$). We present a two step algorithm which reduces the problem to the one studied in the surjective case.

We first work on a *bounded* time interval [0, T] (if we intend to work on $[0, +\infty)$ then we shall let $T \to +\infty$).

We denote $\|\cdot\|_{T,\infty}$ the norm in $L^\infty(0,T)$ while $\|\cdot\|_\infty$ denotes the norm in $L^\infty(0,\infty)$.

The key observation is that the following inequalities hold on [0, T]:

$$\begin{aligned}\|x\|_{T,\infty} &\leq \mathcal{M}\left(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|\right) \\ \|x'\|_{T,\infty} &\leq \mathcal{M}\left(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|\right).\end{aligned} \quad (20)$$

If A is exponentially stable then the previous inequalities hold on *every* bounded interval [0, T], with the same constant, i.e., $\mathcal{M}$ does not depend on T, $\mathcal{M} = \mathcal{C}$.

We put ourselves in a reference frame in which the matrix D is

$$D^T = [\,D_1^T \quad 0\,]$$

and D_1 is invertible.

We decompose the system accordingly

$$\begin{aligned}(x^1)' &= A_{11}x^1 + A_{12}x^2 + D_1(v - u^{(S)}) + D_1f_0 \\ y_1 &= x^1 \\ (x^2)' &= A_{21}x^1 + A_{22}x^2 \\ y^2 &= x^2\end{aligned} \quad (21)$$

and $\xi_k^T = [\,\xi_k^{1T} \quad \xi_k^{2T}\,]$.

The idea is to work exclusively with the first equation, interpreting $A_{12}x^2$ as an approximately known input signal to be assimilated with D_1f_0. Unfortunately, A_{11} may well be unstable even if the global system is stable. This is not a problem if we are working on a finite time interval. If A is exponentially stable and we want to study the system on $[0, +\infty)$, we impose a negative coefficient to x^1 in the first equation in (21) with a further modification (which is not needed but harmless when working on a finite interval [0, T]).

We represent the top subsystem as

$$(x^1)' = -x^1 + D_1(v - u^{(S)}) + \left[D_1f_0 + (A_{11}+I)x^1 + A_{12}x^2\right].$$

We interpret $D_1f_0 + (A_{11}+I)x_1 + A_{12}x^2$ as an approximately known input and we apply the algorithm using the model system

$$\begin{aligned}(\hat{x}^1)' &= -\hat{x}^1 + D_1(\hat{v} - u^{(M)}) + \hat{f}_0(t-\tau) \\ \hat{f}_0 &= (D_1\tilde{f}_0 + (A_{11}+I)\xi^1 + A_{12}\xi^2)\end{aligned} \quad (22)$$

where $\xi^i(t)$, $i = 1, 2$, is given by

$$\xi^i(t) = \xi_k^i, \qquad t \in [\tau_k, \tau_{k+1})$$

and where, we recall, the controls $u^{(M)}$ and $u^{(S)}$ are defined through (10) and (15). We have the following result.

Theorem 5: Let D be any matrix and $C = I$. Let the interval $[\sigma, T]$, $0 \leq \sigma < T < +\infty$ be fixed. For τ, h, $\tilde{h}$ and α converging to zero while respecting the consistency condition (19), the solution $x(t) = (x_1(t), x_2(t))$ of (21) converges *uniformly* on $[\sigma, T]$ to the solution x_{f_0} of the nominal system (2) (i.e., approximate disturbance cancellation is achieved). Moreover, if A is *exponentially stable* convergence holds on $[\sigma, +\infty)$.

Proof: We can repeat the same arguments which led to Theorem 3 with $\tilde{h}$ replaced by $\hat{h}$ which, from inequalities (20), can be estimated as

$$\begin{aligned}\hat{h} &= \|[\hat{f}_0(t-\tau) - (D_1f_0(t-\tau) \\ &\quad + (A_{11}+I)x^1(t-\tau) + A_{12}x^2(t-\tau))]\|_{T,\infty} \\ &\leq \mathcal{M}\left[(\|(x^1)'\|_{T,\infty} + \|(x^2)'\|_{T,\infty})\tau + h + \tilde{h}\right] \\ &\leq \mathcal{M}\left[\left(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|\right)\tau + h + \tilde{h}\right].\end{aligned}$$

Here, $\mathcal{M} = \mathcal{C}$ if A is exponentially stable.

Using now estimation (17), we obtain

$$\begin{aligned}\|\hat{v}\|_{T,\infty} = \|u^{(S)}\|_{T,\infty} &\leq \mathcal{M}\left[\|v\|_\infty + \left(\|v - u^{(S)}\|_{T,\infty}\right.\right. \\ &\quad \left.\left. + \|f_0\|_\infty + \|x_0\|\right)\tau + \tilde{h} + \frac{h+\tau}{\alpha}\right] \\ &\leq \mathcal{M}\left[2\|v\|_\infty + \tau(\|f_0\|_\infty + \|x_0\|) + \tilde{h} + \frac{h+\tau}{\alpha}\right] \\ &\quad + \mathcal{M}\tau\|u^{(S)}\|_{T,\infty}.\end{aligned}$$

Hence, under the assumption

$$\tau < 1, \qquad \frac{1}{2} < 1 - \mathcal{M}\tau$$

$\|\hat{v}\|_{T,\infty} = \|u^{(S)}\|_{T,\infty}$ is less than

$$\mathcal{M}_1\left(\|v\|_\infty + \|f_0\| + \|x_0\| + \tilde{h} + \frac{h+\tau}{\alpha}\right). \quad (23)$$

This inequality holds on every interval [0, T].

In particular

$$\hat{h} \leq \mathcal{M}\left[(\|v\|_\infty + \|f_0\|_\infty + \|x_0\|)\,\tau + \frac{(h+\tau)\tau}{\alpha} + h + \tilde{h}\right]$$

is infinitesimal provided that τ, α, h, and $\tilde{h}$ converge to zero, while respecting the consistency condition (19).

Even more, if the matrix A is exponentially stable, the constants $\mathcal{M} = \mathcal{C}$, $\mathcal{M}_1 = \mathcal{C}_1$ do not depend on T. It follows that *if A is exponentially stable then $\hat{v}$, $u^{(S)}$ and $u^{(M)}$ are bounded for $t \geq 0$.*

Using Theorem 3, we see that $e^1(t) = \hat{x}^1(t) - x^1(t-\tau)$ converges uniformly to zero when τ, α, h, $\tilde{h}$ converge to zero while respecting the consistency condition (19).

Now, $\hat{x}^1$ and x^2 solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{x}^1\\ x^2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\hat{x}^1\\ x_2\end{bmatrix} + \begin{bmatrix}\zeta\\ A_{21}(x^1 - \hat{x}^1)\end{bmatrix}$$

where $\zeta = D_1\tilde{f}_0(t-\tau) + D_1(\hat{v} - u^{(M)}) + (A_{11}+I)(\xi^1 - \hat{x}^1) + A_{12}(\xi^2 - x^2)$.

The matrix of this system is the matrix A (if $T = +\infty$ the matrix A is exponentially stable by assumption).

Notice that $(x^1(t) - \hat{x}^1(t)) = (x^1(t) - x^1(t-\tau)) - e^1(t)$. Since $(x^1)'$ is uniformly bounded and $e^1(t)$ converges uniformly to 0, $(x^1 - \hat{x}^1)$ converges uniformly to 0. The same clearly holds for $(\xi^1 - \hat{x}^1)$ and $(\xi^2 - x^2)$. Repeating the arguments used in the proof of Theorem 4 (with the stronger assumption of asymptotic stability if $T = +\infty$) we obtain that the pair $(\hat{x}^1, x^2)$ converges uniformly to the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{x}^1\\ x^2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\hat{x}^1\\ x^2\end{bmatrix} + \begin{bmatrix}D_1 f_0\\ 0\end{bmatrix}$$

with $\hat{x}^1(0) = x^1(0)$ and $\hat{x}^2(0) = 0$. We know that e^1 converges to 0 so that also the pair (x^1, x^2) converges to the solution $x(t)$ (with $x(0) = x_0$) of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x^1\\ x^2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x^1\\ x^2\end{bmatrix} + \begin{bmatrix}D_1 f_0\\ 0\end{bmatrix}.$$

This is (2) and our goal is achieved. ∎

C. Convergence of $D\hat{v}$ to the Unknown Input Dv

We mentioned that the algorithm that we developed is based on "deconvolution" ideas, and that $D\hat{v}$ should approximate Dv when τ, h, $\tilde{h}$, and α converge to zero, while respecting suitable conditions. In fact, we could prove the following result.

Theorem 6: Let the following conditions hold:

$$\frac{h}{\alpha^2} \to 0 \qquad \frac{\tau}{\alpha^3} \to 0 \qquad \frac{\tilde{h}}{\alpha} \to 0. \tag{24}$$

Under these conditions, $D\hat{v}$ converges to Dv in $L^2(0,T)$ for every $T > 0$.

We do not present a proof of the previous theorem which will not be explicitly used later on. The proof can be found in [9] (in the case $\tilde{h} = 0$). Instead, we prove the theorem in a special case which will be used in Section V. In this case it is also possible to give explicit convergence estimates.

The special case that we consider is: $A = 0$, $C = I$, and D surjective. The disturbance v, which is bounded, is also assumed to be a.e. differentiable with bounded or square integrable derivatives. We note that the DD^* has positive eigenvalues since D is surjective.

Theorem 7: Let $A = 0$, $C = I$, and D be surjective. Let v be bounded and a.e. differentiable. Let ρ be the minimum eigenvalue of DD^* and let $\tau/\alpha < 1$. Then, there exists a number $\mathcal{C}$ such that the following hold.

i) If v' is square integrable, then, for $t \geq \tau$

$$\|Dv(t) - D\hat{v}(t)\| \leq \mathcal{C}\left\{\tilde{h} + e^{-((\rho/\alpha)(t-\tau))}\|Dv(0)\| + \frac{\tau\|v\|_\infty + h + \tau}{\alpha} + \|v'\|_2[\sqrt{\tau} + \sqrt{\alpha}]\right\}. \tag{25}$$

ii) If v' is bounded, then, for $t \geq \tau$

$$\|Dv(t) - D\hat{v}(t)\| \leq \mathcal{C}\left\{\tilde{h} + e^{-((\rho/\alpha)(t-\tau))}\|Dv(0)\| + \frac{\tau\|v\|_\infty + h + \tau}{\alpha} + \|v'\|_\infty[\tau + \alpha]\right\}. \tag{26}$$

Proof: We study the case $t \geq \tau$ since by definition $\hat{v} = 0$ on $[0,\tau)$. We first study the differential equation of $e(t)$ on a fixed interval $[\tau_k, \tau_{k+1})$, $k > 0$. On this interval, we have

$$\begin{aligned} e'(t) &= \hat{x}'(t) - x'(t-\tau)\\ &= D\hat{v}(t) - Dv(t-\tau) + \tilde{f}(t-\tau) - f(t-\tau). \end{aligned}$$

We deduce, from (17) in Remark 2, that e is Lipschitz continuous of constant $\mathcal{C}(\|v\|_\infty + \tilde{h} + (h+\tau)/\alpha)$. Hence, on (τ_k, τ_{k+1}) we have

$$\|e(t) - e(\tau_k) + x(\tau_{k-1}) - \xi_{k-1}\| \leq \tau\left[\mathcal{C}\left(\|v\|_\infty + \tilde{h} + \frac{h+\tau}{\alpha}\right)\right] + h \leq \mathcal{C}(\tau\|v\|_\infty + \tau\tilde{h} + h + \tau).$$

The numbers $\mathcal{C}$ do not depend on τ, h, $\tilde{h}$, α; in particular they do not depend on the interval $[\tau_k, \tau_{k+1})$ on which we are working.

Now, we represent

$$\begin{aligned} e'(t) &= \hat{x}'(t) - x'(t-\tau) = D\hat{v}(t) - Dv(t-\tau)\\ &\quad + [\tilde{f}(t-\tau) - f(t-\tau)]\\ &= -\frac{DD^*}{\alpha}[e(\tau_k) + x(\tau_{k-1})\\ &\quad - \xi_{k-1}] - Dv(t-\tau) + [\tilde{f}(t-\tau) - f(t-\tau)]\\ &= -\frac{DD^*}{\alpha}e(t) - Dv(t-\tau) + \frac{DD^*}{\alpha}\{e(t) - e(\tau_k)\\ &\quad + x(\tau_{k-1}) - \xi_{k-1}\} + [\tilde{f}(t-\tau) - f(t-\tau)]\\ &= -\frac{DD^*}{\alpha}e(t) - Dv(t-\tau) + \Psi(t;\tau,h,\alpha) \end{aligned} \tag{27}$$

where Ψ satisfies on (τ, T)

$$\|\Psi(t;\tau,h,\alpha)\| \leq \mathcal{C}\left[\frac{\tau\|v\|_\infty + h + \tau}{\alpha} + \tilde{h}\right]. \tag{28}$$

We get from (27)

$$DD^*e(t)=e^{-((DD^*/\alpha)(t-\tau))}DD^*e(\tau) + \int_\tau^t e^{-((DD^*/\alpha)(t-s))}DD^*\Psi(s;\tau,h,\alpha)\mathrm{d}s - \int_\tau^t e^{-((DD^*/\alpha)(t-s))}DD^*[Dv(s-\tau)]\mathrm{d}s. \quad (29)$$

Diagonalization of D shows that $\|e^{-((DD^*/\alpha)t)}DD^*\| \le \mathcal{C}\cdot e^{-((\rho/\alpha)t)}$ (because ρ is the *minimum* eigenvalue of J). It follows that

$$\left\|\int_\tau^t e^{-((DD^*/\alpha)(t-s))}DD^*\Psi(s;\tau,h,\alpha)ds\right\| \le \mathcal{C}\left[(\tau\|v\|_\infty + h + \tau) + \alpha\tilde{h}\right].$$

We divide both sides of (29) by α and we integrate by parts the last integral

$$\int_\tau^t \left[\frac{\mathrm{d}}{\mathrm{d}s}e^{-((DD^*/\alpha)(t-s))}\right]Dv(s-\tau)\mathrm{d}s = Dv(t-\tau) - e^{-((DD^*/\alpha)(t-\tau))}Dv(0) - \int_\tau^t e^{-((DD^*/\alpha)(t-s))}Dv'(s)\mathrm{d}s.$$

Since $v \in W^{1,2}(0,T)$, the last integral satisfies

$$\left\|\int_\tau^t e^{-((DD^*/\alpha)(t-s))}Dv'(s)\mathrm{d}s\right\| \le \mathcal{C}\left[\int_\tau^t e^{-2((\rho/\alpha)(t-s))}\mathrm{d}s\right]^{1/2}\|v'\|_{L^2} \le \mathcal{C}\|v'\|_2\sqrt{\alpha} \quad (30)$$

(this holds on $t > \tau$; moreover, if v' is bounded then we can replace $\sqrt{\alpha}$ with α and $\|v'\|_{L^2}$ with $\|v\|_\infty$).

We use this inequality, inequality (28), and $\|e(\tau)\| < h+\tau$ in order to get

$$\left\|Dv(t-\tau) + \frac{1}{\alpha}DD^*e(t)\right\| \le \mathcal{C}\left\{e^{-((\rho/\alpha)(t-\tau))}\|Dv(0)\| + \frac{\tau\|v\|_\infty + h + \tau}{\alpha} + \tilde{h} + \|v'\|_2\sqrt{\alpha}\right\}.$$

We use

$$\|v(t) - v(t')\| \le \begin{cases} \|v'\|_\infty |t-t'| & \text{if } v' \in L^\infty \\ \|v'\|_2\sqrt{|t-t'|} & \text{if } v' \in L^2 \end{cases} \quad (31)$$

to see that

$$\left\|Dv(t) + \frac{1}{\alpha}DD^*e(t)\right\| \le \left\|Dv(t-\tau) + \frac{1}{\alpha}DD^*e(t)\right\| + \|v'\|_2\sqrt{\tau}$$

(the last addendum can be replaced by $\tau\|v'\|_\infty$ if v' is bounded).

We finally estimate $\|Dv(t) - D\hat{v}(t)\|$. Let $t \in [\tau_k, \tau_{k+1})$

$$\begin{aligned} &\|Dv(t) - D\hat{v}(t)\| \\ &= \|Dv(t) + \frac{1}{\alpha}DD^*[\hat{x}(\tau_k) - \xi_{k-1}]\| \\ &= \|Dv(t) + \frac{1}{\alpha}DD^*[e(t) + (e(\tau_k) - e(t) \\ &\quad + x(\tau_{k-1}) - \xi_{k-1})]\| \\ &\le \|Dv(t) + \frac{1}{\alpha}DD^*e(t)\| \\ &\quad + \|\frac{1}{\alpha}DD^*[e(\tau_k) - e(t) + x(\tau_{k-1}) - \xi_{k-1})]\| \\ &\le \mathcal{C}\left\{\|v'\|_2\left(\sqrt{\tau} + \sqrt{\alpha}\right) + e^{-\rho(t-\tau)/\alpha}\|Dv(0)\| \right. \\ &\quad \left. + \frac{\tau\|v\|_\infty + h + \tau}{\alpha} + \tilde{h}\right\} \end{aligned}$$

(obvious modifications if v' is bounded). The result follows. ■

This yields the following.

Corollary 8: Let v be bounded and v' be either bounded or square integrable. Let $\sigma > 0$ be fixed. Let τ, h, $\tilde{h}$, α converge to zero and the consistency condition (19) hold. Let furthermore $A = 0$, $C = I$ and D be surjective. Under these conditions, $D\hat{v}$ converges Dv uniformly on $[\sigma, +\infty]$. We have uniform convergence on $[0, +\infty]$ if, furthermore, $Dv(0) = 0$.

Remark 9: The conditions in Corollary 8 are weaker then (24). They only imply convergence under additional information on the regularity of v.

D. Remarks and Comments

We collect now several comments on the proposed algorithm.

1) If we impose, in our algorithm, $u^{(M)} = 0$ and $u^{(S)} = 0$, the evolution of the system is guided by the input f_0 and by the disturbance. In this case $\hat{x}$ still tracks x and $\hat{v}$ still gives an estimate of v, i.e., $\hat{v}$ solves a *deconvolution problem.* See [8] for an analysis of the deconvolution problem.

   In the particular case that $f_0 = 0$, $A = 0$, and $D = 1$, we have an algorithm of numerical differentiation.
2) Theorem 7 explicitly estimates the overshooting and the transient behavior of the error for $t \to 0$.
3) Equation (27) in the proof of Theorem 7 shows that the underlying structure of our algorithm is a singular perturbation of an ordinary differential equation. Singular perturbations and high gain feedback are strongly related, see [30]. The high-gain feedback of our algorithm is always in terms of the variable e.
4) As in every application of the high gain or deconvolution technique, tuning of the parameters in terms of the *a priori* available information is crucial. This appears from the estimates (25) and (26) which depend on $\|v\|_\infty$, $\|v'\|_\infty$, $\|v'\|_2$ shows up.

V. PARTIAL-STATE OBSERVATION

In this section, we consider the general case $C \ne I$. As always, we assume that if $T = +\infty$ then A is exponentially stable.

In this general case $C \neq I$, it is not always possible to reduce the effect of the disturbance without changing the free dynamic of the system. In order to clarify this point, we use a particular set of transformations, introduced by Morse in [20], which essentially reduce the system to a cascade of integrators. The simplest case is illustrated by the following example (where, for simplicity, we put $f_0 = 0$ and zero initial conditions):

$$x_1' = x_2 \quad x_2' = v - u^{(S)}, \qquad y = x_1.$$

This is treated as follows. We first consider the system

$$x_1' = x_2 \qquad y = x_1. \tag{32}$$

We think of x_2 as a "disturbance" and we apply our algorithm to this system, with $u^{(S)} = 0$, $u^{(M)} = 0$. As noted in observation 1) of Section IV-D we obtain a deconvolution algorithm which produces an estimate $\hat{x}_2$ of x_2. The estimate is *uniform* for $t \geq \sigma > \tau$, see Theorem 7. Actually, the uniform estimations obtained in Section IV-C were derived precisely to be used here.

After time σ, we look at the system

$$x_2' = v - u^{(S)} \qquad y_2 = x_2 \tag{33}$$

with observations

$$(\xi_2)_k = \hat{x}_2(\tau_k), \qquad \tau_k > \sigma$$

and we apply our algorithm to this subsystem. We obtain an estimate $\hat{v}$ of the disturbance v, and the inputs $u^{(S)}$ which is now used to reduce the influence of v.

The previous simple example contains features which are completely general, as seen from an application of Morse canonical form which we now describe.

It is possible to find a feedback matrix F, an output injection K and suitable reference frames in the state, input and output spaces, such that $A_{\rm CL} = A + DF + KC$, D, and C take the form

$$A_{\rm CL} = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \end{bmatrix} D = \begin{bmatrix} 0 & 0 \\ D_1 & 0 \\ 0 & 0 \\ 0 & D_3 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix}.$$

The eigenvalues of A_0 are the transmission zeros of the system. In particular A_0 is stable if the system is minimum phase; the subsystem of the matrices A_1 and D_1 is in *Brunovski canonical form*, i.e., both A_1 and D_1 are block diagonal, the blocks being of the form

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The matrices A_2^* and C_2^* are in Brunovski form and the last subsystem, described by the matrices A_3, D_3 and C_3 is in *prime form*. This means that A_3 and D_3 are in Brunovski form and, at the same time, A_3^* and C_3^* are in Brunovski form. Hence, they consist of independent subsystems and each subsystem is a chain of integrators

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$
$$[1 \quad 0 \quad 0 \quad \dots \quad 0].$$

The geometric interpretation of the blocks is as follows. Let the state space be decomposed as $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ in accordance with Morse form. Then, $\mathcal{X}_1 = \mathcal{R}_*$ is the maximal controllability subspace in $\ker C$ while $\mathcal{V}_* = \mathcal{X}_0 \oplus \mathcal{X}_1$ is the maximal maintainability subspace in $\ker C$, i.e., $\mathcal{V}_*$ is the maximal subspace of $\ker C$ such that for any $x_0 \in \mathcal{V}_*$ there exists $u_{x_0}(t)$ such that the corresponding solution of (1) (with $v = f_0 = 0$) is in $\mathcal{V}_*$ for every $t > 0$. An interesting frequency domain characterization of $\mathcal{V}_*$ is as follows: $\mathcal{V}_*$ is the subspace of those $x_0 \in \ker C$ such that there exists a strictly proper rational $r(z)$ such that $C(zI - A)^{-1}x_0 = C(zI - A)^{-1}Br(z)$. The subspace $\mathcal{R}_*$ is defined as follows: Choose the feedback F and the matrix D such that the reachability subspace $\mathcal{R}_{F,D}$ of the system $(A + BF, BD)$ is in $\ker C$. Clearly, $F = 0$ and $D = 0$ have the required properties. Now, $\mathcal{R}_*$ is the maximal (with respect to the inclusion) of these subspaces $\mathcal{R}_{F,D}$. Algorithms exist which explicitly constructs $\mathcal{V}_*$ and $\mathcal{R}_*$ in a finite number of steps, see [1] and [26].

Now, we write (1) as

$$x' = A_{\rm CL}x + D(v - u^{(S)}) - (DFx + Ky) + Df_0, \; y = Cx$$

and we transform $A_{\rm CL} = (A + DF + KC)$, D and C into Morse form (see [20]). Consequently, we obtain

$$\begin{aligned} x_0' &= A_0x_0 && && -K_0y \\ x_1' &= A_1x_1 && +D_1(v_1 - u_1^{(S)}) && -K_1y \\ & \quad -D_1Fx + D_1f_{0,1} \\ x_2' &= A_2x_2 && && -K_2y \\ x_3' &= A_3x_3 && +D_3(v_3 - u_3^{(S)}) && -K_3y \\ & \quad -D_3Fx + D_3f_{0,3} \end{aligned}$$
$$y = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} C_2x_2 \\ C_3x_3 \end{bmatrix}. \tag{34}$$

Notice that y is directly measured (with errors of known tolerance h) so that Ky is known and its effect on the components of the system can be explicitly computed with an error of known tolerance.

We now consider three cases.

1) $\mathcal{X}_0 \oplus \mathcal{X}_1 = \{0\}$. The idea is, in this case, to recursively estimate every component of x_2 and x_3. At last, we obtain an estimate $\hat{v}_3$ of v_3 and we can use the input $u = \hat{v}_3$ in order to reduce the effect of the disturbance, without affecting the free evolution of the system.
2) $\mathcal{X}_1 = \{0\}$. In this case, we can estimate the components of x_2 and x_3 as in the previous case so that at last we get an estimate of $v_3 - Fx$. Hence, we obtain an estimate $\tilde{v}$ of $v_3 - F_0x_0$; but there is no way to distinguish between

v_3 and F_0x_0 (unless we know the initial condition of the system). Hence, the best that we can do is to use the control $u^{(S)} = \tilde{v}$ which however affects the dynamic of the system even if the disturbance v is zero. We observe that the eigenvalues of A_0 are the transmission zeros of the system. Hence, for minimum phase systems, the dynamic effect of $u^{(S)}$ decays exponentially with time.

3) $\mathcal{X}_1 = \mathcal{R}_* \neq \{0\}$. In this case, we can repeat the previous considerations. We identify $v_3 - Fx$, i.e., we obtain an estimate $\tilde{v}$ of $(v_3 - F_0x_0 - F_1x_1)$; but we cannot distinguish among the three terms and we are forced to use a compensator $u^{(S)} = \tilde{v}$, which affects the free evolution of the system. It may well be that the system is minimum phase and the matrix $A_1 - D_1F_1$ is stable. In this case, again the dynamic effect will fade exponentially, but there is no simple characterization of the stability of the matrix $A_1 - D_1F_1$.

We now describe in some detail the recursive estimation of the components of x_2 and x_3 which is the key step in the three cases shown previously, and we will give a final more precise result on cases 1) and 2).

In the recursive estimation of the components, we face a difficulty: The use of the input $u = \hat{v}$ inside the system will change x_2 and x_3, and therefore also their recursive estimations which have been used to construct $\hat{v}$. This could *a priori* lead to nonconvergence phenomena. We can see that, under suitable assumptions on the convergence of the various parameters $\tau, h, \tilde{h}$, and α, this can always be avoided and convergence is insured. For simplicity, we will illustrate this in full detail in the case when also $\mathcal{X}_2 = \{0\}$ and only one block of the variables of type x_3 is present; afterwards we will comment on the general case. Thus, we have the following system:

$$\begin{aligned}(x^1)' &= x^2 + k_1 y \qquad y = x_1\\ (x^2)' &= x^3 + k_2 y\\ &\vdots\\ (x^n)' &= v - u^{(S)} + k_n y + \sum_{i=1}^{n} g_i x^i + f_0.\end{aligned} \tag{35}$$

We recall that we have at our disposal the observation ξ_k approximating $y(\tau_k)$ with an error bounded by h. We work first on a compact interval [0, T] and we will eventually make $T \to +\infty$ in the case when A is exponentially stable. As before, $\|\cdot\|_{T,\infty}$ will denote the sup norm on [0, T]. We have

$$\begin{aligned}\|x^i\|_{T,\infty} &\le \mathcal{M}(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|)\\ \|(x^i)'\|_{T,\infty} &\le \mathcal{M}(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|).\end{aligned} \tag{36}$$

This, in particular, yields

$$\tilde{h}_1 = \|y - \xi\|_{T,\infty} \le \tau\mathcal{M}_1(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|) + h \tag{37}$$

where $\xi(t) = \xi_k$ if $t \in [\tau_k, \tau_{k+1})$.

If the original system matrix A is exponentially stable, then the constants $\mathcal{M}$ in (36) and (37) can be chosen independent of T since x^i are the components of the solution of the original system (1) just written in new coordinates.

We now consider the first equation in (35) with observations ξ_k (with observation error $h_1 = h$) and we think of the signal y entering the left-hand side as an approximately known signal, the approximation being given by ξ, see (37). Applying the estimation algorithm, with penalization parameter α_1, we obtain an estimation $\hat{x}^2$ of x^2 which, by Theorem 7 ii), satisfies the following estimate on an interval $[\sigma_2, T]$ with $\sigma_2 > 0$ and $\tau < \sigma_2/2$:

$$\begin{aligned}h_2 = \|\hat{x}^2 - x^2\|_{T,\infty} \le \mathcal{M}\Bigg[&\frac{\tau}{\alpha_1}\|x^2\|_{T,\infty} + \frac{h+\tau}{\alpha_1} + \tilde{h}_1\\ &+(\tau+\alpha_1)\|(x^2)'\|_{T,\infty} + \alpha_1\|x^2(0)\|\Bigg]\end{aligned} \tag{38}$$

(we used $e^{-\rho\sigma_2/2\alpha_1} < \mathcal{C}\alpha_1$). Using (36)–(38) we obtain (under the assumption $\alpha_1 < 1$)

$$\begin{aligned}h_2 = \|\hat{x}^2 - x^2\|_{T,\infty} \le \mathcal{M}_2\Bigg[&\left(\frac{\tau}{\alpha_1} + \alpha_1\right)(\|v - u\|_{T,\infty}\\ &+\|f_0\|_\infty + \|x_0\|) + \frac{h+\tau}{\alpha_1}\Bigg].\end{aligned} \tag{39}$$

We now proceed to the second equation, considering the output $y^2 = x^2$ and observations $\xi_k^2 = \hat{x}^2(\tau_k)$. Applying again the estimation algorithm after time σ_2 but with an a priori different penalization constant $\alpha_2 < 1$, we obtain an estimation $\hat{x}^3$ of x^3 which, again by Theorem 7 ii) and by (39), satisfies on an interval $[\sigma_3, T]$ with $\sigma_3 > \sigma_2$ and $\tau < (\sigma_3 - \sigma_2)/2$:

$$\begin{aligned}h_3 = \|\hat{x}^3 - x^3\|_{T,\infty} &\le \mathcal{M}\Bigg[\left(\frac{\tau}{\alpha_2} + \alpha_2\right)(\|v - u^{(S)}\|_{T,\infty}\\ &\quad+\|f_0\|_\infty + \|x_0\|) + \frac{\|\hat{x}^2 - x^2\|_\infty + \tau}{\alpha_2}\Bigg]\\ &\le \mathcal{M}_3\Bigg[\left(\frac{\tau}{\alpha_1\alpha_2} + \frac{\alpha_1}{\alpha_2} + \alpha_2\right)(\|v - u\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|)\\ &\quad+\frac{h+\tau}{\alpha_1\alpha_2}\Bigg].\end{aligned}$$

Now, we apply the estimation algorithm, successively, to the first $n-1$ equations (with penalization constants $\alpha_j < 1$). We obtain a sequence $0 < \sigma_2 < \sigma_3 < \cdots \sigma_{n-1}$ and estimations $\hat{x}^j$ such that, on each $[\sigma_j, T]$ and for $\tau < (\sigma_j - \sigma_{j-1})/2$, we have

$$\begin{aligned}h_j = \|\hat{x}^j - x^j\|_{T,\infty} \le \mathcal{M}_j\Bigg[&\Bigg(\frac{\tau}{\alpha_1\alpha_2\cdots\alpha_{j-1}}\\ &+\sum_{i=1}^{j-1}\frac{\alpha_i}{\alpha_{i+1}\cdots\alpha_{j-1}}\Bigg)\cdot(\|v - u^{(S)}\|_{T,\infty} + \|f_0\|_\infty + \|x_0\|)\\ &+\frac{h+\tau}{\alpha_1\alpha_2\cdots\alpha_{j-1}}\Bigg], \qquad j = 2, \ldots, n.\end{aligned}$$

We finally consider the last equation (after time σ_{n-1}) which we rewrite as

$$(x^n)' = -x^n + v - u + \left[k_n y + \sum_{i=1}^{n} g_i x^i + x^n + f_0\right]. \tag{40}$$

Here, we take observations $\xi^n = \hat{x}^n(\tau_k)$ and we interpret $[k_n y + \sum_{i=1}^n g_i x^i + x^n + f_0]$ as an approximately known input, the approximation being given by $[k_n\xi + \sum_{i=1}^n g_i\hat{x}^i + \hat{x}^n + \tilde{f}_0]$.

We are in the full state observation case with D onto and $A = -1$ exponentially stable. We apply our basic algorithm (with penalization constant α_n) and we obtain an input $u^{(S)}$. From Remark 2 and (40), $\|\hat{v}\|_{T,\infty} = \|u^{(S)}\|_{T,\infty}$ is less than

$$\mathcal{M}_{n+1}\left[\left(\frac{\tau}{\alpha_1\alpha_2\cdots\alpha_n}+\sum_{i=1}^{n-1}\frac{\alpha_i}{\alpha_{i+1}\cdots\alpha_n}\right)(\|v-u^{(S)}\|_{T,\infty}+\|f_0\|_\infty+\|x_0\|)+\frac{h+\tau}{\alpha_1\alpha_2\cdots\alpha_n}+\tilde{h}+\|v\|_\infty\right]. \quad (41)$$

Choose now $0 < \alpha < 1$ and $\alpha_j = \alpha^{2^{n-j}}$ for $j = 1,\ldots,n$. With this choice, we have that

$$\frac{\alpha_i}{\alpha_{i+1}\cdots\alpha_n} = \alpha \qquad \alpha_1\cdots\alpha_n = \alpha^{2^n-1}.$$

We can now state the following convergence result:

Theorem 10: Consider the system (35) with the input $u^{(S)}$ defined above. Let the interval $[\sigma, T]$, $0 \le \sigma < T < +\infty$ be fixed. For τ, h, $\tilde{h}$, and α converging to zero while respecting the consistency condition

$$\frac{\tau}{\alpha^{(2^n-1)}} \to 0 \quad \frac{h}{\alpha^{(2^n-1)}} \to 0 \quad (42)$$

the solution $x(t)$ of (35) converges *uniformly* on $[\sigma, T]$ to the solution x_{f_0} of the nominal system (i.e., (35) with $v - u^{(S)} = 0$). Moreover, if A is *exponentially stable* convergence holds on $[\sigma, +\infty)$.

Proof: Choose all $\sigma_j < \sigma$. From (41), under the conditions

$$\frac{\tau}{\alpha^{(2^n-1)}} < \frac{1}{4\mathcal{M}_{n+1}} \quad (n-1)\alpha < \frac{1}{4\mathcal{M}_{n+1}}$$
$$\frac{h}{\alpha^{(2^n-1)}} < 1 \quad \tilde{h} < 1 \quad (43)$$

we obtain the following bound (on $[\sigma_{n-1}, T]$):

$$\|\hat{v}\|_{T,\infty} = \|u^{(S)}\|_{T,\infty} \le \tilde{\mathcal{M}}_{n+1}[\|v\|_\infty + \|f_0\|_\infty + \|x_0\| + 1]. \quad (44)$$

In particular, we have that $u(t)$ remains uniformly bounded under (43).

It now follows from (40) and (44) that each $\hat{x}^j$ converges uniformly to x^j on $[\sigma_{n-1}, T]$ if the parameters τ, h, α all converge to 0 while satisfying conditions (42). We can now apply Theorem 4 and we obtain that x^n converges uniformly on $[\sigma_{n-1}, T]$ to the solution of

$$(x^n)' = k_n y + \sum_{i=1}^{n} g_i x^i + f_0.$$

This proves the first part of the result.

Consider now the case when A is exponentially stable. In this case, the constants $\mathcal{M}_{n+1}$, $\tilde{\mathcal{M}}_{n+1}$ are *independent* of T. In fact, if A is exponentially stable, the constant $\mathcal{M}$ in (36) and in (37) can be chosen to be independent of T. As a consequence, see Theorem 7, also the constants $\mathcal{M}_j$ in (40) can be taken independent of T. Finally, since (40) is exponentially stable in x^n, Remark 2 shows that also $\mathcal{M}_{n+1} = \mathcal{C}_{n+1}$ will be independent of T. Conditions (43) therefore become independent of T and (44) can be reformulated as

$$\|\hat{v}\|_\infty = \|u^{(S)}\|_\infty \le \tilde{\mathcal{C}}_{n+1}[\|v\|_\infty + \|f_0\|_\infty + \|x_0\| + 1] \quad (45)$$

which in particular shows that $\hat{v}$ is bounded also on $[0, +\infty)$, a fact which was not *a priori* evident.

With this stronger bound we can now easily prove convergence on $[\sigma, +\infty)$, repeating the above arguments. ∎

When more than one block is present, the previous considerations can be repeated for each block: in each block we will first perform the recursive estimations of the state variables of every block and we will finally apply the disturbance reduction algorithm simultaneously to the final equations of each block (where all the state variables in principle may appear). Finally if $\mathcal{X}_2 \neq \{0\}$, we will also need to perform recursive estimations of the state variables in those blocks to be used in $\mathcal{X}_3$. We omit the lengthy but straightforward formal convergence results.

We sum up as follows.

1) If $\mathcal{V}_* = 0$, the proposed algorithm approximately cancels the disturbance v, without changing the free dynamics of the system.
2) If we only have $\mathcal{R}_* = 0$ and the system is minimum-phase, the proposed algorithm approximately cancels v while introducing a dynamical modification, exponentially decaying for $t \to +\infty$.

A comment about the robustness of the method: we proved robustness in the case when every component of the state is observed; hence, every single step in the general case is robust. In spite of this, lack of robustness can derive from the representation of the system in Morse form, but algorithms have been proposed for the identifications of the subspaces which are relevant for Morse form [13], [6]. However, in many cases the intrinsic properties of the subspaces $\mathcal{X}_i$ imply that the reduction of the system to a chain of integrators is not so difficult, the simplest case being the case in which the system is directly obtained as an n–order differential equation.

VI. Few Simulations

In this section, we present few numerical simulations. We are not going to present a numerical discussion of the method. We simply note that the recursivity of the algorithm avoids the inversion of large Toeplitz matrices. Moreover, in practical cases numerical instability and accumulation of errors can occur which can be obviated with some tricks, like the introduction of saturations on $u^{(M)}$. In spite of this, we just use the proposed algorithm, without any adaptation since we believe that this is matter for practitioners. The simulations concerns the two guiding examples of Section II.

A. Robot Motor

We consider the example illustrated in Section II-A. As we noted, the acceleration control insure tracking of the input signal if there is no disturbance, or if the disturbance fades away. Otherwise, the error $X^{\mathrm{cmd}} - X$ persists. We apply our method in order to reduce this error. We consider the case that we can read the velocity signal sX and we consider the disturbance due to the addition of transported load. For simplicity we assume that

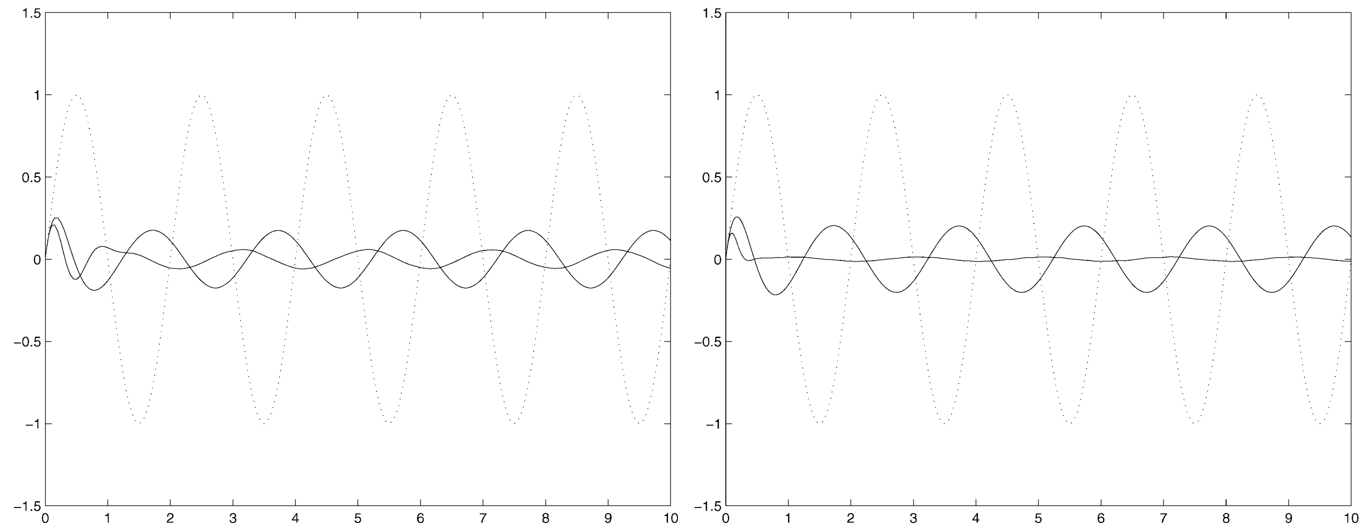


Fig. 1. Motor. $\tau = 0.05, \alpha = 0.06, h = 0.1$ (left), $\tau = 0.01, \alpha = 0.01, h = 0.1$(right).

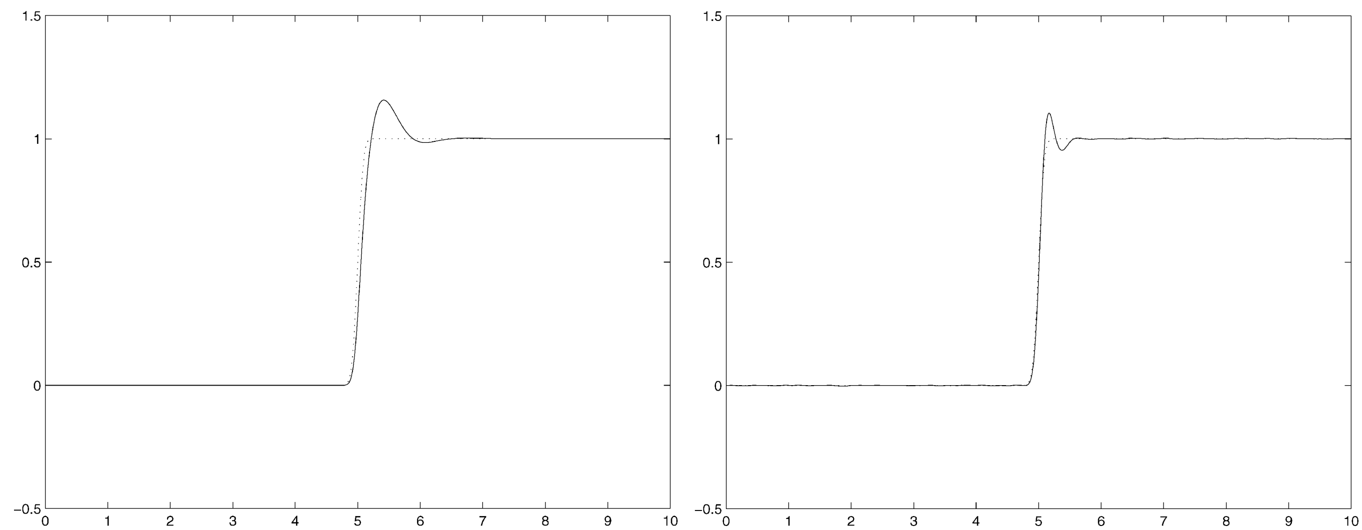
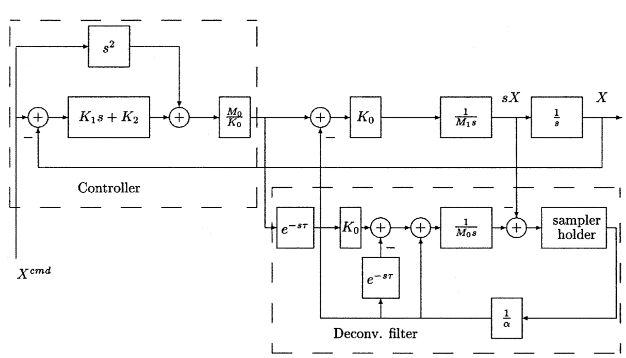


Fig. 2. Motor. $\tau = 0.01, \alpha = 0,01, h = 0.1$.

the mass is added at time $t = 0$, so that the system and its model have different masses (mass $M_0 = 1$ for the nominal mass, used in the model equation, mass $M_1 = 3$ for the "real" mass transported by the harm).

The addition of our disturbance reduction device transforms the system described in Section II-A as follows:



The dashed box marked "controller" produces the control input f_0, a feedback control, which is fed both to the system and, with one step delay, to the compensator (which is the dashed box marked "deconv. filter").

We recall that the input signal is now the track to be followed.

We present plots of the simulations. In this example it is most effective to plot the desired track (dotted) against the errors committed by the motor with and without compensator.

Fig. 1 presents X^{cmd} and the errors $X^{\text{cmd}} - X$ without and with the compensator, for different values of τ (and consequently of α).

We see that, in the first simulation, after a short transient, the action of the compensator reduces the error of about 50%. In the second one, tracking is quite good, after a short transient.

Fig. 2 refers to the track of a path X^{cmd} exhibiting an abrupt change, essentially $\text{sign}(t-5)$. We show the path X^{cmd} (dotted) and the paths X without compensator on the left, and with compensator on the right. In order to insure a fast tracking in the rising part of the trajectory, we choose $\tau = 0.01$.

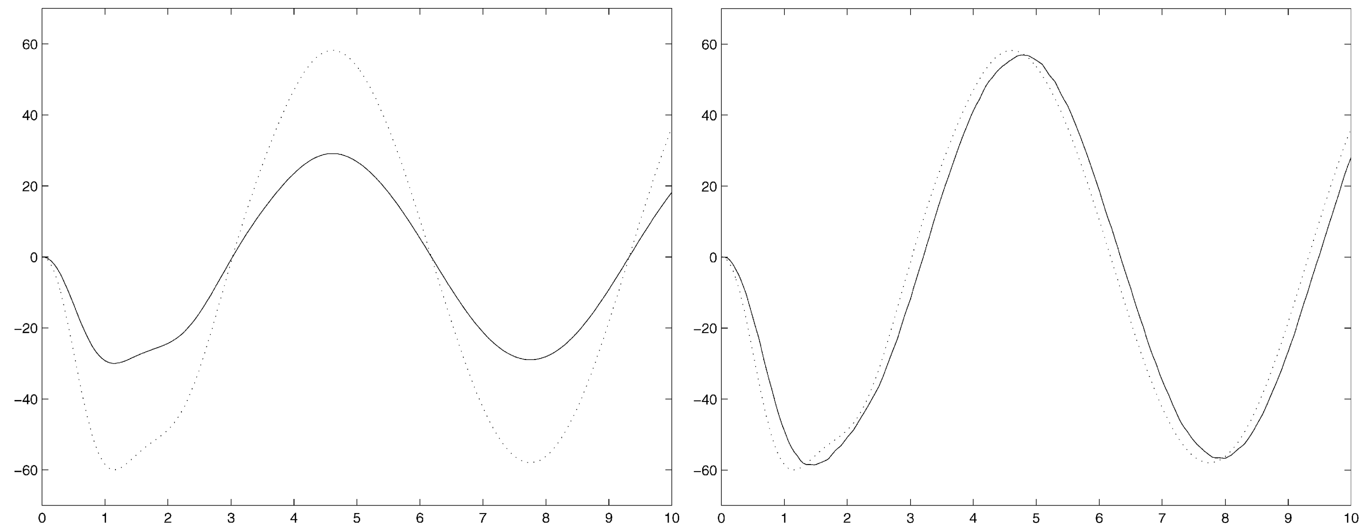


Fig. 3. Actuator fault. $\tau = 0.1$, $\alpha = 0.324$, $h = 1$.

The effect of the observation noise is filtered out by the system.

B. Fault Detection and Compensation

Now, we examine the second example, from fault detection. It is easily seen that the transfer function is

$$T(s) = \frac{-175.6s^2 - 3950s - 14\,540}{s^3 + 15.74s^2 + 43.6s + 269.1}.$$

Hence, the system is both controllable and observable; it is stable and it has *two stable zeros* which are -17.8747 and -4.6196. Hence it is minimum phase. We use this and the fact that inputs and outputs are scalar and we see that its Morse form has two blocks: the 2×2 block of the stable zeros and the last block which is 1×1. Hence, we are in **case 2** described in Section V and we can correct the effect of the fault in the actuator, at the expenses of a dynamical modification of the system, whose effect decays exponentially.

It is easy to reduce the system to its Morse form: we apply the coordinate transformations

$$T = \begin{bmatrix} -0.0057 & 0.0708 & 0 \\ 0 & 1 & 0 \\ 1 & -12.43 & -12.5429 \end{bmatrix}.$$

We have $CT^{-1} = [-175.6 \quad 0 \quad 0]$ while $T^{-1}AT, TB$ are

$$\begin{bmatrix} 6.8 & -0.7 & 0.1 \\ -304.2 & 7.3 & -2.4 \\ -3644.7 & 124.7 & -29.8 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We then apply the feedback F and the output injections K given by

$$F = [0 \quad 0.7101 \quad -0.0902] \qquad K = \begin{bmatrix} 0.0385 \\ -1.7325 \\ -20.7556 \end{bmatrix}.$$

These transformations reduce the system to the following form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 7.3467 & -2.4221 \\ 0 & 124.6962 & -29.8436 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$[-175.6 \quad 0 \quad 0].$$

Once the system is in this form we apply the procedure described in Section V **case 2**. Now, the model system has two blocks, as follows:

$$\hat{x}_0' = \begin{bmatrix} 7.3467 & -2.4221 \\ 124.6962 & -29.8436 \end{bmatrix} \hat{x}_0 - K_0 y$$
$$\hat{x}_3' = \{\omega_3 - u_3^M\} - K_3 y + f_0(t - \tau).$$

The signal y enters into the first block and the output is applied to the second one, throughout the term D_3Fx, indistinguishable from the disturbance. The second block receives also the injection $-K_3y$ which is known. Hence, our method gives an estimate of $v - F_3x$ where F_3 is the last row of the feedback F. This is fed to the system, so to remove the effect of the disturbance v.

Now, we present the following simulations. We want to track the output produced by an input signal f_0 in spite of the fault of the actuator ($\zeta = 1/2$, constant for simplicity). We consider the case when f_0 is a sinusoidal signal and when it is a piecewise linear signal.

The case that the input is sinusoidal, $f_0(t) = \sin t$, is in Fig. 3. The plot on the left shows $y^{\rm nom}$ (dotted) and the actual track y, without compensator. The plot on the right shows instead $y^{\rm nom}$ and y when the compensator is active.

Fig. 4 presents the analogous plots, in the case that $f_0(t) = t/3$ for $0 \le t \le 3$ and $f_0(t) = 1$ for $t \ge 3$.

VII. Conclusion

In this paper, we presented an algorithm for the reduction of the effect of disturbances on the output of a linear system. We used a recursive deconvolution method to estimate the disturbance which is then subtracted from the input of the system. An

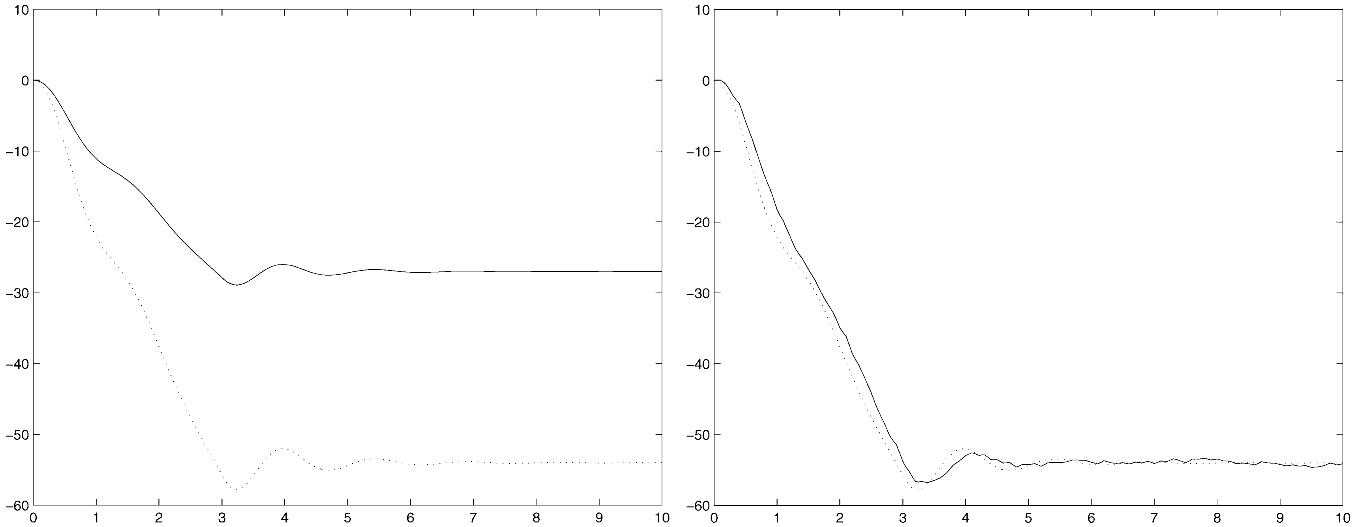


Fig. 4. Actuator fault. $\tau = 0.1$, $\alpha = 0.324$, $h = 1$.

adaptive compensator is thus obtained which, in particular, does not act in the absence of the disturbance.

The system that we studied evolves in continuous time and we considered the case that the data are available only at discrete time instants. We presented an analysis of the effect that sampling has on the algorithm.

The simulations that we presented show that the method is effective, in accordance which many similar simulations which can be found in the literature, see for example the references that we cited.

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