Democracy and the role of minorities
in Markov chain models
Non-reversible perturbations of Markov chains models

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Perturbation of dynamical networks

The stability of a complex large-scale dynamical network under localized perturbations is one of the paradigmatic problem of these decades.

Key issues:

- **Correlation:** understand how local perturbation affect the overall behavior.
- **Resilience** find bounds on the perturbation ’size’ which the network can tolerate.
- **Phase transitions**
Perturbation of dynamical networks

State of the art:

- Most of the results available in the literature are on connectivity issues.
- Analysis of how the perturbation is altering the degree distribution of the network.
- Degrees are in general not sufficient to study dynamics.
- Example: non-reversible Markov chain models.
Perturbation of dynamical networks

What type of perturbations:

▶ Failures in nodes or links in sensor or computer networks. Sensor with different technical properties.
▶ Heterogeneity in opinion dynamics models: minorities, leaders exhibiting a different behavior
▶ A subset of control nodes in the network...

In this talk:

▶ Non-reversible perturbations of Markov chain models
▶ Applications to consensus dynamics
Outline

- Perturbation of consensus dynamics.
- The general setting: perturbation of Markov chain models.
- An example: heterogeneous gossip model.
- Results on how the perturbation is affecting the asymptotics.
- Conclusions and open issues.
Consensus dynamics

\[ G = (V, E) \text{ connected graph} \]

\[ y_v \text{ initial state (opinion) of node } v \]

Dynamics: \[ y(t + 1) = Py(t), \quad y(0) = y \]
\[ P \in \mathbb{R}^{V \times V} \text{ stochastic matrix on } G \quad (P_{uv} > 0 \iff (u, v) \in E) \]

Consensus: \[ \lim_{t \to +\infty} (P^t y)_u = \pi^* y \text{ for all } u \quad (* \text{ means transpose}) \]
\[ \pi \in \mathbb{R}^V_+, \quad \pi^* P = \pi^*, \quad \sum_u \pi_u = 1 \quad \text{(invariant probability)} \]
Consensus dynamics

\[ G = (V, E) \text{ connected graph} \]

**Example:** (SRW) \( P_{uv} = \frac{1}{d_u}, d_u \text{ degree of node } u \)

Explicit expression for \( \pi \): \( \pi_u = \frac{d_u}{2|E|} \)

\( \pi \) essentially depends on local properties of \( G \).

This holds true for general reversible Markov chains.
Consensus dynamics

\[ G = (V, E) \text{ connected graph} \]

Dynamics: \[ y(t + 1) = Py(t), \ y(0) = y \]
Consensus: \[ \lim_{t \to +\infty} (P^t y)_u = \pi^* y \text{ for all } u \]

Two important parameters:
- the invariant probability \( \pi \) responsible for the asymptotics
- the mixing time \( \tau \) responsible for the transient behavior (speed of convergence)
Perturbation of consensus dynamics

\[ G = (V, E) \] connected graph

- \( P \in \mathbb{R}^{V \times V} \) stochastic matrix on \( G \)

- Perturb \( P \) in a small set of nodes:
  \( \tilde{P}_{uv} = P_{uv} \) if \( u \notin W = \{w_1, w_2, w_3\} \).
Perturbation of consensus dynamics

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\[ P \in \mathbb{R}^{V \times V} \text{ stochastic matrix on } G \]

Perturb \( P \) in a small set of nodes:

\[ \tilde{P}_{uv} = P_{uv} \text{ if } u \notin W = \{w_1, w_2, w_3\}. \]

Cut edges
Perturbation of consensus dynamics

\[ G = (V, E) \text{ connected graph} \]

- \( P \in \mathbb{R}^{V \times V} \) stochastic matrix on \( G \)
- Perturb \( P \) in a small set of nodes:
  \[ \tilde{P}_{uv} = P_{uv} \text{ if } u \notin W = \{w_1, w_2, w_3\}. \]
- Cut edges. Add new edges.
A heterogeneous gossip model
(Acemoglu et al. 2009)

\[ G = (V, E), \ W \subset V \text{ a minority of \textit{influent} (stubborn) agents} \]

- At each time \( t \) choose an edge \( \{u, v\} \) at random.
- If \( u, v \in V \setminus W \),
  \[ y_u(t + 1) = y_v(t + 1) = (x_u(t) + x_v(t))/2 \] (reg. interaction)
A heterogeneous gossip model
(Acemoglu et al. 2009)

\[ G = (V, E), \ W \subset V \text{ a minority of influential (stubborn) agents} \]

- At each time \( t \) choose an edge \( \{u, v\} \) at random.
- If \( u \in W, v \in V \setminus W \),
  - \( y_u(t + 1) = y_u(t), y_v(t + 1) = \varepsilon y_v(t) + (1 - \varepsilon)y_u(t) \)
    with probability \( p \) (forceful interaction)
  - \( y_u(t + 1) = y_v(t + 1) = (x_u(t) + x_v(t))/2 \)
    with probability \( 1 - p \) (reg. interaction)
A heterogeneous gossip model

(Acemoglu et al. 2009)

\[ G = (V, E), \ W \subset V \] a minority of influential (stubborn) agents

- At each time \( t \) choose an edge \( \{u, v\} \) at random.
- If \( u, v \in W \) nothing happens.
A heterogeneous gossip model

- \( y(t + 1) = P(t)y(t) \)
- \( y(t) \) converges to a consensus almost surely if \( p \in [0, 1) \).
  But what type of consensus?
- If no forceful interaction is present (\( p = 0 \)),
  \( y(t)_u \rightarrow N^{-1} \sum_v y(0)_v \) for every \( u \).
- \( \mathbb{E}(P(t)) = P_p \)
- \( P_p \) and \( P_0 \) only differ in the rows having index in \( W \cup \partial W \).
Perturbation of Markov chain models

The abstract setting:

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on $G$. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \tilde{\pi}^*, \tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.
- Study $||\pi - \tilde{\pi}||_{TV} := \frac{1}{2} \sum_v |\pi_v - \tilde{\pi}_v|$ (as a function of $N$)
  Notice that $|\tilde{\pi}^* y - \pi^* y| \leq ||\pi - \tilde{\pi}||_{TV} ||y||_\infty$

The ideal result: $\pi(W) \to 0 \Rightarrow ||\tilde{\pi} - \pi||_{TV} \to 0$
A counterexample

\[ P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi \text{ uniform} \]
A counterexample

\[ P_{u,u+1} = P_{u,u-1} = 1/2, \quad \pi \text{ uniform} \]

\[ \tilde{P}_{1,2} = 1, \quad \tilde{P}_{1,n} = 0, \quad \tilde{\pi}_1 = 1/n, \quad \tilde{\pi}_j = \frac{2(n-j+1)}{n^2} \text{ for } j \geq 1 \]

\[ ||\pi - \tilde{\pi}||_{TV} \lesssim \text{cost.} \]
Perturbation of Markov chain models

- \( G = (V, E) \) family of connected graphs. \( N = |V| \to +\infty \).
- \( W \subseteq V \) perturbation set
- \( P, \tilde{P} \) stochastic matrices on \( G \). \( \tilde{P}_{uv} = P_{uv} \) if \( u \notin W \)
- \( \pi^*P = \tilde{\pi}^*, \tilde{\pi}^*\tilde{P} = \tilde{\pi}^* \).

If the chain mixes slowly, the process will pass many times through the perturbed set \( W \) before getting to equilibrium. \( \tilde{\pi} \) will be largely influenced by the perturbed part.

Consequence: \( ||\pi - \tilde{\pi}||_{TV} \not\to 0 \)
Perturbation of Markov chain models

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on $G$. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^*P = \pi^*$, $\tilde{\pi}^*\tilde{P} = \tilde{\pi}^*$.

A more realistic result:

$P$ mixes suff. fast, $\pi(W) \to 0 \Rightarrow \tilde{\pi} - \pi \to 0$

Recall: mixing time $\tau := \min\{t | ||\mu^*P^t - \pi^*||_{TV} \leq 1/e \ \forall \mu\}$

SRW on $d$-grid with $N$ nodes, $\tau \asymp N^{2/d} \ln N$

SRW on Erdos-Renji, small world, configuration model $\tau \asymp \ln N$
Perturbation of Markov chain models: the literature

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on $G$. $\tilde{P}_{uv} = P_{uv}$ if $u \not\in W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

$$||\tilde{\pi} - \pi||_{TV} \leq C\tau||\tilde{P} - P||_1$$ (Mitrophanov, 2003)

To measure perturbations of $P$, the 1-norm is not good to treat localized perturbations: if $P$ and $\tilde{P}$ differ just in one row $u$ and $|P_{uv} - \tilde{P}_{uv}| = \delta$, then, $||P - \tilde{P}||_1 \geq \delta$ and will not go to 0 for $N \to \infty$. In our context, the bound will always blow up.
Perturbation of Markov chain models

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on $G$. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*, \tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

A more realistic result:

$P$ mixes suff. fast, $\pi(W) \to 0 \Rightarrow \tilde{\pi} - \pi \to 0$

There is another problem: if $P$ mixes rapidly, nobody guarantees that $\tilde{P}$ will also do...
Perturbation of Markov chain models: first result

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $W \subseteq V$ perturbation set
- $P, \tilde{P}$ stochastic matrices on $G$. $\tilde{P}_{uv} = P_{uv}$ if $u \notin W$
- $\pi^* P = \pi^*$, $\tilde{\pi}^* \tilde{P} = \tilde{\pi}^*$.

Theorem

$$||\tilde{\pi} - \pi||_{TV} \leq \tau\tilde{\pi}(W) \log \frac{e^2}{\tau\tilde{\pi}(W)}.$$ (1)

or, symmetrically,

$$||\tilde{\pi} - \pi||_{TV} \leq \tilde{\pi}(W) \log \frac{e^2}{\tilde{\tau}\pi(W)}.$$ (2)

Proof: Coupling technique.
Perturbation of Markov chain models: first result

Corollary
\[ \tau \pi(W) \to 0, \ \tilde{\tau} = O(\tau) \implies ||\tilde{\pi} - \pi||_{TV} \to 0 \]
\[ \tau \pi(W) \to 0, \ \tilde{\pi}(W) = O(\pi(W)) \implies ||\tilde{\pi} - \pi||_{TV} \to 0 \]

The perturbation, in order to achieve a modification of the invariant probability, necessarily has to

- slow down the chain
- increase the probability on the perturbation subset \( W \).

\( \pi \) and \( \tau \) are intimately connected to each other!
Perturbation of Markov chain models: first result

Slowing down the chain and putting weight on $W$ look quite connected to each other and essentially amounts to decrease the probability of exiting $W$:

$$\tilde{P}_{ww} = 1 - 1/N$$

$$\tilde{\pi}_w = \mathbb{E}(\tilde{T}_w^+)^{-1} = \frac{1}{1 - N^{-1} + N^{-1}\mathbb{E}(T_w^+)} = \frac{\pi_w}{(1-N^{-1})\pi_w + N^{-1}}$$

$$\pi_w \sim \frac{k}{N} \Rightarrow \tilde{\pi}_w \sim \frac{k}{k+1}$$
A deeper analysis

Lemma

\[ \tilde{\pi}(W) \leq \frac{1}{1 + \tilde{\phi}_W^* \tau_W^*}, \]

where

\[ \tau_W^* := \min \{ \mathbb{E}_v [T_W] : v \in V \setminus W \}, \]

minimum entrance time to \( W \)

it depends on \( P \)

\[ \tilde{\phi}_W^* := \frac{\sum_{w \in W} \sum_{v \in V \setminus W} \tilde{\pi}_w \tilde{P}_{wv}}{\tilde{\pi}(W)}, \]

bottleneck ratio of \( W \)

it depends on \( \tilde{P} \)

Proof From Kac's lemma

\[ \tilde{\pi}(W)^{-1} = \mathbb{E}_{\tilde{\pi}_W} [T_W^+] = 1 + \sum_w \sum_v \frac{\tilde{\pi}_w}{\tilde{\pi}(W)} \tilde{P}_{wv} \mathbb{E}_v [T_W] \geq 1 + \tilde{\phi}_W^* \tau_W^*. \]
A deeper analysis

- bottleneck ratio $\leftrightarrow$ exit probability from $W$:

$$\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha \ \forall w \in W \Rightarrow \tilde{\phi}^*_W \geq d/\alpha$$

If $\mathbb{P}_w(T_{V \setminus W} \leq d) \geq \alpha$ for fixed $d, \alpha > 0$ and for every $w \in W$, then

$$\tilde{\pi}(W) \leq \frac{1}{1 + \tilde{\phi}^*_W \tau^*_W} \asymp (\tau^*_W)^{-1}, \quad \tau \tilde{\pi}(W) = O\left(\frac{\tau}{\tau^*_W}\right)$$

Hence,

$$\frac{\tau}{\tau^*_W} \to 0 \Rightarrow ||\tilde{\pi} - \pi||_{TV} \to 0$$
A deeper analysis

- minimum entrance time $\leftrightarrow \pi(W)$:

  \[
  \tau^*_W := \min \{E_v[T_W]\} \sim \pi(W)^{-1} \quad \text{(Conjecture)}
  \]

(Kac’s lemma: $\pi(W)^{-1} = 1 + \sum_w \sum_v \frac{\pi_w}{\pi(W)} P_{vw} E_v[T_W]$
\[\Rightarrow E_v[T_W] \sim \pi(W)^{-1} \text{ for some } v\ldots\])

\[
\mathbb{P}_w(T_{V\setminus W} \leq d) \geq \alpha \text{ for every } w \in W \text{ plus conjecture imply}
\]

\[
\frac{\tau}{\tau^*_W} \sim \tau \pi(W) \to 0 \Rightarrow ||\tilde{\pi} - \pi||_{TV} \to 0
\]
Examples

The conjecture

\[ \tau^*_W := \min \{ \mathbb{E}_v [T_W] \} \preceq \pi(W)^{-1} \]

holds if \( P \) is the simple random walk SRW on

- \( d \)-grids with \( d \geq 3 \), \( |W| \) bounded.
- Erdos-Renji, configuration model (w.p. 1) if \( |W| = o(N^{1-\epsilon}) \)

(techniques: electrical network interpretation, effective resistance; locally tree-like graphs)

Recall that

- \( d \)-grid, \( \tau \preceq N^{d/2} \ln N \)
- Erdos-Renji, configuration model (w.p. 1) \( \tau = O(\ln N) \)
Examples

Theorem

- $G = (V, E)$ family of connected graphs. $N = |V| \to +\infty$.
- $\mathcal{W} \subseteq V$ perturbation set

- $\mathcal{P}$ SRW on $G$, $\tilde{P}_{uv} = P_{uv}$ if $u \notin \mathcal{W}$
- $\pi^* \mathcal{P} = \pi^*$, $\tilde{\pi}^* \tilde{\mathcal{P}} = \tilde{\pi}^*$.
- $\mathbb{P}_w( T_{V \setminus \mathcal{W}} \leq d ) \geq \alpha$ for fixed $d$, $\alpha > 0$ and for every $w \in \mathcal{W}$,

If $G$ and $\mathcal{W}$ are:

- $d$-grids with $d \geq 3$, $|\mathcal{W}|$ bounded
- Erdos-Renji, configuration model (w.p. 1), $|\mathcal{W}| = o(N^{1-\epsilon})$

then,

$$||\tilde{\pi} - \pi||_{TV} \to 0$$
Application to the heterogeneous gossip model

- \( G = (V, E), \ W \subseteq V \) forceful agents (with prob. \( p \)).
- \( y(t + 1) = P(t)y(t), \quad \mathbb{E}(P(t)) = P_p \)
- \( P_p \) and \( P_0 \) only differ in the rows having index in \( W \cup \partial W \).

A specific example: \textit{d-regular (toroidal) grid}.

\( P_0 = (1 - N^{-1})\text{Id} + N^{-1}d^{-1}A_G \) is a lazy simple random walk,

\( \pi_0 \) uniform probability, \( \tau_0 \asymp N^{2/d+1} \ln N, \quad \tau^*_W \asymp \frac{|W|}{N^2} \)

\( \frac{\tau_0}{\tau^*_W} \asymp \frac{|W|N^{2/d-1} \ln N \to 0 \text{ if } d \geq 3, \ |W| \text{ bounded}} \)

\[ \|\pi_p - \pi_0\|_{TV} \to 0 \]

- the minority has a \textbf{vanishing effect} on the global population
- \( \max_v(\pi_p)_v \to 0 \) democracy is preserved ('wise society' in Jackson’s terminology)
Gossip with stubborn agents

Take $p = 1$ in the heterogeneous gossip model.

$G = (V, E)$, $W \subset V$ a minority of influential (stubborn) agents

- At each time $t$ choose an edge $\{u, v\}$ at random.
- If $u, v \in V \setminus W$,
  \[ y_u(t+1) = y_v(t+1) = \frac{x_u(t) + x_v(t)}{2} \]
- If $u \in W$, $v \in V \setminus W$,
  \[ y_u(t+1) = y_u(t), \ y_v(t+1) = \frac{y_v(t) + y_u(t)}{2} \]
Gossip with stubborn agents

(Acemoglu, Como, F., Ozdaglar)

- \( y(t) \to y(\infty) \) in distribution. (\( y_w(\infty) = y_w(0) \ \forall w \in \mathcal{W} \))

- If \( \exists w, w' \in \mathcal{W} : y_w(0) \neq y_{w'}(0) \), then,
  \[ \mathbb{P}(y_\mathcal{V}(\infty) \neq y_{\mathcal{V}'}(\infty)) > 0 \] (asymptotic disagreement)

- However \( \frac{1}{n} \left| \left\{ \mathcal{V} : \left| \mathbb{E}[y_\mathcal{V}(\infty)] - \xi \right| \geq \varepsilon \right\} \right| \leq C_\varepsilon \tau_\Pi(W) \)

  If \( \tau_\Pi(W) \to 0 \), then approximate consensus!

This can also be read as a sort of lack of controllability: constraints on the shape of the final state configuration achievable by the global system.
Conclusions and open issues

- Perturbations of Markov chain models and their effect on the invariant probabilities.
- If the mixing time is sufficiently small w.r. to the size of the perturbation, the effect on the invariant probability becomes negligible in the large scale limit.
- Applications to consensus dynamics
  - Find more general estimation of the minimum entrance time parameter $\tilde{\tau}_W^*$.
  - Find estimation of type $c_1 \leq \tilde{\pi}_v / \pi_v \leq c_2$. They would permit to obtain estimations of $|\tilde{\tau} - \tau|$.
- Study phase transitions.
- Consider perturbations of non linear models (consensus versus epidemic, threshold models).