On the achievable capacity of group codes over symmetric channels

ITA Workshop 2007

Open problems session

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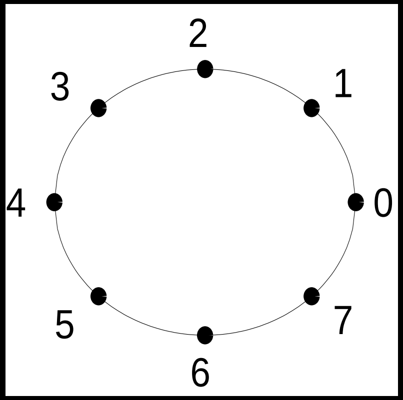
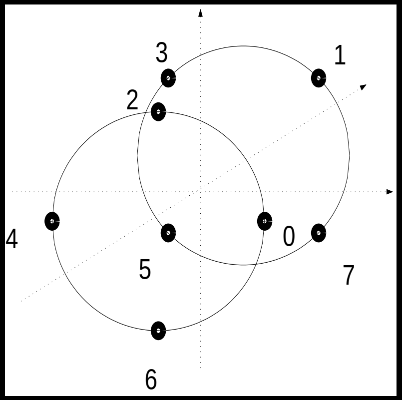
Symmetric Gaussian channels

- $S \subseteq \mathbb{R}^q$ GEOMETRICALLY UNIFORM if

$$s_0, s_1 \in S \Rightarrow \exists g \in \text{Iso}(S) : gs_0 = s_1$$

- Example: 8-PSK and $8h$-PSK constellations





- S-AWGN CHANNEL

$$s \in S \mapsto s + N(0, \sigma^2 I)$$

GU codes

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GU CODE over S of length N: $\mathcal{C} \subseteq S^N$ GU

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PROBLEM A:

Do GU codes suffice to achieve the capacity of the S-AWGN channel?

Generating groups

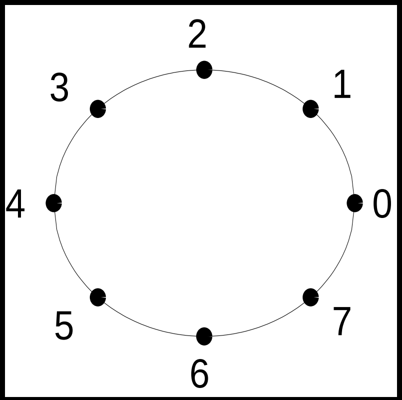
$S \subseteq \mathbb{R}^q$ GU

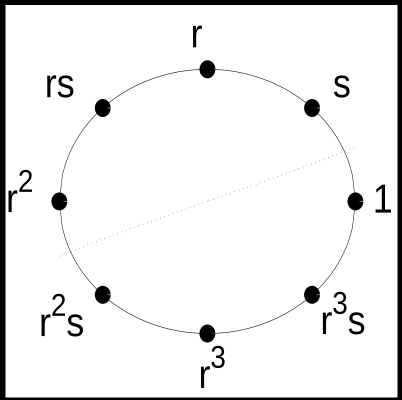
- $G \leq \mathrm{Iso}(S)$ GENERATING GROUP for S if

$$s_0, s_1 \in S \Rightarrow \exists! g \in G : gs_0 = s_1$$

- ISOM. LABELLING: $\mu : G \leftrightarrow S,\ \mu(g) = gs_0;$

- Ex: 8–PSK: $G = \mathbb{Z}_8$ OR $G = D_4$





G-codes

$S \subseteq \mathbb{R}^q$ GU, G gen. group, $\mu : G \to S$ labelling.

G-CODE: $\mathcal{X} \leq G^N$ subgroup

G-codes are special GU codes:

$\mathcal{X} \leq G^N, \quad \mu : \mathcal{X} \mapsto \mu(\mathcal{X}) \subseteq S^N$ GU code.

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PROBLEM B:

Does there exist a generating group G for S such that G-codes allow to achieve the capacity of the S-AWGN channel?

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PROBLEM C:

For a fixed generating group G for S, do G-codes allow to achieve the capacity of the S-AWGN channel?

Some history

- Group codes appear in 1968 (Slepian)
- Widely studied during the 90's (Forney, Loeliger, Trott)
- They yield distance profiles independent of the reference point
- They yield the UEP
- Recent research on their use for high performance coding schemes
- Problem C was conjectured by Loeliger in 1991

Some answers to problem C

- 2-PAM AWGN channel and $G = \mathbb{Z}_2$:
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- p^r-PSK AWGN channel and $G = \mathbb{Z}_{p^r}$:
  YES (G. Como's bach. thesis 2004)

Some answers to problem C

- S-PSK AWGN channel and G Abelian:

Formula for capacity of G-codes (Como-F. 2005)

(Case $G = \mathbb{Z}_{p^r}$)

$$C_{\mathbb{Z}_{p^r}} := \min_{l=1}^{r} \frac{r}{l} C_l \leq C_r \,.$$

S_l sub-constellation associated with $p^{r-l}\mathbb{Z}_{p^r}$

C_l Shannon capacity of the S_l-AWGN channel.

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- $8h$-PSK AWGN channel and $G = \mathbb{Z}_8$:
  - $C_{\mathbb{Z}_{p^r}} < C_r \;\Rightarrow\;$ NO
  - What about D_4?

Open problems

- Analysis for non-Abelian generating groups.
- Semidirect group structures.
- Error exponents, minimum distances.