On the achievable capacity of group codes over symmetric channels

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Open problems session

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Symmetric Gaussian channels

- $S \subseteq \mathbb{R}^q$ **GEOMETRICALLY UNIFORM** if
  
  \[ s_0, s_1 \in S \Rightarrow \exists g \in \text{Iso}(S) : gs_0 = s_1 \]

- **Example:** 8-PSK and 8h-PSK constellations

- **$S$-AWGN CHANNEL**
  
  \[ s \in S \mapsto s + N(0, \sigma^2 I) \]
GU codes

$S \subseteq \mathbb{R}^q$ GU constellation.

GU CODE over $S$ of length $N$: $\mathcal{C} \subseteq S^N$ GU
GU codes

$S \subseteq \mathbb{R}^q$ GU constellation.

GU CODE over $S$ of length $N$: $C \subseteq S^N$ GU

PROBLEM A:

Do GU codes suffice to achieve the capacity of the $S'$-AWGN channel?
Generating groups

$S \subseteq \mathbb{R}^q$ GU

- $G \leq \text{Iso}(S)$ **GENERATING GROUP** for $S$ if
  
  $s_0, s_1 \in S \Rightarrow \exists! g \in G : gs_0 = s_1$

- **ISOM. LABELLING**: $\mu : G \leftrightarrow S, \mu(g) = gs_0$

- **Ex**: 8–PSK: $G = \mathbb{Z}_8$ OR $G = D_4$

- \[ \begin{align*}
  &3 \quad 2 \quad 1 \\
  &4 \quad 5 \quad 6 \\
  &7 \quad 0
\end{align*} \quad \begin{align*}
  &r \quad rs \\
  &r^2 \quad r^3 \\
  &r^2s \quad r^3s
\end{align*} \]
$G$-codes

$S \subseteq \mathbb{R}^q$ GU, $G$ gen. group, $\mu : G \rightarrow S$ labelling.

**$G$-CODE:** $\mathcal{X} \leq G^N$ subgroup

$G$-codes are special GU codes:

$\mathcal{X} \leq G^N, \quad \mu : \mathcal{X} \mapsto \mu(\mathcal{X}) \subseteq S^N$ GU code.
\(G\text{-codes}\)

\[ S \subseteq \mathbb{R}^q \text{ GU, } G \text{ gen. group, } \mu : G \rightarrow S \text{ labelling.} \]

\(G\text{-CODE:} \; \mathcal{X} \leq G^N \text{ subgroup}\)

\(G\text{-codes are special GU codes:}\)

\[ \mathcal{X} \leq G^N, \; \mu : \mathcal{X} \mapsto \mu(\mathcal{X}) \subseteq S^N \text{ GU code.} \]

**PROBLEM B:**

Does there exist a generating group \(G\) for \(S\) such that \(G\text{-codes}\) allow to achieve the capacity of the \(S\text{-AWGN channel}\)?
**$G$-codes**

$S \subseteq \mathbb{R}^q$ GU, $G$ gen. group, $\mu : G \rightarrow S$ labelling.

**$G$-CODE:** $\mathcal{X} \leq G^N$ subgroup

$G$-codes are special GU codes:

$\mathcal{X} \leq G^N$, $\mu : \mathcal{X} \mapsto \mu(\mathcal{X}) \subseteq S^N$ GU code.

**PROBLEM C:**

For a fixed generating group $G$ for $S$, do $G$-codes allow to achieve the capacity of the $S$-AWGN channel?
Some history

- Group codes appear in 1968 (Slepian)
- Widely studied during the 90’s (Forney, Loeliger, Trott)
- They yield distance profiles independent of the reference point
- They yield the UEP
- Recent research on their use for high performance coding schemes
- Problem C was conjectured by Loeliger in 1991
Some answers to problem C

- 2-PAM AWGN channel and $G = \mathbb{Z}_2$: YES (Classical 60’s).
Some answers to problem C

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- \( p \)-PSK AWGN channel (\( p \) prime) and \( G = \mathbb{Z}_p \): YES (Easy extension)
Some answers to problem C

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  - YES (Easy extension)

- **\( p^r \)-PSK AWGN** channel and \( G = \mathbb{Z}_{p^r} \):
  - YES (G. Como’s bach. thesis 2004)
Some answers to problem C

- **$S$-PSK AWGN channel and $G$ Abelian:**

  Formula for capacity of $G$-codes (Como-F. 2005)

  \[
  (\text{Case } G = \mathbb{Z}_{p^r}) \quad C_{\mathbb{Z}_{p^r}} := \min_{l=1}^{r} \frac{r}{l} C_l \leq C_r. 
  \]

  $S_l$ sub-constellation associated with $p^{r-l}\mathbb{Z}_{p^r}$

  $C_l$ Shannon capacity of the $S_l$-AWGN channel.
Some answers to problem C

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  $S_l$ sub-constellation associated with $p^{r-l}\mathbb{Z}_{p^r}$
  $C_l$ Shannon capacity of the $S_l$-AWGN channel.

- **$8h$-PSK AWGN channel and $G = \mathbb{Z}_8$:**
  - $C'_{\mathbb{Z}_{p^r}} < C_r \Rightarrow \text{NO}$
  - What about $D_4$?
Open problems

• Analysis for non-Abelian generating groups.
• Semidirect group structures.
• Error exponents, minimum distances.