TWO-SCALE CONVERGENCE AND
OPTIMIZATION WITH COUPLE-WISE FORCES.

The aim of the works [MSA05] and [NARVIK04] is to apply the two-scale convergence method to a problem with coupled-wise applied forces. The problem has been considered in a classical work of E.Sanchez-Palencia (Lecture Notes in Physics N.127), using asymptotic development. We add a periodic non homogeneity of the medium. The external ordinary force has a macroscopic and microscopic behavior. The microscopic scale is periodic with small period of order ε. The coupled forces have the intensity depending on $1/\varepsilon$. The interest of the problem is related to the influence of two type of perturbation on the homogenized problem. The two scale convergence method gives the convergence of the gradient and an immediate estimate of the energy.

Let Ω be a regular domain of $\mathbf{R^3}$ and let $a_{ijkh}(y)$ be the symmetric, coercive and Y-periodic elasticity tensor. We note $\vec{F}(y)$ the couple force Y-periodic and such that:

$$\frac{1}{|Y|}\int_Y \vec{F}(y)\mathrm{d}y = 0$$

and $f(x)$ a continuous scalar function representing the intensity of the couples. We indicate $G(x,y)$ the external force Y-periodic in the y variable. The problem consists of the following equation:

$$-\frac{\partial}{\partial x_j}\left(a_{ijkh}(\frac{x}{\varepsilon})e_{kh}(\vec{u}^{\,\varepsilon})\right) - \frac{f(x)}{\varepsilon}F_i(\frac{x}{\varepsilon}) - G_i(x,\frac{x}{\varepsilon}) = 0.$$

with the boundary condition $\vec{u}^{\,\varepsilon}\,|_{\partial\Omega} = 0$. We prove that the solutions $u^\varepsilon(x)$ are uniformly bounded:

$$\|\vec{u}^{\,\varepsilon}(x)\|_{\left(H_0^1(\Omega)\right)^3} \leq C.$$

Then taking as test function:

$$\vec{v}(x,y) = \vec{v}_0(x) + \varepsilon\vec{v}_1(x,\frac{x}{\varepsilon})$$

the two-convergence limit gives us:

$$\int_\Omega\int_Y a_{ijkh}(y)\left[e_{khx}(\vec{u}_0(x)) + e_{khy}(\vec{u}_1(x,y))\right]\left[e_{khx}(\vec{v}_0(x)) + e_{khy}(\vec{v}_1(x,y))\right]\mathrm{d}x\mathrm{d}y +$$

$$-\int_\Omega\int_Y f(x)\vec{F}(y)\vec{u}_1(x,y)\mathrm{d}x\mathrm{d}y - \int_\Omega\int_Y \vec{G}(x,y)\vec{u}_0(x)\mathrm{d}x\mathrm{d}y = 0.$$

An integration by parts of the preceding formula gives the limit problem:

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left[a_{ijkh}(y)\left[e_{khx}(\vec{u}_0(x)) + e_{khy}(\vec{u}_1(x,y))\right]\right] - f(x)F_i(y) = 0 \\ -\dfrac{\partial}{\partial x_j}\left[\displaystyle\int_Y a_{ijkh}(y)\left[e_{khx}(\vec{u}_0(x)) + e_{khy}(\vec{u}_1(x,y))\right]\mathrm{d}y\right] - \displaystyle\int_Y G_i(x,y)\mathrm{d}y = 0 \end{cases}$$

The paper [TO05], in the framework of a problem related to an elastic non homogeneous medium, deals with a periodic couple force $(f(x)/\varepsilon^\alpha)\,\vec{F}\,(x/\varepsilon)$ with intensity of order $1/\varepsilon^\alpha$. The parameter ε is connected with the period of the non homogeneity of the medium and with the periodicity of the coupled force. The determination of the parameter α is the target of our study to obtain an effect in the microscopic limit equation. The homogenization technique as used in order to study the equation: $-\left(\partial/\partial x_j\right)\left(a_{ijkh}\,(x/\varepsilon)\,e_{kh}\,(\vec{u}^\varepsilon)\right) = \left(f(x)/\varepsilon^\alpha\right)F_i\,(x/\varepsilon) + G_i\,(x, x/\varepsilon)$, where $G_i\,(x, x/\varepsilon)$ is the volume applied force. The limit, when $\varepsilon \to 0$, of $\vec{u}^\varepsilon(x)$, in the meaning of two scale convergence, is $\left(\vec{u}^0(x), \vec{u}^1(x,y)\right)$ and the microscopic equation becomes: $-\left(\partial/\partial y_j\right)\left(a_{ijkh}\,(y)\,e_{khx}\left(\vec{u}^0(x)\right) + e_{khy}\left(\vec{u}^1(x,y)\right)\right) = f(x)F_i\,(y)$ if $\alpha = 1$, $-\left(\partial/\partial y_j\right)\left(a_{ijkh}\,(y)\,e_{khx}\left(\vec{u}^0(x)\right) + e_{khy}\left(\vec{u}^1(x,y)\right)\right) = 0$ if $0 < \alpha < 1$. When $\alpha > 1$ the solutions are not uniformly bounded respect to ε.