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THE SPHERICAL PALEY-WIENER THEOREM ON THE COMPLEX GRASSMANN MANIFOLDS $\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q)$

ROBERTO CAMPORESI

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ABSTRACT. We prove the Paley-Wiener theorem for the spherical transform on the complex Grassmann manifolds $U/K = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$. This theorem characterizes the K-biinvariant smooth functions f on the group U that are supported in the K-invariant ball of radius R, with R less than the injectivity radius of U/K, in terms of holomorphic extendability, exponential growth, and Weyl invariance properties of the spherical Fourier transforms $\hat{f}$, originally defined on the discrete set Λ_{sph} of highest restricted spherical weights.

1. INTRODUCTION

The Paley-Wiener theorem for the spherical transform on noncompact Riemannian symmetric spaces G/K was proved over 30 years ago by Helgason and Gangolli [8, 4].

This theorem characterizes the radial (i.e., K-invariant) smooth functions with compact support on G/K in terms of holomorphic extendability, exponential growth, and Weyl invariance of their spherical Fourier transforms.

The analogous problem for the compact dual spaces U/K is to describe the image under spherical Fourier transformation of the space of smooth radial functions f on U/K which are supported in the ball of radius R, where R is less than the injectivity radius of U/K. Since U/K is compact, the spherical transforms $\hat{f}$ are defined on a discrete set, namely the set Λ_{sph} of highest restricted spherical weights, which parameterizes the spherical representations of U.

The right candidate for the spherical Paley-Wiener space is known (see Theorem 2.1 below), and one would expect the result on U/K to follow from that on G/K by analytic continuation. However, it has not been possible, so far, to give a general proof of the compact result using this method. Particular cases where the spherical Paley-Wiener theorem has been proved are:

1) U/K of rank one (see [12], Theorem 5.1; see also [15, 1] for the case of spheres);

2) $U/K = K \times K/K \simeq K$, with K a compact simply connected semisimple Lie group embedded diagonally in $K \times K$, i.e., the group (or complex) case [6];

3) U/K of split-rank type, i.e., $\mathrm{rk}\, U = \mathrm{rk}\, U/K + \mathrm{rk}\, K$, or equivalently, when all restricted roots have even multiplicity [3].

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Case 3) generalizes case 2) and includes, in particular, all compact semisimple Lie groups. The split-rank case is rather special since there exists a differential operator intertwining the radial Laplacian on U/K to the ordinary Laplacian on the maximal torus of U/K, and yielding an explicit formula for the spherical functions in terms of exponential functions.

In this paper we prove the Paley-Wiener theorem for the complex Grassmann manifolds $U/K = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$. This case is, in a sense, a combination of the complex and rank one cases. There exists a differential operator intertwining the radial Laplacian on U/K to the radial Laplacian on a product of rank one symmetric spaces, and yielding an explicit formula for the spherical functions in terms of products of Jacobi polynomials. This formula (as well as the explicit description of the invariant differential operators) was given in a paper by Berezin and Karpelevic [2]. A proof of their result can be found in [11]. Using this formula Meaney [13] computed the inverse Abel transform on the noncompact dual spaces $G/K = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$.

By the same methods of [13], we shall obtain the following formula for the spherical transform $\hat{f}$ of $f \in C^\infty(U)_{rad}$:

$$C(\mu+\rho)\hat{f}(\mu) = \text{const.}\, \mathcal{F}\left(\left(\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^{1}(f\,\omega) \circ \exp(i\cdot)|_{\overline{\mathfrak{a}_0}}\right)^{\sim}\right)(\mu+\rho) \qquad (\mu \in \Lambda_{sph}),$$

where C is a suitable polynomial function, $\mathcal{F}$ is the usual Fourier transform on $\mathfrak{a} \simeq \mathbb{R}^p$, $\mathbf{W}_\nu^\sigma$ is a tensor product of one-dimensional fractional operators of Riemann-Liouville type, ω^2 is a part of the density function for the Cartan decomposition of U/K, and ρ is half the sum of the positive restricted roots. Moreover, $\mathfrak{a}_0$ is the maximal open domain of $\mathfrak{a}$ such that the map $X \to \exp(iX)$ is a diffeomorphism, and given a function g on $\mathfrak{a}_0$ we denote by $\tilde{g}$ the extension of g to $\mathfrak{a}$ that takes the value 0 outside $\mathfrak{a}_0$.

The Paley-Wiener theorem then follows by combining the classical Paley-Wiener theorem on $\mathbb{R}^p$ with the results of Koornwinder in [12] about the Jacobi (polynomial) transform.

This paper is organized as follows. In section 2 we briefly recall the definition of the spherical transform and the inversion formula for a compact symmetric space U/K. Then we state the spherical Paley-Wiener theorem. For simplicity we assume U is simply connected, but the statement can be generalized to any Riemannian symmetric space of the compact type U/K (see, e.g., [3], for the split-rank case).

In section 3 we recall the basic facts about restricted roots and spherical weights for the complex Grassmann manifolds, and give the formula for the spherical functions.

In section 4 we compute the spherical transform as outlined above, and prove the Paley-Wiener theorem.

2. The spherical transform on compact symmetric spaces

Let U be a compact semisimple simply connected Lie group, K the (necessarily connected) fixed point group of a (nontrivial) involutive automorphism of U, and U/K the corresponding Riemannian symmetric space of the compact type. Let G/K be the noncompact dual symmetric space, with G and U analytic subgroups of the complex semisimple simply connected Lie group $G^{\mathbb{C}} = U^{\mathbb{C}}$ whose Lie algebra is the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G.

We regard the Lie algebra $\mathfrak{u}$ of U as the subspace of $\mathfrak{g}^{\mathbb{C}}$ given by $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{p}$ being the orthogonal complement of $\mathfrak{k} = \mathrm{Lie}(K)$ in $\mathfrak{g}$ with respect to the Killing form.

Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$, let $A = \exp \mathfrak{a} \subset G$, and $A' = \exp i\mathfrak{a} \subset U$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of restricted roots, and $W = W(\mathfrak{g}, \mathfrak{a})$ the associated Weyl group. Fix a Weyl chamber in $\mathfrak{a}$, and let Σ^+ be the corresponding system of positive restricted roots. We denote by $\langle , \rangle$ the inner products on $\mathfrak{a}, \mathfrak{a}^*$ induced by the Killing form, and by $| \cdot |$ the corresponding norm. We define the inner product on $\mathfrak{p}' = i\mathfrak{p}$ so that $|iX| = |X|$, $\forall X \in \mathfrak{p}$, and induce the corresponding Riemannian structure on U/K.

Let Λ_{sph} be the set of highest restricted spherical weights, given by

$$\Lambda_{sph} = \{\mu \in \mathfrak{a}^* : \quad \frac{\langle \mu, \alpha \rangle}{|\alpha|^2} \in \mathbb{Z}^+, \ \text{ for } \alpha \in \Sigma^+\}.$$

For each $\mu \in \Lambda_{sph}$ let δ_μ be an irreducible spherical representation of U with highest restricted weight μ, acting in a space H_μ with inner product $\langle , \rangle$. Let ψ_μ be the associated spherical function on U given by $\psi_\mu(u) = \langle \delta_\mu(u^{-1}) v_K, v_K \rangle$ $(u \in U)$, where $v_K \in H_\mu$ is a unit vector fixed under $\delta_\mu(K)$.

The space $C^\infty(U)_{rad}$ of smooth K-biinvariant functions on U may be identified with the space of smooth radial (i.e., K-invariant) functions on U/K. Define the spherical transform of $f \in C^\infty(U)_{rad}$ by

$$\hat{f}(\mu) = \int_U f(u) \psi_\mu(u^{-1}) du \qquad (\mu \in \Lambda_{sph}), \tag{2.1}$$

where du is the normalized Haar measure on U. Then one has the inversion formula

$$f = \sum_{\mu \in \Lambda_{sph}} d_\mu \, \hat{f}(\mu) \psi_\mu, \tag{2.2}$$

where $d_\mu = \dim H_\mu$. Let R_0 be the injectivity radius of U/K. (If U/K is irreducible, then it follows from [9], Thm. 11.2, p. 334, that $R_0 = \pi/|\delta|$, where δ is the highest restricted root.) For $0 < R < R_0$, let $C^\infty_R(U)_{rad}$ be the space of functions $f \in C^\infty(U)_{rad}$ that are supported in the (closed) K-invariant ball of radius R in U, namely the set

$$B_R(e) = \{k_1 a k_2 : \ \ k_1, k_2 \in K, \ \ a = \exp(iH), \ \ H \in \mathfrak{a}, \ \ |H| \leq R\}.$$

Let $\mathfrak{a}^*_{\mathbb{C}}$ be the set of complex-valued linear functions on $\mathfrak{a}$, and let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, where m_α is the multiplicity of the restricted root α. The Paley-Wiener theorem states that a function f is in $C^\infty_R(U)_{rad}$ if and only if its spherical Fourier transform $\hat{f}$, originally defined on Λ_{sph}, extends to a holomorphic function $\hat{f}$ on $\mathfrak{a}^*_{\mathbb{C}}$ of exponential type R (cf. [10], p. 15) satisfying

$$\hat{f}(s(\lambda + \rho) - \rho) = \hat{f}(\lambda), \quad \forall s \in W, \ \forall \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{2.3}$$

More precisely, letting $PW_R(\mathfrak{a}^*_{\mathbb{C}})_W$ be the space of holomorphic functions of exponential type R on $\mathfrak{a}^*_{\mathbb{C}}$ satisfying (2.3), we have

Theorem 2.1 (Spherical Paley-Wiener theorem)**.** *Let $0 < R < R_0$. For each $f \in C^\infty_R(U)_{rad}$ the spherical transform $\hat{f}$ extends to a holomorphic function $\hat{f} \in PW_R(\mathfrak{a}^*_{\mathbb{C}})_W$ such that the map $f \to \hat{f}$ is a bijection of $C^\infty_R(U)_{rad}$ onto $PW_R(\mathfrak{a}^*_{\mathbb{C}})_W$.*

As mentioned in the Introduction, this theorem has not yet been proved for general symmetric spaces U/K, but only in some particular cases (rank-one, complex, split-rank). In section 4 we shall give a proof for $U/K = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$.

3. Spherical functions on complex Grassmann manifolds

Let $U = \mathrm{SU}(p+q)$, let $K = \mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$, and let U/K be the associated complex Grassmann manifold. As in [13], we fix $q \geq p \geq 2$ and let $k = q - p \geq 0$. The noncompact dual space is G/K, where $G = \mathrm{SU}(p,q)$, with Lie algebra $\mathfrak{g} = \mathfrak{su}(p,q)$. We fix the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ as in [11]. We identify the Lie algebra $\mathfrak{u} = \mathfrak{su}(p+q)$ of U with the subspace $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$, and let $A = \exp \mathfrak{a} \subset G$, $A' = \exp i\mathfrak{a} \subset U$.

Since $\operatorname{rk} G/K = p$, we can identify $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ with $\mathbb{R}^p$ as follows. For a suitable definition of the linear functionals β_j $(1 \leq j \leq p)$ on $\mathfrak{a}$, we have the following standard choice of positive restricted roots:

$$\Sigma^+ = \{\beta_j,\, \tfrac{1}{2}\beta_j \quad (1 \leq j \leq p), \qquad \tfrac{1}{2}(\beta_i \pm \beta_j) \quad (1 \leq i < j \leq p)\},$$

with multiplicities $m_{\beta_j} = 1$, $m_{\beta_j/2} = 2k$, and $m_{(\beta_i \pm \beta_j)/2} = 2$ (see, e.g., [11]). Given the longer roots $\beta_1, \dots, \beta_p$ in Σ^+, let $X_1, \dots, X_p$ be the elements of $\mathfrak{a}$ such that $\beta_i(X_j) = \delta_{ij}$. We then identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}^p$ by identifying $\{\beta_j\}$ and $\{X_j\}$ with the canonical basis of $\mathbb{R}^p$. We normalize the inner product on $\mathfrak{a}, \mathfrak{a}^*$ (induced by a multiple of the Killing form) so that

$$\langle \beta_i, \beta_j \rangle = \langle X_i, X_j \rangle = \delta_{ij} \qquad (1 \leq i, j \leq p).$$

Let $|\cdot|$ be the corresponding norm on $\mathfrak{a}, \mathfrak{a}^*$, and let the inner product on $\mathfrak{p}' = i\mathfrak{p}$ be such that $|iX| = |X|$ $(X \in \mathfrak{p})$. This corresponds to the normalization of the Riemannian metric on U/K in which the injectivity radius R_0 is just π (the highest restricted root being $\delta = \beta_1$, so that $|\delta| = 1$).

The positive Weyl chamber of $\mathfrak{a} \simeq \mathbb{R}^p$ corresponding to the given choice of positive roots is

$$\mathcal{C}^+ = \{X = (x_1, \dots, x_p) : \quad x_1 > x_2 > \cdots > x_p > 0\}.$$

The Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ is given by [11]

$$W = \{s : \;\; s(x_1, \dots, x_p) = (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_p x_{\sigma(p)}), \;\; \varepsilon_j = \pm 1, \;\; \sigma \in S_p\},$$

i.e., W is the semidirect product of the permutation group S_p, acting by $x_j \to x_{\sigma(j)}$, and the group $\{\pm 1\}^p$, acting by $x_j \to \varepsilon_j x_j$. Note that $|W| = p! 2^p$.

As is well known, the set Λ_{sph} is given by

$$\Lambda_{sph} = \{\, \mu = \sum_{j=1}^{p} r_j \pi_j, \;\; r_j \in \mathbb{Z}^+, \;\; \forall 1 \leq j \leq p\},$$

where the fundamental spherical weights $\pi_1, \dots, \pi_p$ are given by

$$\pi_j = (\overbrace{1, 1, \dots, 1}^{j}, 0, \dots, 0) = \beta_1 + \cdots + \beta_j \qquad (1 \leq j \leq p).$$

We can then write any $\mu \in \Lambda_{sph}$ in the equivalent forms

$$\mu = \sum_1^p r_j \pi_j = (r_1 + \cdots + r_p\,,\, r_2 + \cdots + r_p\,, \ldots,\, r_{p-1} + r_p\,,\, r_p)$$
$$\equiv (m_1, m_2, \ldots, m_p) = \sum_1^p m_j \beta_j,$$

where the numbers $m_j \in \mathbb{Z}^+$ satisfy $m_1 \geq m_2 \geq \cdots \geq m_p \geq 0$.

Let ρ, ρ_0 be the half sums of the positive restricted roots for, respectively, U/K, and the symmetric space which is the product of p copies of the complex projective space $P^{2k+2}(\mathbb{C})$. This is a reducible symmetric space with positive roots $\beta_j, \frac{1}{2}\beta_j$ and multiplicities $m_{\beta_j} = 1$, $m_{\beta_j/2} = 2k$ $(1 \leq j \leq p)$. We easily get

$$\rho_0 = (\tfrac{k+1}{2}, \ldots, \tfrac{k+1}{2}), \qquad \rho = \rho_0 + (p-1, p-2, \ldots, 1, 0).$$

In order to write the Berezin and Karpelevic formula, consider the spherical functions on $P^{2k+2}(\mathbb{C})$ in the normalization in which the diameter is π. These are normalized Jacobi polynomials of indices $(k, 0)$:

$$\phi_n(\exp ixH) = \tilde{P}_n(\cos x) = \frac{P_n^{(k,0)}(\cos x)}{P_n^{(k,0)}(1)} = F(-n, n+k+1, k+1, \sin^2 \frac{x}{2}),$$

where $n \in \mathbb{Z}^+$, x is the Riemannian distance from the point $\exp(ixH)x_0$ to the origin x_0, and $F(a, b, c, z)$ is the usual hypergeometric function. (See [10], Thm. 4.5, p. 543. Here $H \in \mathfrak{a}_1$ is the element such that $\beta(H) = 1$, where β is the longer root in Σ_1^+, the subscript 1 denoting rank one.) The functions $\tilde{P}_n(\cos x)$ are eigenfunctions of the radial Laplacian on $P^{2k+2}(\mathbb{C})$ with eigenvalue $-c(n)$, where

$$c(n) = n(n+k+1) = (n + \tfrac{k+1}{2})^2 - (\tfrac{k+1}{2})^2. \tag{3.1}$$

The Berezin and Karpelevic formula gives the spherical functions on U/K, $\psi_\mu(\exp iX)$, $X \in \mathcal{C}^+$, in terms of a determinant whose entries are the polynomials $\tilde{P}_{n_i}(\cos x_j)$, namely

$$\psi_\mu(\exp iX) = c\, \frac{\det\left(\tilde{P}_{n_i}(\cos x_j)\right)_{1 \leq i,j \leq p}}{\omega(\exp iX) \prod_{i<j}(c(n_i) - c(n_j))}, \tag{3.2}$$

where

$$\omega(\exp iX) = \prod_{i<j}(\cos x_i - \cos x_j), \tag{3.3}$$

$$c = 2^{p(p-1)/2} \prod_{j=1}^{p-1} j!(j+k)^{p-j},$$

and the indices $n_1, \ldots, n_p$ satisfy $n_1 > n_2 > \cdots > n_p \geq 0$ and are related to the indices m_j of $\mu = \sum m_j \beta_j$ by $n_j = m_j + p - j$. That is,

$$(n_1, n_2, \ldots, n_p) = (m_1 + p - 1,\, m_2 + p - 2, \ldots,\, m_p) = \mu + \rho - \rho_0. \tag{3.4}$$

The value of the constant c is determined by the requirement that $\psi_\mu(e) = 1$ using, e.g., [11], Lemma 4.1. One can also check that for $\mu = 0$ one gets $\psi_0(\exp iX) = 1$, $\forall X \in \mathfrak{a}$.

4. The spherical transform and the Paley-Wiener theorem

We keep the notations of section 3. The spherical transform of $f \in C^\infty(U)_{rad}$ is defined in (2.1). In our case the spherical functions are real, so $\psi_\mu(u^{-1}) = \overline{\psi_\mu(u)} = \psi_\mu(u)$. Using the integral formula for the polar decomposition $U = KA'K$ (see [7], Prop. 1.19, p. 385) we obtain

$$\hat{f}(\mu) = c \int_{A'} (f\,\psi_\mu)(a')|D(a')|da',$$

where c is a fixed constant depending only on the normalization of the Haar measures, and the density function $D(a')$ ($a' \in A'$) is given by

$$D(\exp iX) = \prod_{\alpha\in\Sigma^+} (\sin\alpha(X))^{m_\alpha} = 2^{-p(p-1)}\,\omega^2(\exp iX) \prod_{j=1}^{p} \Delta(x_j) \qquad (X \in \mathfrak{a}).$$

Here ω is given by (3.3), and $\Delta(x_j) = \sin x_j (\sin\frac{x_j}{2})^{2k}$ is the density function for the space $P^{2k+2}(\mathbb{C})$ corresponding to the j-th coordinate on $\mathfrak{a}$.

Let $\mathfrak{a}_0$ be the maximal open domain of $\mathfrak{a}$ such that the map $X \to \exp(iX)$ is a diffeomorphism of $\mathfrak{a}_0$ onto $A' = \exp(i\mathfrak{a})$ minus a zero measure set. Then

$$\int_{A'} F(a')da' = \int_{\mathfrak{a}_0} F(\exp iX) dx_1 \cdots dx_p$$

for all $F \in C(A')$, and we can rewrite $\hat{f}(\mu)$ as

$$\hat{f}(\mu) = \text{const.} \int_{\mathfrak{a}_0} (f\,\omega^2\,\psi_\mu)(\exp iX) \prod_{j=1}^{p} |\Delta(x_j)|\, dx_1 \cdots dx_p. \tag{4.1}$$

The domain $\mathfrak{a}_0$ can be described explicitly as follows. Since $U = \mathrm{SU}(p+q)$ is simply connected and U/K is irreducible, we have $\overline{\mathfrak{a}_0} = W\overline{Q_0}$, where Q_0 is the polyhedron

$$\begin{aligned} Q_0 =& \{X \in \mathfrak{a} : \ \ \mu_j(X) > 0,\ \forall 1 \le j \le p,\ \ \delta(X) < \pi\} \\ =& \{X = (x_1, \ldots, x_p) \in \mathbb{R}^p : \ \ \pi > x_1 > x_2 > \cdots > x_p > 0\}. \end{aligned}$$

(See [10], p. 191. Here $\{\mu_j\}$ are the simple restricted roots.) Thus $\overline{\mathfrak{a}_0}$ is the union of $p!2^p$ copies of $\overline{Q_0}$, and we have $\mathfrak{a}_0 \simeq (-\pi,\pi)^p$. Note that if F is a Weyl invariant function on $\mathfrak{a}_0$, then $\int_{\mathfrak{a}_0} F = p!2^p \int_{Q_0} F$. If the function F is invariant under the subgroup $\{\pm 1\}^p$ of W, then $\int_{\mathfrak{a}_0} F = 2^p \int_{C_0} F$, where $C_0 \simeq (0,\pi)^p$ is the domain such that $\overline{\mathfrak{a}_0} = \{\pm 1\}^p \overline{C_0}$, namely $\overline{C_0} = S_p\,\overline{Q_0}$.

Substitution of (3.2) in (4.1) gives

$$\begin{aligned} \prod_{i<j} (c(n_i) - c(n_j))\, \hat{f}(\mu) =&\ \text{const.} \int_{\mathfrak{a}_0} (f\,\omega)(\exp iX) \\ &\times\ \det\left(\tilde{P}_{n_i}(\cos x_j)\right) \prod_{j=1}^{p} |\Delta(x_j)|\, dx_1 \cdots dx_p. \end{aligned} \tag{4.2}$$

The right-hand side of (4.2) is a sum over all permutations $\sigma \in S_p$ of integrals

$$\int_{\mathfrak{a}_0} (f\,\omega)(\exp iX)(-1)^\sigma \prod_{j=1}^{p} \left(\tilde{P}_{n_j}(\cos x_{\sigma(j)})|\Delta(x_{\sigma(j)})|\right) dx_1 \cdots dx_p$$
$$= \int_{\mathfrak{a}_0} (f\,\omega)(\exp iX) \prod_{j=1}^{p} \left(\tilde{P}_{n_j}(\cos x_j)|\Delta(x_j)|\right) dx_1 \cdots dx_p,$$

where we have used the fact that $\omega \circ \exp(i\cdot)$ is skew under S_p:

$$\omega(\exp i\sigma X) = (-1)^\sigma \omega(\exp iX), \quad \forall \sigma \in S_p, \quad X \in \mathfrak{a}.$$

The spherical transform of any $f \in C^\infty(U)_{rad}$ is then given by

$$\prod_{i<j}(c(n_i) - c(n_j))\,\hat{f}(\mu) = \text{const.} \int_{\mathfrak{a}_0} (f\,\omega)(\exp iX) \prod_{j=1}^{p} \left(\tilde{P}_{n_j}(\cos x_j)|\Delta(x_j)|\right) dx_1 \cdots dx_p. \tag{4.3}$$

Since the integrand is invariant under $\{\pm 1\}^p$, we can restrict the integration in (4.3) to the domain $C_0 \simeq (0,\pi)^p$.

We now recall some results of Koornwinder [12]. Let f be a smooth even function on $[-\pi,\pi]$, and define its Fourier-Jacobi transform with respect to $\tilde{P}_n(\cos\theta)$ by

$$\hat{f}(n) = \int_0^\pi f(\theta)\tilde{P}_n(\cos\theta)\Delta(\theta)d\theta \qquad (n \in \mathbb{Z}^+). \tag{4.4}$$

The polynomials $\tilde{P}_n(\cos\theta)$ admit the integral representation given in [12], formula (5.7), with $(\alpha,\beta) = (k,0)$, where we take $k > 0$ for now. (Note the misprint in the inner integral in $d\varphi$, which should be from 0 to ψ rather than from 0 to φ. Compare with formula (11) of [5].) Using (5.7) of [12] in (4.4) we get (cf. [12], formula (5.10))

$$\hat{f}(n) = \tfrac{2^{k+1/2}k!}{\sqrt{\pi}} \int_0^\pi \cos\left((n + \tfrac{k+1}{2})\varphi\right) \left(W_k^{1/2} \circ W_{1/2}^1 f\right)(\varphi)d\varphi, \tag{4.5}$$

$\forall f \in C^\infty([-\pi,\pi])_{even}$, where the operator W_ν^σ ($\sigma = 1/2, 1$) is defined for all $\nu > 0$ by

$$(W_\nu^\sigma f)(\varphi) = \Gamma(\nu)^{-1} \int_\pi^\varphi f(\theta)(\cos\sigma\varphi - \cos\sigma\theta)^{\nu-1} d(\cos\sigma\theta) \quad (0 \le \varphi < \pi). \tag{4.6}$$

This is a fractional integral of Riemann-Liouville type (see, e.g., [14]). It is analogous to the operator in (3.10) of [12], with the hyperbolic functions replaced by trigonometric ones, and with the boundary point at infinity in the upper limit of integration replaced by π. Note that $\lim_{\varphi\to\pi} W_\nu^\sigma f(\varphi) = 0, \forall f \in C([-\pi,\pi]), \forall \nu > 0$.

If f is supported in $[-R,R]$ with $0 < R < \pi$, i.e., $f \in C_R^\infty([-\pi,\pi])_{even}$, then we can define W_ν^σ ($\sigma = 1/2, 1$) for $\nu \le 0$ by

$$(W_\nu^\sigma f)(\varphi) = \Gamma(\nu+n)^{-1} \int_\pi^\varphi \partial^n_{\cos\sigma\theta} f(\theta)(\cos\sigma\varphi - \cos\sigma\theta)^{n+\nu-1} d(\cos\sigma\theta), \tag{4.7}$$

where $n \in \mathbb{Z}^+$ is such that $\nu + n > 0$, and $\partial_x^n = (\partial/\partial x)^n$. This amounts to integrating by parts n times in (4.6). In particular, we get $W_0^\sigma f = f$ and $W_{-j}^\sigma f(\varphi) = \partial^j_{\cos\sigma\varphi} f(\varphi)$. As in the noncompact case one has the semigroup property

$$W_\nu^\sigma \circ W_{\nu'}^\sigma f = W_{\nu+\nu'}^\sigma f, \qquad \forall \nu,\nu', \ \forall f \in C_R^\infty([-\pi,\pi])_{even}. \tag{4.8}$$

It follows that W_ν^σ is a bijection of $C_R^\infty([-\pi,\pi])_{even}$ onto itself, with inverse given by $W_{-\nu}^\sigma$ (cf. [12], remark 8). For general functions $f \in C^\infty([-\pi,\pi])_{even}$ one defines $W_\nu^\sigma f$ for $\nu<0$ by the same procedure, and one gets n boundary terms in the right-hand side of (4.7). The semigroup property (4.8) then holds for all ν,ν' except when $\nu'=-j$ ($j\in\mathbb{Z}^+$); see [14], p. 14.

Note that (4.5) also holds for $k=0$ (with $W_0^{1/2}=\mathrm{Id}$). Indeed for $k=0$ the polynomials $\tilde{P}_n(\cos\theta)$ reduce to the Legendre polynomials $P_n(\cos\theta)$. Using the Dirichlet-Mehler integral representation of $P_n(\cos\theta)$ (see [5], formula (9)) in (4.4), one gets precisely (4.5) with $k=0$.

We now go back to (4.3). Using (4.4) and (4.5) for each variable $x_1,\dots,x_p$ in the integral over $C_0=(0,\pi)^p$, we get for all $f\in C^\infty(U)_{rad}$

$$\prod_{i<j}(c(n_i)-c(n_j))\,\hat{f}(\mu)=\text{const.}\int_{(0,\pi)^p}\mathbf{W}_k^{1/2}\circ\mathbf{W}_{1/2}^1((f\,\omega)\circ\exp(i\cdot))(X)$$
$$\times\prod_{j=1}^p\cos((n_j+\tfrac{k+1}{2})x_j)\,dx_1\cdots dx_p, \tag{4.9}$$

where $\mathbf{W}_\nu^\sigma$ is the tensor product of the one-dimensional operators $W_\nu^\sigma(x_j)$ ($1\le j\le p$). It is defined for $\sigma=1/2,1$ and $\nu>0$ by the formula

$$\begin{aligned}\mathbf{W}_\nu^\sigma g(X)&=W_\nu^\sigma(x_1)\otimes\cdots\otimes W_\nu^\sigma(x_p)\,g(x_1,\dots,x_p)\\&=\Gamma(\nu)^{-p}\int_\pi^{x_1}\cdots\int_\pi^{x_p}g(y_1,\dots,y_p)\prod_{j=1}^p(\cos\sigma x_j-\cos\sigma y_j)^{\nu-1}d(\cos\sigma y_j),\end{aligned}$$

where $0\le x_j<\pi$ and g is a smooth function on $\overline{\mathfrak{a}_0}\simeq[-\pi,\pi]^p$ invariant under $\{\pm1\}^p$, i.e., $g\in C^\infty([-\pi,\pi]^p)_{\{\pm1\}^p}$. Again if g is supported in the ball

$$B_R(0)=\{X\in\mathfrak{a}\simeq\mathbb{R}^p:\ \ |X|\le R\},$$

with $0<R<\pi$, we can define $\mathbf{W}_\nu^\sigma$ for $\nu<0$ by

$$\begin{aligned}\mathbf{W}_\nu^\sigma g(X)=\Gamma(\nu+n)^{-p}\int_\pi^{x_1}\cdots\int_\pi^{x_p}&\left(\prod_{j=1}^p\partial^n_{\cos\sigma y_j}g(Y)\right)\\&\times\prod_{j=1}^p(\cos\sigma x_j-\cos\sigma y_j)^{n+\nu-1}d(\cos\sigma y_j),\end{aligned}$$

where $\nu+n>0$. Then the semigroup property holds, and $\mathbf{W}_\nu^\sigma$ is a bijection of $C_R^\infty([-\pi,\pi]^p)_{\{\pm1\}^p}$ onto itself, with inverse $\mathbf{W}_{-\nu}^\sigma$, for all $0<R<\pi$.

Let $C:\mathfrak{a}^*\to\mathbb{R}$ be the polynomial function defined by

$$C(\lambda)=\prod_{i<j}(\lambda_i^2-\lambda_j^2),\quad\text{ where }\ \lambda=\sum_{j=1}^p\lambda_j\beta_j\in\mathfrak{a}^*.$$

Then $\prod_{i<j}(c(n_i)-c(n_j))=C(\mu+\rho)$, since $(\mu+\rho)_j=n_j+\frac{k+1}{2}$ (cf. (3.1) and (3.4)). The function $C(\lambda)$ is skew under S_p and invariant under $\{\pm1\}^p$.

For any function g on $\mathfrak{a}_0$ define the function $\tilde{g}$ on $\mathfrak{a}$ by

$$\tilde{g}(X) = \begin{cases} g(X), & \text{for } X \in \mathfrak{a}_0, \\ 0, & \text{for } X \notin \mathfrak{a}_0. \end{cases}$$

We can then rewrite formula (4.9) as follows: for all $\mu \in \Lambda_{sph}$,

$$C(\mu+\rho)\hat{f}(\mu) = \text{const.}\, \mathcal{F}\left(\left(\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^1(f\,\omega) \circ \exp(i\cdot)|_{\overline{\mathfrak{a}_0}}\right)^{\sim}\right)(\mu+\rho), \tag{4.10}$$

where we have used the invariance of the integrand under $\{\pm 1\}^p$ to replace the multidimensional cosine transform by the usual Fourier transform $\mathcal{F}$ on $\mathfrak{a} \simeq \mathbb{R}^p$. Formula (4.10) holds for any $f \in C^\infty(U)_{rad}$. Note that the function which is acted upon by $\mathcal{F}$ in (4.10) is skew under S_p and invariant under $\{\pm 1\}^p$. We can now prove the Paley-Wiener theorem (Theorem 2.1).

Let $f \in C_R^\infty(U)_{rad}$, with $0 < R < \pi$. Then $f|_{A'}$ is supported in the ball $B_R(e) \cap A'$, and the function $(f\omega) \circ \exp(i\cdot)|_{\overline{\mathfrak{a}_0}}$ is supported in $B_R(0) \subset \overline{\mathfrak{a}_0}$. Since $\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^1$ is a bijection of $C_R^\infty(\overline{\mathfrak{a}_0})_{\{\pm 1\}^p}$ onto itself, the function

$$\left(\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^1(f\,\omega) \circ \exp(i\cdot)|_{\overline{\mathfrak{a}_0}}\right)^{\sim}$$

is in $C_R^\infty(\mathfrak{a})_{\{\pm 1\}^p}$. By the "easy" part of the Paley-Wiener theorem in $\mathbb{R}^p$, the function on the right-hand side of (4.10) extends to a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}^p$ of exponential type R in $\mu + \rho$, hence in μ, which is invariant under $\{\pm 1\}^p$ and skew under S_p (these properties being preserved by $\mathcal{F}$). Let us denote this function by F, so that the right-hand side of (4.10) is $F(\mu+\rho)$. Clearly $F(\lambda)$ vanishes at the hyperplanes $\lambda_i = \pm\lambda_j$, and it is divisible by $C(\lambda)$. Then the function $\hat{f}(\mu)$ itself extends to a holomorphic function $\hat{f}$ on $\mathfrak{a}_{\mathbb{C}}^*$ such that $C(\lambda)\hat{f}(\lambda-\rho) = F(\lambda)$, $\forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Since $C(\lambda)$ is a polynomial, the function $\hat{f}$ is of exponential type R (see, e.g., [3], Lemma 3.3). Finally, since the functions $C(\lambda)$ and $F(\lambda)$ are both skew under S_p and invariant under $\{\pm 1\}^p$, the function $\lambda \to \hat{f}(\lambda-\rho)$ must be Weyl invariant, or equivalently, (2.3) holds.

We have proved that the map $f \to \hat{f}$ maps $C_R^\infty(U)_{rad}$ into $PW_R(\mathfrak{a}_{\mathbb{C}}^*)_W$. This map is clearly injective (by the inversion formula (2.2)). There remains to prove the surjectivity.

Let $h \in PW_R(\mathfrak{a}_{\mathbb{C}}^*)_W$, with $0 < R < \pi$, and define the function f on U by

$$f = \sum_{\mu \in \Lambda_{sph}} d_\mu\, h(\mu)\, \psi_\mu.$$

It is easy to see that f is smooth and K-biinvariant, i.e., $f \in C^\infty(U)_{rad}$, and moreover $\hat{f}(\mu) = h(\mu)$, $\forall \mu \in \Lambda_{sph}$ (cf. [3], Lemma 3.10). To prove that $\operatorname{supp} f \subset B_R(e)$ we reason as follows. The function $\lambda \to h(\lambda-\rho)$ is W-invariant. Since C is a polynomial, the function $\lambda \to C(\lambda)h(\lambda-\rho)$ is holomorphic of exponential type R, and it is skew under S_p and invariant under $\{\pm 1\}^p$. By the "difficult" part of the Paley-Wiener theorem in $\mathbb{R}^p$, there exists a function $g_1 \in C_R^\infty(\mathfrak{a})_{\{\pm 1\}^p}$ such that $\mathcal{F}g_1 = C(\cdot)h(\cdot - \rho)$. Since $\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^1$ is a bijection of $C_R^\infty(\overline{\mathfrak{a}_0})_{\{\pm 1\}^p}$ onto itself, there exists a function $g_2 \in C_R^\infty(\mathfrak{a})_{\{\pm 1\}^p}$ (skew under S_p like g_1) such that $\left(\mathbf{W}_k^{1/2} \circ \mathbf{W}_{1/2}^1\, g_2|_{\overline{\mathfrak{a}_0}}\right)^{\sim} = g_1$. Define a skew function g_3 on A' by

$$g_3(\exp iX) = \sum_{\gamma \in \mathfrak{a}_K} g_2(X+\gamma) \qquad (X \in \mathfrak{a}), \tag{4.11}$$

where $\mathfrak{a}_K = \{\gamma \in \mathfrak{a} : \ \exp i\gamma \in K\}$. Since U is simply connected, by [9], Thm. 8.5, p. 322, we have $\mathfrak{a}_K = \mathfrak{a}_\Sigma = \frac{1}{2}\mathfrak{a}_e$, where $\mathfrak{a}_e = \{\gamma \in \mathfrak{a} : \ \exp i\gamma = e\}$ is the unit lattice, and $\mathfrak{a}_\Sigma$ is the lattice in $\mathfrak{a}$ spanned by the vectors $2\pi A_\alpha/|\alpha|^2$ ($\alpha \in \Sigma$), where $A_\alpha \in \mathfrak{a}$ is determined as usual by $\langle X, A_\alpha\rangle = \alpha(X)$, $\forall X \in \mathfrak{a}$. In our case one easily computes $\mathfrak{a}_\Sigma = \{2\pi \sum n_j\beta_j, \ n_j \in \mathbb{Z}\} = 2\pi\mathbb{Z}^p$. Since $\mathfrak{a}_e = 4\pi\mathbb{Z}^p \subset 2\pi\mathbb{Z}^p$, g_3 is a well-defined function on $A' \simeq i\mathfrak{a}/i\mathfrak{a}_e$. Moreover, g_3 defines a function on the "maximal torus" $A' \cdot x_0 \simeq i\mathfrak{a}/i\mathfrak{a}_K$ of U/K (x_0 the origin of U/K). Since $g_3 \circ \exp(i\cdot)$ is skew under S_p and invariant under $\{\pm 1\}^p$ and under translations in the lattice $\mathfrak{a}_K$, the function g_3 must be divisible by the function ω in A'. This can be proved as in [10], p. 504.

We then define a W-invariant function g on A' by $g_3 = g\,\omega$, or more precisely

(4.12)
$$(g\,\omega)(\exp iX) = \text{const.} \sum_{\gamma \in \mathfrak{a}_K} \left(\mathbf{W}^1_{-1/2} \circ \mathbf{W}^{1/2}_{-k}\left(\mathcal{F}^{-1}\left(C(\cdot)h(\cdot - \rho)\right)\right)|_{\overline{\mathfrak{a}_0}}\right)^{\sim}(X + \gamma),$$

where the constant is the same as in the right-hand side of (4.10). In (4.11) and (4.12) only one term in the $\sum_\gamma$ is nonzero for each $X \in \mathfrak{a}$, the function g_2 in the sum being supported in $B_R(0)$. Clearly g is supported in $B_R(e) \cap A'$, and we can define $g \in C^\infty_R(U)_{rad}$ by $g(k_1 a' k_2) = g(a')$ ($k_1, k_2 \in K$, $a' \in A'$). The spherical transform of g is easily computed to be $\hat{g}(\mu) = h(\mu) = \hat{f}(\mu)$, using the fact that $C(\mu + \rho) \neq 0$, $\forall \mu \in \Lambda_{sph}$. Thus in fact $g = f$.

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Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail address: camporesi@polito.it